Thèse de doctorat
Pour obtenir le grade de Docteur de
l'UNIVERSITÉ POLYTECHNIQUE HAUTS-DE-FRANCE
et de l'INSA HAUTS-DE-FRANCE
et du titre de Docteur en Sciences de
l'UNIVERSITÉ DE MONS

Discipline : **Mathématiques et leurs interactions**

Présentée et soutenue par Damien GALANT

Le 09/12/2024, à Valenciennes

École doctorale :

École Doctorale Polytechnique Hauts-de-France (ED PHF n°635)

Unité de recherche :

Laboratoire de Matériaux Céramiques et de Mathématiques (CERAMATHS), département DMATHS

L'équation de Schrödinger non-linéaire sur les graphes métriques

JURY

Président du jury

- Menet, Quentin. Chargé de cours. Université de Mons, Mons, Belgique.

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PhD Thesis

Submitted for the degree of Doctor of Philosophy from

UNIVERSITÉ POLYTECHNIQUE HAUTS-DE-FRANCE

And INSA HAUTS-DE-FRANCE

and the title of Docteur en Sciences from the

UNIVERSITÉ DE MONS

Subject : **Mathematics and their interactions**

Presented and defended by Damien GALANT

On 09/12/2024, at Valenciennes

Doctoral school :

Doctoral School Polytechnique Hauts-de-France (ED PHF n°635)

Research unit :

Laboratory of Ceramic Materials and Mathematics (CERAMATHS), DMATHS Department

The nonlinear Schrödinger equation on metric graphs

JURY

Jury president

- Menet, Quentin. “Chargé de cours”. Université de Mons, Mons, Belgium.

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Résumé

Dans cette thèse, nous étudions l'équation de Schrödinger non-linéaire

$$-\Delta u + \lambda u = |u|^{p-2}u \tag{NLS}$$

où $\Delta := \sum_{1\leq i\leq N} \partial_{ii}$ est le laplacien, $p, \lambda \in \mathbb{R}$ et $p > 2$. L'équation est considérée sur des ouverts de $\mathbb{R}^N$ ou, dans l'essentiel des chapitres, sur des graphes métriques.

Tout d'abord, nous plantons le décor dans lequel les chapitres suivants se déroulent. Ainsi, nous présentons l'équation elliptique superlinéaire (NLS), les graphes métriques et la formulation de (NLS) sur ceux-ci.

Ensuite, nous introduisons plusieurs notions. En particulier, nous considérons deux façons d'aborder le problème d'une façon variationnelle : l'une basée sur les points critiques de la fonctionnelle d'action sur la variété de Nehari, ce qui mène aux (nodal) action ground states, l'autre basée sur les points critiques de la fonctionnelle d'énergie sur une contrainte de masse L^2, ce qui mène aux solutions normalisées. Suivent cinq chapitres, consacrés à :

1. un théorème d'existence de solutions de (NLS) sur les graphes métriques qui permet de construire des exemples où l'on peut comparer les notions d'action ground state et de solution d'action minimale sur des domaines non-compacts ;
2. des résultats d'existence et de non-existence d'action ground states et de nodal action ground states sur plusieurs classes de graphes métriques ;
3. une nouvelle méthode pour démontrer l'existence de solutions (positives et nodales) L^2-normalisées de (NLS) pour la condition au bord de Dirichlet sur des ouverts bornés de $\mathbb{R}^N$, y compris dans le régime masse-supercritique ;
4. la multiplicité infinie de solutions normalisées, sur des graphes métriques et dans le régime L^2-supercritique, à l'équation de Schrödinger non-linéaire à non-linéarité localisée ;
5. l'analyse asymptotique de (NLS) sur des graphes compacts dans le régime asymptotique $p \to 2$ grâce à une réduction de Lyapunov-Schmidt, l'étude des nodal ground states s'annulant identiquement sur des arêtes sur des graphes compacts en étoile et l'étude détaillée de solutions sur le « graphe tétraèdre » à l'aide d'une preuve assistée par ordinateur utilisant des calculs certifiés grâce à l'arithmétique d'intervalles.

Mots clés : équations elliptiques superlinéaires, NLS, graphes métriques, action ground states, energy ground states, nodal ground states, domaines non-compacts, solutions normalisées, régime masse-supercritique, indices de Morse, réduction de Lyapunov-Schmidt, unicité et symétries, arithmétique d'intervalles, preuves assistées par ordinateur

Abstract

In this thesis, we investigate the nonlinear Schrödinger equation

$$-\Delta u + \lambda u = |u|^{p-2}u \tag{NLS}$$

where $\Delta := \sum_{1\leq i\leq N} \partial_{ii}$ is the Laplacian, $p, \lambda \in \mathbb{R}$ and $p > 2$. The equation is considered on open domains of $\mathbb{R}^N$ or, in most chapters, on metric graphs.

To begin with, we set the stage in which the following chapters take place. Thus, we present the superlinear elliptic equation (NLS), metric graphs and the formulation of (NLS) on them.

Then, we introduce several notions. In particular, we consider two ways to tackle the problem variationally: one based on the critical points of the action functional on the Nehari manifold, leading to (nodal) action ground states, the other based on critical points of the energy functional on a L^2-mass constraint, leading to normalized solutions. Five chapters follow, dedicated to:

1. an existence theorem of solutions to (NLS) on metric graphs which allows to construct examples where one may compare the notions of action ground state and of minimal action solution on noncompact domains;
2. existence and non-existence results for action ground states and nodal action ground states on several classes of metric graphs;
3. a new method to prove the existence of (positive and nodal) L^2-normalized solutions to (NLS) with the Dirichlet boundary condition on bounded open sets of $\mathbb{R}^N$, including in the L^2-supercritical regime;
4. the infinite multiplicity of normalized solutions, on metric graphs and in the L^2-supercritical regime, to the nonlinear Schrödinger equation with localized nonlinearity;
5. the asymptotic analysis of (NLS) on compact graphs in the asymptotic regime $p \to 2$ thanks to a Lyapunov-Schmidt reduction, the study of nodal ground states vanishing identically on edges on compact star graphs as well as the detailed study of the "tetrahedron graph" thanks to a computer-assisted proof using computations certified by interval arithmetic.

Keywords: superlinear elliptic equations, NLS, metric graphs, action ground states, energy ground states, nodal ground states, noncompact domains, normalized solutions, mass-supercritical regime, Morse indices, Lyapunov-Schmidt reduction, uniqueness and symmetries, interval arithmetic, computer-assisted proofs

Publications and preprints

Here are the publications and preprints of the author (as of December 19, 2024[1]), listed in chronological order.

The two first elements correspond to works performed before the author's PhD.

1. Frédéric Bourgeois, D. G.

   Geography of bilinearized Legendrian contact homology (preprint on arXiV in 2019, the journal version was published in 2024).
   Algebraic & Geometric Topology. Vol. 24, issue 7.

   Available on the publisher website at the address
   `https://msp.org/agt/2024/24-7/p01.xhtml`

2. Cédric Pilatte, D. G.

   A note on optimal degree-three spanners of the square lattice, 2022.
   Discrete Mathematics, Algorithms and Applications. Vol. 14, No. 03, 2150124.

   Available on the publisher website at the address
   `https://www.worldscientific.com/doi/abs/10.1142/S179383092150124X`

3. Colette De Coster, Simone Dovetta, D. G., Enrico Serra.

   On the notion of ground state for nonlinear Schrödinger equations on metric graphs, 2023. Calculus of Variations and Partial Differential Equations, Vol. 62, No. 159.

   Available on the publisher website at the address
   `https://link.springer.com/article/10.1007/s00526-023-02497-4`

   This open access article is licensed under a Creative Commons Attribution 4.0 International License,[2] see
   `http://creativecommons.org/licenses/by/4.0/`.

   This article corresponds to the first chapter of this thesis.

4. Colette De Coster, Simone Dovetta, D. G., Enrico Serra, Christophe Troestler.

   Constant sign and sign changing NLS ground states on noncompact metric graphs, 2023.

   arXiV preprint 2306.12121, hal preprint hal-04145190/.

   This article corresponds to the second chapter of this thesis.

[1] The date of the final version of the manuscript after small corrections and updates.

[2] The open access funding was provided by Politecnico di Torino within the CRUI-CARE Agreement.

5. Pablo Carrillo, D. G., Louis Jeanjean, Christophe Troestler.

   Infinitely many normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities, 2024.

   arXiV preprint 2403.10959, hal preprint hal-04590083.

   This article corresponds to the fourth chapter of this thesis.

6. Colette De Coster, Simone Dovetta, D. G., Enrico Serra.

   An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations, 2024.

   arXiV preprint 2411.10317, hal preprint hal-04788478.

   This article corresponds to the third chapter of this thesis.

Minor changes were performed while inserting the four last works in this manuscript.

Summary of changes between this document and the manuscript submitted to the jury:

- *the list of publications of the author was updated;*
- *minor typos were fixed and the suggestions from the jury were implemented;*
- *Chapter 3 was updated to correspond to the preprint 6 mentioned above.*

Remerciements, acknowledgments

Tout d'abord, j'aimerais remercier mes directeurs de thèse, le Pr. Colette De Coster et le Pr. Christophe Troestler. Un immense merci pour votre disponibilité dès mon M2 et pour ma candidature auprès du FNRS, votre enthousiasme, vos précieux conseils et toutes les maths que nous avons faites ensemble (et ce n'est pas fini !).

Colette, merci pour les heures de discussion au tableau (on en aura fait des plans de phase !), ton sens inimitable du « Pourquoi ? » sans lequel nous serions passés à côté de beaucoup de choses[3], tes nombreuses relectures de mes notes qui m'ont grandement fait progresser dans mes rédactions, ton sens de la précision et des arguments « à la main », ta confiance pour m'avoir intégré dans tes projets et collaborations, tes mots justes quand il le fallait.[4] *Merci Colette !*

Christophe, c'est avec Stéphanie Bridoux et toi que j'ai pu faire mes premiers pas en analyse. À présent, me voilà en train d'écrire les remerciements d'une thèse... À croire que vous avez bien fait cela ! Merci pour ton enthousiasme débordant, ta disponibilité dès le début de mon bachelier, ta vision des belles constructions variationnelles, tes inimitables discussions sur les mathématiques, les ordinateurs et tout le reste.[5] *Merci Christophe !*

Colette, Christophe: encore merci pour tout.[6]

It is a great honor to have Prof. François Hamel and Prof. Tobias Weth as reviewers for this thesis, as well as Prof. Angela Pistoia and Prof. Quentin Menet as jury members. I really hope you will enjoy the journey through metric graphs. *Merci beaucoup, danke schön, grazie mille!*

This thesis was greatly influenced by several collaborations. First of all, I would like to thank Prof. Enrico Serra for introducing Colette and myself to the world of graphs. You have made a fantastic job at explaining us the "hows and whys" of the field at the beginning of my thesis. I really admire your sense of beautiful "low tech" arguments which are underlying many of our joint mathematics. I would say you are quite responsible for making me a *minimizer*. *Grazie mille!*

I also want to thank Prof. Simone Dovetta who played an important role in this metric graph journey.

[3]Et qui ne te laisse jamais être bernée par un argument un peu trop cavalier... ;-)

[4]Bien sûr, un grand merci pour les *biscuits*, le *thé* et le *chant*. J'attends avec impatience des nouvelles de l'atelier lyrique !

[5]Un grand merci pour les *biscuits* et évidemment pour *le stylo*. Je ne suis pas sûr d'arriver à ton (impressionnant !) niveau de calligraphie, mais je vais faire de mon mieux !

[6]Et pas *seulement* pour les moments passés au resto! ;-)

Thanks for your unbounded enthusiasm, your legendary *heuristic arguments* that led to many of the nice results in our joint works, your bravery to go ahead and show the way through computations before the situation got as clear as in the final papers. This was very inspiring to me as a young researcher. And thanks again for your wonderful course given at the NQG summer school. *Grazie mille!*

So, Enrico, Simone, thanks for coming to Valenciennes, welcoming us warmly in Torino and letting us have a lot of fun on graphs with you!

Je tiens à remercier chaleureusement le Pr. Louis Jeanjean. Un grand merci pour votre accueil à Besançon et pour m'avoir permis de présenter un exposé au séminaire, vos belles constructions variationnelles, votre vision des directions de recherches intéressantes à poursuivre, votre ouverture sur de nouveaux horizons mathématiques, l'aperçu historique sur les solutions normalisées et votre confiance pour m'avoir intégré au sein du projet ANR NQG. Je suis heureux d'avoir pu apporter ma petite pierre à ce bel édifice qu'est la théorie des solutions normalisées. J'espère vraiment pouvoir continuer à y contribuer. *Encore merci !*

Merci à Pablo Carillo pour ton accueil chaleureux et nos discussions à Besançon et ailleurs. *¡Muchas gracias Pablo !*

Je tiens également à remercier le Pr. Serge Nicaise. Merci de nous avoir présenté tes beaux résultats sur les graphes et d'avoir partagé ton enthousiasme, ton énergie sans limites, tes conseils et ta vision des mathématiques. Ce fut un plaisir de covoiturer ensemble entre la Belgique et Valenciennes ! *Merci Serge !*

Je remercie également le Pr. Félix Ali Mehmeti pour ses conseils sur la pratique des mathématiques, les discussions des problèmes sur les réseaux et sa vision de leur importance du point de vue physique. Merci également au Pr. Luc Paquet pour les discussions mathématiques et les échanges de références. *Encore merci !*

La recherche n'est pas possible sans être dans un environnement accueillant. Cette thèse a été réalisée en cotutelle entre le département DMATHS du laboratoire CERAMATHS de l'Université Polytechnique Hauts-de-France et le département de mathématique de l'Université de Mons. Un grand merci aux membres de ces deux laboratoires ! C'est un plaisir de passer du temps avec vous.

Merci aux membres du DMATHS pour les opportunités de présenter un exposé lors du séminaire et de m'investir dans les activités de vulgarisation. Merci aussi pour tous les bons moments en salle de convivialité ! Merci à Mateo, mon collègue de bureau valenciennois.

Merci à tous les membres du département de mathématique de l'UMONS, où l'on se sent si bien et où j'ai appris ce que sont les mathématiques. Merci à mes collègues de bureau montois, Noémie, Dimitrios et Madeline, pour les bons moments partagés. Noémie, Madeline : bon courage pour vos thèses ! Naomi, j'ai été ravi de te rencontrer lors de ta BIR de l'été 2022. Bon amusement avec l'IA !

Lors de la préparation de cette thèse, j'ai eu l'opportunité d'échanger avec le Pr. Charles A. Stuart. Professeur, un grand merci pour nos échanges de courriels, avoir transmis vos travaux et avoir éclairci plusieurs aspects historiques. Ce fut un plaisir d'en apprendre davantage sur les problèmes de bifurcation depuis le spectre essentiel. *Encore merci !*

I also want to thank Prof. Riccardo Adami. *Thanks for giving me access to one of your papers!*

J'ai eu la chance de pouvoir assister à des conférences et de donner plusieurs exposés lors de cette thèse.

Many thanks to Félix del Teso, Antonio J. Fernández and María Medina for organizing and giving me the opportunity to talk at the "Young Researchers in PDEs" summer school. I learned a lot and made many friends in Madrid. Thanks to all participants: we did a lot of maths and had a great time! *¡Gracias a todos!*

Merci à toutes les personnes m'ayant donné l'opportunité de présenter mes travaux lors de séminaires.

En particulier, merci aux laboratoires de mathématiques d'Amiens et de Calais pour votre remarquable sens de l'accueil et pour les discussions mathématiques qui ont suivi les exposés. Merci à Alberto Farina, à Nicolas Beuvin et à Mabel Cuesta pour vos invitations. *Encore merci !*

Un grand merci à Stefan Le Coz et à Romain Duboscq pour leur invitation au congrès *Nonlinear Quantum Graphs* à Toulouse. *Encore merci !*

Thanks to all the professors and participants to the NQG Summer School in Valenciennes for making this event a success!

J'ai eu la chance de pouvoir m'initier à la recherche avant le début de cette thèse. Je ne peux que remercier le département de mathématique de l'UMONS de promouvoir cette dynamique et en particulier le Pr. Christian Michaux pour son soutien dès le (tout !) début de mon parcours mathématique et pour m'avoir permis de participer à de nombreux évènements. *Merci Christian !*

Thanks to all the participants to the "Modern Mathematics" summer school in Bremen in 2017 and to the ones of the "Mathematical summer in Paris" in 2018.

Cette entrée en la matière a eu la chance de se concrétiser. Un très grand merci au Pr. Frédéric Bourgeois pour avoir eu l'audace de proposer un sujet impliquant de l'algèbre homologique à l'étudiant de deuxième année de bachelier que j'étais ! Mais bien guidé, rien n'est impossible. J'ai passé un superbe été 2018 à Orsay et je suis heureux d'avoir pu apporter ma petite pierre à la théorie des nœuds legendriens.[7] *Encore merci !*

[7] Si je me suis éloigné de l'algèbre depuis lors, je suis heureux d'être resté dans un voisinage de la théorie de Morse (qui est d'ailleurs bien visible à la page 334). Il est amusant de noter que les objets unidimensionnels étaient déjà bien présents : *nœuds legendriens et leurs diagrammes.*

Merci aussi à Cédric Pilatte avec qui nous nous sommes beaucoup amusés grâce à la géométrie combinatoire avec les problèmes de *dilation*. Je suis content de notre petite note sur la grille entière[8]. *Merci pour les maths et le reste ! Bonne continuation à Oxford !*

Cette thèse n'aurait pas été possible sans les enseignements de mes professeurs.

Un tout grand merci à tous les professeurs et assistants du département de mathématique de l'UMONS pour m'*avoir appris à faire des maths* et pour avoir chacun transmis votre passion. Je n'aurais pas pu rêver mieux comme cadre pour débuter mon aventure dans le monde de la recherche. Merci aussi aux étudiants de ma promotion lors de mes études à l'UMONS. Il y avait vraiment une superbe ambiance au sein de notre classe. *Au plaisir de se revoir bientôt !*

Merci également à mes professeurs de physique. Sans vous, je n'aurais pas pu discuter de mécanique quantique. De plus, c'est au cours de mécanique analytique que j'ai rencontré pour la première fois des méthodes variationnelles, au cœur de cette thèse.

Et bien sûr, merci à mes professeurs d'informatique. L'outil informatique est employé pour la preuve assistée par ordinateur du dernier chapitre. Il a permis de faire des simulations d'équations différentielles et de réaliser un certain nombre des dessins de cette thèse.[9]

J'ai eu la chance d'effectuer ma première année de master en Erasmus au sein de l'Institut de Mathématique d'Orsay. Merci aux professeurs pour la qualité de leur enseignement.

Il est clair que le cours de théorie des distributions du Pr. Patrick Gérard a contribué à me faire définitivement basculer vers l'analyse, dont les travaux dirigés assurés par le Pr. Konstantin Pankrashkin m'ont permis de progresser dans ma façon de *faire* de l'analyse. Un grand merci au Pr. Stéphane Nonnenmacher et à Rémy Rodiac pour leur cours de problèmes d'évolution et leur remarquable implication à distance lors du sombre mois de *mars 2020*[10].

Merci à tous les amis que j'ai rencontrés lors d'évènements mathématiques.

En particulier, merci au groupe des *Journées Montoises*. Chers Alexandre, Antoine, Célia, France, Manon, Pierre, Savinien: merci pour le bon temps passé ensemble à Prague.

Plus généralement, merci aux membres du département de mathématique de l'ULiège pour votre très chaleureux accueil lors de chacune de nos rencontres. J'en profite pour remercier à nouveau Manon pour son précieux *soutien moral* lors de la rédaction de ce manuscrit.

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[8]Où nous avons étudié des *graphes métriques quasi-périodiques*, comme quoi...

[9]Et n'oublions pas LaTeX !

[10]Pendant la pandémie de COVID-19.

Un grand merci aux personnes présentes dans mon parcours mathématique "depuis le début", comme mes professeurs de mathématiques du secondaire. Je pense en particulier à la superbe année de rhétorique passée avec M. Marcel Cornez.

Merci à tous les amis rencontrés lors des divers stages d'entraînement aux olympiades de mathématiques et d'informatique, en particulier la grande "famille de Wépion". Un grand merci à toutes les personnes qui rendent tout cela possible.

Merci également à l'équipe des Problémaths, en particulier au Pr. Jean Doyen. Un tout grand merci pour vos encouragements depuis le début de mon parcours!

Il me reste à remercier chaleureusement mes amis: Alexis, Alexandre, Elora, Fiona, Luca, Mehdi, Tatiana et Valentin & Valentin, pour tous les bons moments passés ensemble et d'être là pour moi en toutes circonstances.

Merci à Ingrid, Juline, Morine et Pascal pour votre amitié depuis toutes ces années. *On se revoit bientôt!*

Many thanks to my *English consultants*: James and Simon. Thanks a lot for your expert advice and your friendship. *All the best for your own research!*

N'oublions pas l'essentiel. Ainsi, un tout grand merci à mes parents et à mes grands-parents, pour avoir pris soin de moi, m'avoir encouragé à me dépasser et avoir toujours cru en moi.

Maman, on en aura traversé des galères en 25 ans ! Tu m'as toujours appris à ne pas baisser les bras. Me voici en train de conclure les remerciements de ma thèse ! *Merci pour tout !* Et bien sûr, merci pour tes nombreuses relectures[11] de portions de ce manuscrit.

Papa, tu as toujours cru en moi. C'est notamment grâce à toi que l'aventure des olympiades a débuté. Merci de m'avoir appris tant de choses en grandissant, et ce n'est pas fini. *Merci pour tout !*

Merci à « Papy Laurent » et à « Mamy Rosy » pour tous vos encouragements et pour les bons moments passés ensemble. Cette fois, *on y est* !

Merci à « Papy Claude ». À croire que tes *défis* de mathématiques lorsque j'étais à l'école primaire m'ont bien plu ! Nous en revenons au message de départ : ne pas oublier l'essentiel. Ainsi, bien que la belle théorie de la mesure clarifie la notion d'aire grâce à la définition de la mesure de Lebesgue, je pense que tout cela serait bien vain si nous n'avions jamais enfilé nos bottes et fait le tour d'un hectare de terrain dans un champ. Après tout, *l'aire, c'est ça. Merci pour tes enseignements et à bientôt pour de nouvelles aventures !*

Et bien sûr, merci à « Mamy Annie ». Merci de t'être si bien occupée de moi depuis mon plus jeune âge et pour ton délicieux *support logistique*.

[11] Il serait plus pertinent de parler d'un véritable *travail éditorial* !

Aspects financiers - funding

This thesis was realized while the author was an F.R.S.-FNRS Research Fellow[12].

The thesis is under joint supervision between the mathematics department in the faculty of science of the Université de Mons (UMONS, Belgium) and the DMATHS department of the CERAMATHS laboratory of the Université Polytechnique Hauts-de-France (UPHF) and INSA Hauts-de-France (Valenciennes, France). Both departments and the faculty of science of UMONS provided support (financial and/or material) to the author during his studies and his PhD.

The author is a part of the ANR Project NQG (ANR-23-CE40-0005-01), funded by the French National Research Agency (ANR)

The collaboration with the colleagues from Politecnico di Torino was funded thanks to IEA Project "Nonlinear Schrödinger Equations on Metric Graphs".

Travel expenses for a stay to Besançon were funded by the COMPLEX doctoral school.

The author, Damien Galant, wants to thank all the aforementioned funding sources for allowing him to conduct his research and to perform several research stays abroad.

Use of Artifical Intelligence (AI)

The author, Damien Galant, has followed the charter of UMONS relative to the use of AI: `https://web.umons.ac.be/app/uploads/sites/34/2024/05/CharteIA-UMONS.pdf`

The author used sometimes generative AI to help during translations between French and English and to improve the formulation of some sentences.

[12]See `https://www.frs-fnrs.be/en/financements-resp/chercheur-doctorant`.

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Appendices

I. Introduction (en français)

Avant d'aborder les questions précises étudiées dans cette thèse et les résultats obtenus, il convient de présenter des notions ainsi que des éléments historiques. Cette introduction se veut, dans la mesure du possible, plus accessible que les chapitres ultérieurs. Les dessins, exemples et intuitions seront de rigueur et on ne trouvera généralement ni énoncés précis, ni preuves.

La table ci-dessous indique quels thèmes des sections suivantes sont présents dans les différents chapitres.

Sujets (sections) \ Chapitres	1	2	3	4	5
Équation de Schrödinger non-linéaire (I.1)	✓	✓	✓	✓	✓
Graphes métriques (I.2 et I.3)	✓	✓		✓	✓
(Nodal) action ground states (I.4)	✓	✓	✓		✓
Solutions nodales (I.4)		✓	✓	(✓)	✓
Solutions normalisées (I.5)			✓	✓	
Domaines non-compacts (I.6)	✓	✓		✓	
Rôle de la géométrie et de la topologie du domaine (I.6)	✓	✓	✓		(✓)
Solutions concentrées (I.7)	✓				
Problèmes avec non-linéarité localisée (I.8)				✓	
Solutions nulles sur des arêtes (I.9)		✓		(✓)	✓
Solutions versus minimiseurs (I.10)	✓		✓		
Régime $p \approx 2$ (I.11)					✓
Unicité, symétries et preuves assistées par ordinateur (I.11)					✓

I.1 L'équation de Schrödinger non-linéaire

Cette thèse a pour objet l'étude d'équations aux dérivées partielles elliptiques non-linéaires, essentiellement l'équation de Schrödinger non-linéaire[1]

$$-\Delta u + \lambda u = |u|^{p-2}u, \qquad \text{(NLS)}$$

où $\Delta := \sum_{1\le i\le N} \partial_{ii}$ désigne l'opérateur laplacien, $p > 2$ et λ sont des paramètres réels, $u : \Omega \to \mathbb{R}$ et $\Omega \subseteq \mathbb{R}^N$ est un domaine ouvert, borné ou non.

[1]Par la suite, l'équation (NLS) sera systématiquement appelée « équation de Schrödinger non-linéaire ». Dans la littérature, cette expression désigne également l'équation d'évolution $\boldsymbol{i}\partial_t\Psi(t,x)+\Delta\Psi(t,x)+|\Psi(t,x)|^{p-2}\Psi(t,x) = 0$. L'ansatz des *ondes stationnaires* $\Psi(t,x) = e^{\boldsymbol{i}\lambda t}u(x)$ assure le lien entre l'équation d'évolution et l'équation elliptique (voir l'annexe F).

Reste alors à spécifier des conditions au bord, comme celle de Dirichlet (qui impose que u s'annule au bord de Ω) ou celle de Neumann (qui impose que la dérivée normale de u s'annule au bord de Ω).

Si le domaine Ω n'est pas borné, par exemple si $\Omega = \mathbb{R}^N$, les conditions à l'infini jouent également un rôle. Dans ce cas, on s'intéressera aux solutions « suffisamment petites » en l'infini, comme celles de carré intégrable.

Une propriété importante de l'équation (NLS) est d'admettre une *formulation variationnelle*. En effet, si on définit la *fonctionnelle*[2] *d'action*

$$J_\lambda(u) := \frac{1}{2}\int_\Omega \|\nabla u\|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x,$$

alors les points critiques de J_λ sur l'espace de Sobolev $H^1(\Omega)$ correspondent aux solutions de l'équation avec la condition de Neumann et les points critiques de J_λ sur $H_0^1(\Omega)$ correspondent aux solutions de l'équation avec la condition de Dirichlet. Le point de vue variationnel jouera un grand rôle dans la plupart des chapitres.

La littérature sur le sujet est vaste et il serait vain de tenter de la présenter de façon exhaustive. Néanmoins, mentionnons les articles fondamentaux [65, 66, 213, 307, 332] qui étudient les propriétés de l'équation (NLS) sur $\mathbb{R}^N$.

Le lecteur qui souhaite s'introduire au sujet peut par exemple consulter les articles d'exposition [52, 214] et le livre [35].

Les ouvrages [284, 310, 335] sont de précieuses références en ce qui concerne le point de vue variationnel.

Mentionnons également les traités [98, 152, 317] qui étudient l'équation de Schrödinger au sens large, y compris la théorie des problèmes d'évolution non-linéaires associés.

Résumons l'objet de nos travaux (hormis le chapitre 3) en une phrase.

> *Cette thèse est dédiée à l'étude d'équivalents de l'équation* (NLS) *sur des graphes métriques.*

Le chapitre 3 est, quant à lui, consacré à l'étude de solutions (normalisées au sens L^2, notion que nous préciserons dans la section I.5) pour l'équation (NLS) sur des domaines Ω bornés.

Avant d'aller plus loin, précisons ce que sont les graphes métriques.

[2] Dans le texte, nous appelons *fonctionnelle* une fonction définie sur un espace de fonctions.

I.2 Pourquoi utiliser les graphes métriques comme domaines ?

I.2.1 Que sont les graphes métriques ?

Un *graphe métrique* est un domaine *unidimensionnel* formé de *sommets* (ou *nœuds*) et d'*arêtes* joignant les sommets entre eux ou reliant un sommet et l'infini.

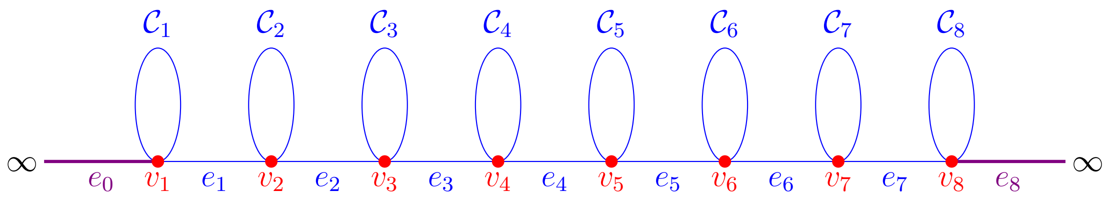


FIGURE I.1 : Un premier exemple de graphe métrique avec 8 sommets $v_1, \ldots, v_8$; 7 arêtes de longueur finie $e_1, \ldots, e_7$ reliant des sommets distincts ; 8 boucles (arêtes) $\mathcal{C}_1, \ldots, \mathcal{C}_8$ reliant un sommet à lui-même ; deux demi-droites e_0 et e_8.

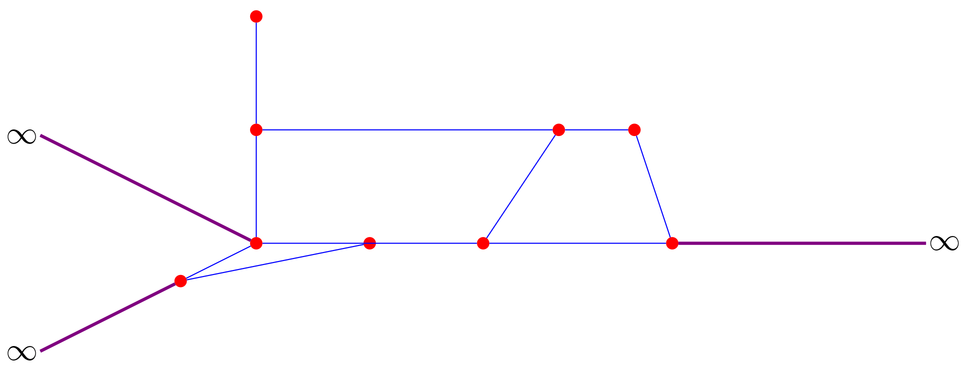


FIGURE I.2 : Second exemple de graphe métrique

Une définition formelle de la notion de graphe métrique est disponible dans l'annexe A. Remarquons dès à présent que :

- les graphes étudiés sont *métriques* : les longueurs des arêtes ont de l'importance et joueront parfois un rôle dans les résultats ;
- les arêtes des graphes allant jusqu'à l'infini sont des demi-droites et ont une *longueur infinie* ;
- dans cette thèse, nous dirons qu'un graphe est *compact* s'il est formé d'un nombre fini d'arêtes de longueurs finies (en effet, dans ce cas, le graphe est compact en tant qu'espace métrique, voir la proposition A.1).

En utilisant uniquement des demi-droites, on peut former la famille des *graphes en étoile*, exemples de graphes non-compacts n'ayant qu'un seul sommet.

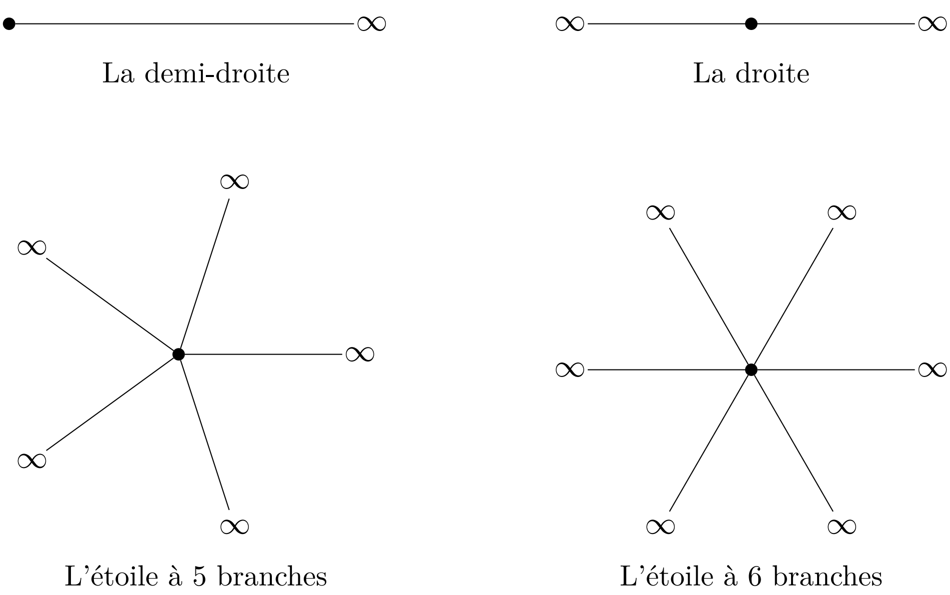


Figure I.3 : Constructions basées sur des demi-droites

La classe des graphes métriques possède une grande richesse. On peut par exemple considérer des *graphes périodiques* (voir les figures I.4 et I.5).

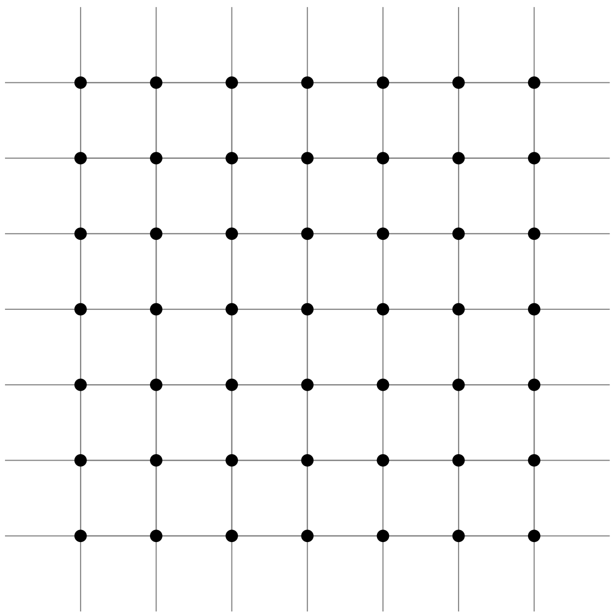


Figure I.4 : La grille infinie dans le plan

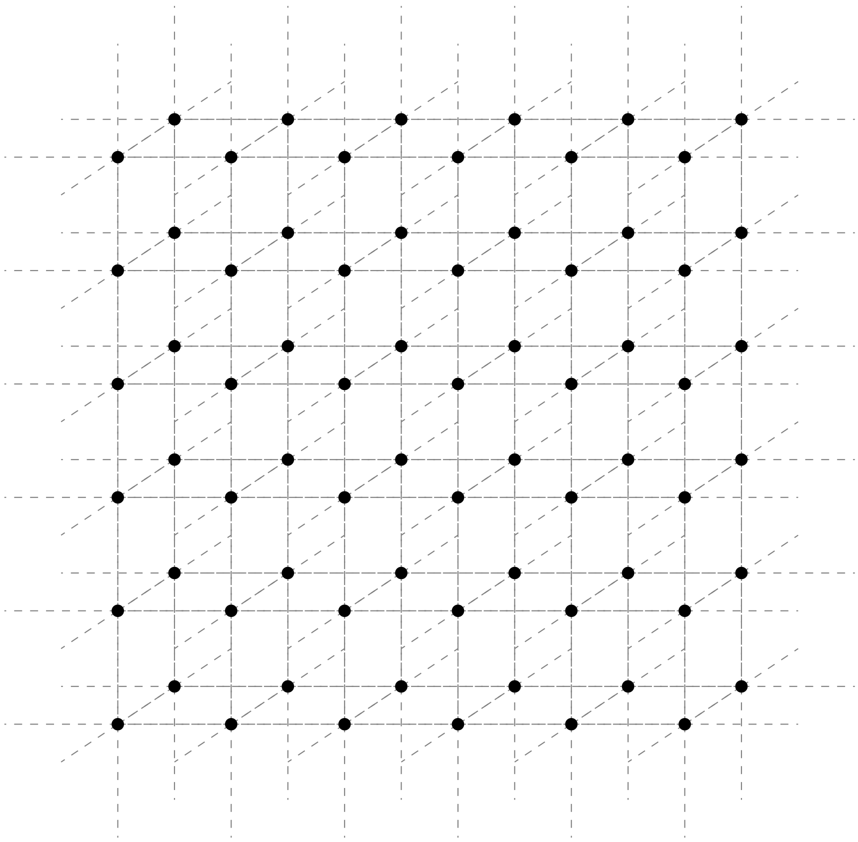


FIGURE I.5 : La grille infinie dans l'espace

Nous considérerons également des *arbres infinis* (voir la figure I.6).

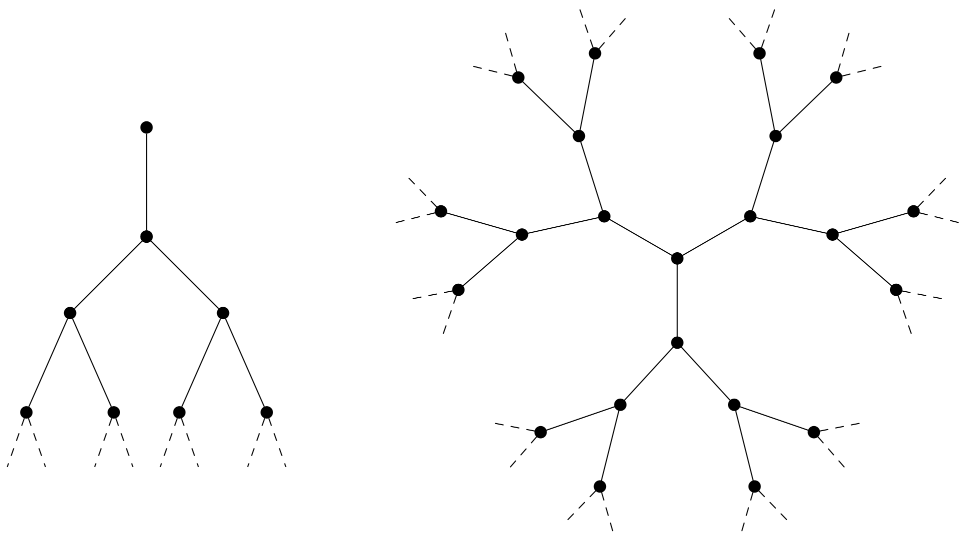


FIGURE I.6 : Arbres infinis

Dans cette thèse, les graphes métriques joueront le rôle de *domaines* sur lesquels les problèmes non-linéaires seront étudiés. Nos objets d'études principaux seront donc des *fonctions* définies sur des graphes métriques.

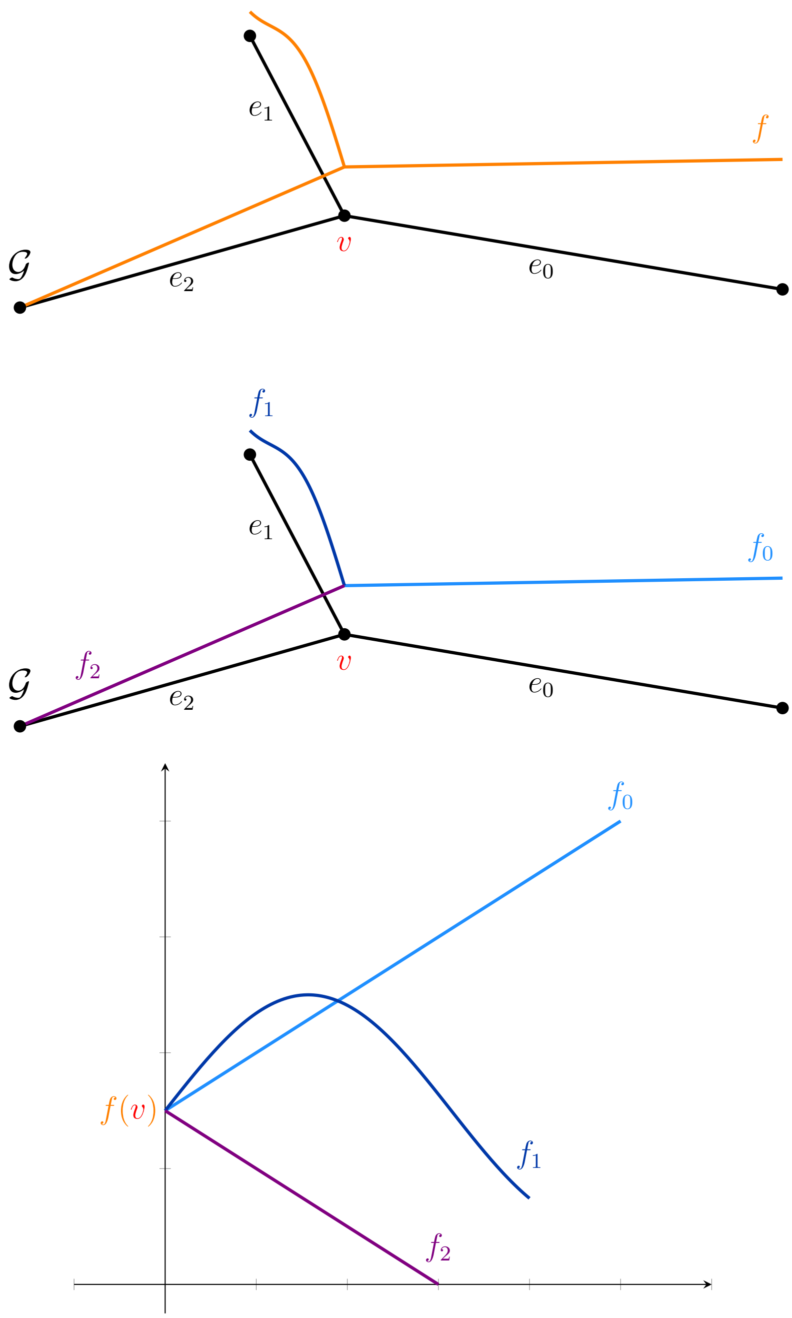


FIGURE I.7 : Un graphe métrique $\mathcal{G}$ avec trois arêtes e_0 (de longueur 5), e_1 (de longueur 4), e_2 (de longueur 3) et une fonction $f : \mathcal{G} \to \mathbb{R}$ avec les trois fonctions réelles f_0, f_1, f_2 associées. *Remarquons que* $f_0(0) = f_1(0) = f_2(0) = f(v)$.

Les opérations classiques de l'analyse peuvent être définies naturellement sur les graphes en travaillant arête par arête. Par exemple, dans la figure I.7, on a

$$\int_{\mathcal{G}} f \,\mathrm{d}x := \int_0^5 f_0(x)\,\mathrm{d}x + \int_0^4 f_1(x)\,\mathrm{d}x + \int_0^3 f_2(x)\,\mathrm{d}x.$$

Disposer d'une théorie de l'intégration permet de définir de façon usuelle les espaces de Lebesgue, que nous noterons $L^p(\mathcal{G})$. Nous pouvons également munir $\mathcal{G}$ d'une structure d'espace métrique en disant que la distance entre deux points du graphe est donnée par la plus petite longueur d'un chemin continu joignant ces deux points. Ainsi, les graphes métriques forment une classe d'exemples d'*espaces métriques mesurés* (voir les sections A.2 et A.3 pour des définitions précises des structures dont on munit les graphes).

À présent, voyons quelques raisons « pragmatiques » d'étudier des problèmes sur les graphes.

I.2.2 Réduction de dimension

Les graphes métriques sont pertinents dans l'étude de problèmes pour lesquels *une seule direction spatiale est importante.*

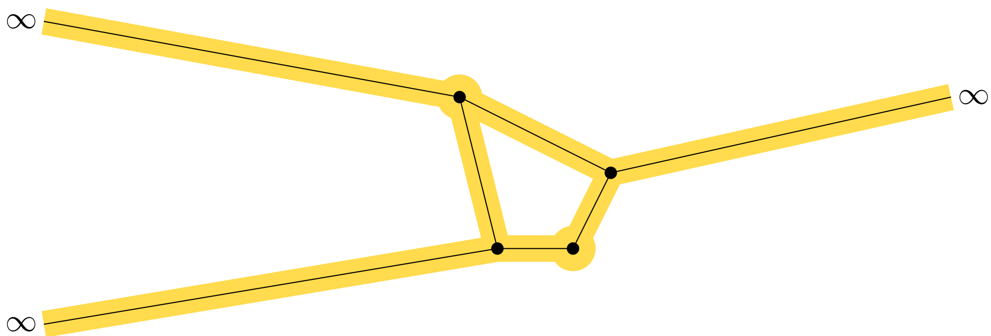


FIGURE I.8 : Un « graphe épais » et le graphe métrique sous-jacent

C'est le cas dans l'exemple illustré par la figure I.8 où l'on peut voir un domaine de $\mathbb{R}^2$ (un « graphe épais »). La situation est analogue pour les câbles dans $\mathbb{R}^3$ (fibres optiques, câbles électriques, etc.). Le bon sens veut que, lorsque l'épaisseur du domaine est petite et que les directions transverses au câble ne jouent pas de rôle important, le problème « devient unidimensionnel ».

Ainsi, dans les cours de physique dédiés à l'étude des circuits électriques, on peut introduire plus ou moins heuristiquement des *modèles réduits* des circuits en partant des principes fondamentaux de l'électromagnétisme. Dans de tels modèles, la forme géométrique précise des câbles dans $\mathbb{R}^3$ ne joue aucun rôle (voir par exemple [151, Volume II, Chapter 22]).

À ce sujet, rappelons la *loi des nœuds de Kirchhoff*, stipulant que la somme des intensités électriques en chaque nœud du circuit est nulle, en employant une bonne convention de signe (voir par exemple [64, Chapitre 7]).

I.2.3 Genèse des graphes quantiques

Une réduction de dimension telle que celle présentée dans les exemples de la section I.2.2 est également pertinente pour des modèles quantiques. Présentons quelques cas historiques illustrant notre propos. Ils se basent[3] sur l'article [113, Section 8], dans lequel le lecteur trouvera plus de détails sur les débuts de la théorie des graphes quantiques.

Dès 1930, E. Hückel [180] a mis en évidence le fait que la description quantique des hydrocarbures pouvait se réduire à l'étude d'un modèle posé sur le graphe associé à la structure de la molécule.

Dans les années 1950, K. Ruedenberg et C.W. Scherr [290] ont utilisé la même démarche afin d'étudier la dynamique des électrons de valence de la naphtalène $C_{10}H_8$ (voir la figure I.9). Dans le même esprit, on pourra également citer les travaux de C.A. Coulson [112] et des travaux postérieurs de Ruedenberg (voir par exemple [289]).

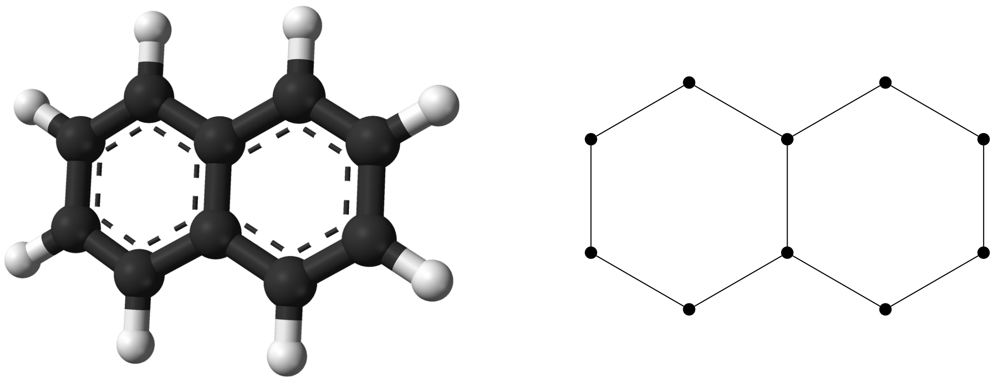


FIGURE I.9 : Représentation[4] d'une molécule de naphtalène $C_{10}H_8$ et le graphe métrique associé.

Les travaux de chimie quantique susmentionnés mettent en évidence le lien entre les valeurs propres d'opérateurs hamiltoniens[5] associés à une molécule avec le *spectre du graphe métrique correspondant à la molécule*.

Précisons ce que signifie cette notion.

[3]Un grand merci au Pr. Delio Mugnolo d'avoir présenté l'historique de la théorie lors de son cours donné à l'école d'été *Nonlinear Quantum Graphs* à Valenciennes en juin 2024 (voir `https://nqg.sciencesconf.org/`), en particulier d'avoir mis en évidence l'ouvrage [113].

[4]Image issue de `https://commons.wikimedia.org/wiki/File:Naphthalene-from-xtal-3D-balls.png`, domaine public.

[5]Qui correspondent à des *niveaux d'énergie*, d'après la théorie quantique.

I.2.4 Le problème spectral et la condition de Kirchhoff

Si $\mathcal{G}$ désigne un graphe métrique, le problème spectral sur $\mathcal{G}$ consiste à déterminer les couples (u, γ) pour lesquels la *fonction propre* $u : \mathcal{G} \to \mathbb{R}$ et la *valeur propre* $\gamma \in \mathbb{R}$ fournissent une solution au système différentiel

$$\begin{cases} -u'' = \gamma u & \text{sur chaque arête } e \text{ du graphe } \mathcal{G}, \\ u \text{ est continue} & \text{en chaque sommet } v \text{ de } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{en chaque sommet } v \text{ de } \mathcal{G}. \end{cases} \tag{Spec$_\mathcal{G}$}$$

La notation $e \succ v$ signifie que la somme porte sur toutes les arêtes de sommet v et $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ est la dérivée sortante de u en v. Si l'on paramétrise e par une coordonnée $x_e \in [0, |e|]$ (où $|e|$ désigne la longueur de e), on a donc

$$\frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = \begin{cases} u'(v) & \text{si } v \text{ correspond à } x_e = 0, \\ -u'(v) & \text{si } v \text{ correspond à } x_e = |e|. \end{cases}$$

Par analogie avec la loi des nœuds de Kirchhoff, la condition

$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0$$

est désignée comme la *condition de Kirchhoff*.

L'observation ci-dessous nous suivra tout au long de notre étude des problèmes posés sur les graphes.

Un problème différentiel posé sur un graphe consiste en la donnée :
1) *d'une équation différentielle arête par arête ;*
2) *de conditions de compatibilité en les sommets.*

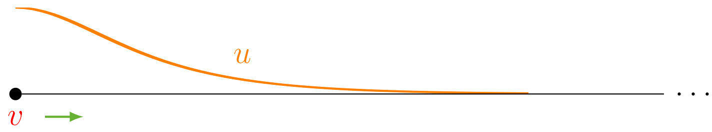
Dans le cas de (Spec$_\mathcal{G}$), la condition de continuité et la condition de Kirchhoff sont les conditions en les sommets.

Dans la section I.3, nous rencontrerons les mêmes conditions lors de l'étude de l'équation de Schrödinger non-linéaire sur les graphes.

À présent, regardons ce que signifie la condition de Kirchhoff en l'illustrant grâce à quelques exemples simples. Par la suite, nous désignerons par *degré de v* le nombre d'arêtes adjacentes à un nœud v donné.

Cas d'un nœud de degré un

Considérons un nœud v de degré un et une fonction u à valeurs réelles. Localement, la situation est la suivante :



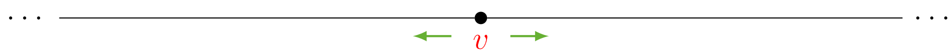
La condition devient

$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(v+t)-u(v)}{t} = 0.$$

Autrement dit, la condition de Kirchhoff impose que la dérivée de u s'annule en le sommet v : on retrouve la *condition de Neumann* usuelle.

Cas d'un nœud de degré deux

Cette fois, la situation est (localement) illustrée comme suit :

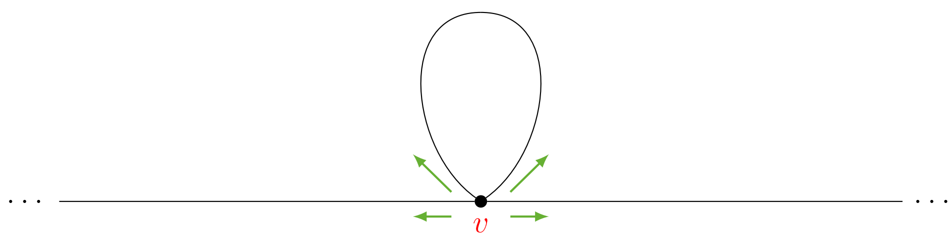


Rappelons que nous considérons des fonctions *continues en les nœuds*. Ainsi, nous pouvons considérer la fonction u définie sur le graphe comme une fonction définie sur un intervalle réel contenant v en son intérieur.

Dans ce cas, la condition de Kirchhoff impose que les dérivées à gauche et à droite de u en v soient égales, ce qui signifie que u (toujours considérée comme une fonction définie sur un intervalle réel) est dérivable en v. Cette propriété de différentiabilité est vérifiée par les solutions d'équations différentielles d'ordre 2, ce qui explique pourquoi on identifiera généralement deux graphes qui ne diffèrent que par la présence de nœuds de degré deux.

Illustration du cas général

Considérons une situation où le graphe possède un nœud v de degré quatre :



Dans ce cas, quatre dérivées sont évaluées en le sommet v et la condition de Kirchhoff affirme que la somme de ces quatre dérivées est nulle.

I.2.5 Les graphes quantiques[6]

Deux articles fondateurs des problèmes d'analyse sur les graphes métriques sont ceux de B.S. Pavlov–M.D. Faddeev [266] et de S. Nicaise [254][7]. Dans ces travaux, les auteurs montrent que le laplacien sur les graphes est un opérateur auto-adjoint dans L^2 si on le munit des conditions de continuité et de Kirchhoff en les sommets. Le lecteur désireux de s'initier à l'étude d'opérateurs différentiels sur les graphes pourra consulter le livre [68] de G. Berkolaiko et P. Kuchment. On peut notamment y trouver la définition d'un *graphe quantique* ([68, Definition 1.4.1]), notion adaptée à l'étude des problèmes physico-chimiques mentionnés précédemment :

> A **quantum graph** is a metric graph equipped with a differential operator $\mathcal{H}$ (Hamiltonian), accompanied by "appropriate" vertex conditions. That is, a quantum graph Γ **is a triple** **{metric graph** Γ**, Hamiltonian** $\mathcal{H}$**, vertex conditions}**.

En particulier, si on couple l'opérateur de dérivée seconde avec les conditions de continuité et de Kirchhoff sur un graphe, le problème spectral du graphe quantique associé prend la forme ($\text{Spec}_{\mathcal{G}}$).

Hormis [68], plusieurs ouvrages sont dédiés à l'étude de problèmes différentiels sur les graphes métriques, voir par exemple [246] de D. Mugnolo, [108] d'Y. Colin de Verdière ou [210] de P. Kurasov.

Parmi les questions relevant du domaine des graphes quantiques, citons l'étude du spectre des graphes compacts (formules de Weyl, inégalités de Cheeger et de Faber-Krahn, etc.) dans [24, 158, 254], les problèmes isospectraux (« *Can one hear the shape of a network ?* ») dans [59], l'obtention d'estimées spectrales par des techniques de « chirurgie » dans [67], le « chaos quantique » dans [165], etc. Un aperçu plus détaillé du sujet se trouve dans les notes de cours [202].

Remarque. Nous considérerons uniquement des modèles uni-dimensionnels, où les grandeurs d'intérêt sont définies sur les *arêtes* du graphe qui sont identifiées à des intervalles. Certains modèles « zéro-dimensionnels », où les grandeurs d'intérêt sont définies sur les *sommets*, existent également, voir par exemple [246, Section 2.1.4] pour une définition du *laplacien discret*. Si toutes les arêtes d'un graphe ont même longueur, il existe un lien profond entre le modèle unidimensionnel et le modèle discret, comme démontré par J. von Below [57] et S. Nicaise [253] au milieu des années 1980 (voir aussi [264] et [68, Section 3.6]). Nous utiliserons ces considérations lors de l'étude du « graphe tétraèdre » dans le chapitre 5.

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[6] À nouveau, nous remercions le Pr. Mugnolo grâce à qui nous avons eu un meilleur aperçu de la littérature *ad hoc*.

[7] Mentionnons également la note [229] de G. Lumer aux Comptes Rendus de l'Académie des Sciences de Paris. L'auteur y considère les graphes (qu'il appelle « réseaux topologiques ») comme étant des exemples d'*espaces ramifiés*, espaces obtenus par recollement de structures plus élémentaires, en l'occurrence des intervalles réels dans le cas des graphes. Nous renvoyons le lecteur intéressé vers [26, 60, 251, 252] et aux références s'y trouvant pour plus d'informations.

I.3 L'équation de Schrödinger non-linéaire sur les graphes métriques

La démarche de réduction de dimension s'applique également à des situations régies par l'équation de Schrödinger *non-linéaire* (NLS).

Étant donné un graphe métrique $\mathcal{G}$ et deux nombres réels $p > 2$ et λ, on couple l'équation de Schrödinger non-linéaire

$$-u'' + \lambda u = |u|^{p-2}u$$

sur chaque arête avec la condition de continuité et la condition de Kirchhoff.

Le problème s'écrit alors

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{sur chaque arête } e \text{ du graphe } \mathcal{G}, \\ u \text{ est continue} & \text{en chaque sommet } v \text{ de } \mathcal{G}, \\ \sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{en chaque sommet } v \text{ de } \mathcal{G}. \end{cases} \tag{NLS$_\mathcal{G}$}$$

Le problème (NLS$_\mathcal{G}$) *(et ses variantes) est le principal objet d'étude de cette thèse.* Il possède des applications en physique, nous les décrirons dans la section I.3.2.

Dans un premier temps, considérons quelques exemples.

I.3.1 Exemples : la (demi-)droite, les graphes en étoile

Dans cette section, étudions l'équation de Schrödinger non-linéaire sur des graphes simples : la droite, la demi-droite et les graphes en étoile. Ces exemples serviront d'importants points de référence par la suite. *Ci-dessous, nous prenons* $\lambda > 0$.

Les affirmations énoncées dans cette section sont prouvées dans [10, Section 2], voir aussi la section 1.4.3. Les arguments sont élémentaires et se basent sur le fait que les solutions non-nulles convergeant vers 0 de l'équation différentielle

$$-u'' + \lambda u = |u|^{p-2}u \tag{I.1}$$

sont (au signe près) des portions translatées du *soliton*[8] ϕ_λ (voir la proposition C.2 et la figure I.10) dont l'expression explicite est

$$\phi_\lambda(x) = \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}}x\right)^{\frac{-2}{p-2}}.$$

[8] Le terme *soliton* provient du terme « onde solitaire » et est utilisé dans l'étude de plusieurs équations dispersives (voir [320, Section 2]). Ici, nous l'utiliserons uniquement pour désigner ϕ_λ.

La droite : $\mathcal{G} = \mathbb{R}$

L'ensemble des solutions non-nulles de ($\mathrm{NLS}_{\mathcal{G}}$) sur la droite réelle est donné par $\left\{\pm\phi_\lambda(x+a) \mid a \in \mathbb{R}\right\}$, où ϕ_λ est le soliton. Dans ce cas, l'ensemble des solutions du problème est *invariant par*[9] *la fonction* $u \mapsto -u$ et *invariant par les translations spatiales*.

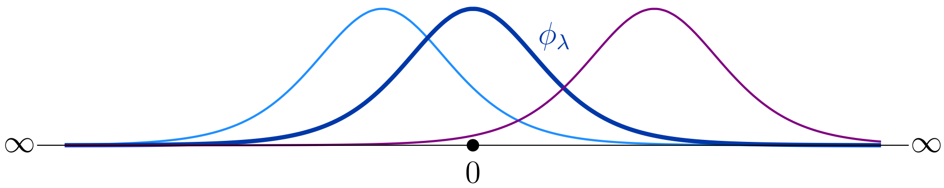

FIGURE I.10 : Trois solutions de ($\mathrm{NLS}_{\mathcal{G}}$) sur la droite réelle

La demi-droite : $\mathcal{G} = \mathbb{R}^+ := [0, +\infty)$

Sur $\mathbb{R}^+$, il n'y a que deux solutions opposées, données par des demi-solitons. En particulier, il n'y a pas de famille continue de solutions.

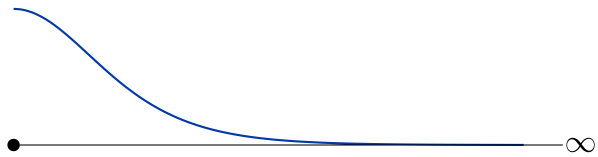

FIGURE I.11 : La solution positive de ($\mathrm{NLS}_{\mathcal{G}}$) sur la demi-droite

Les graphes en étoile avec un nombre impair de demi-droites

On peut montrer que, sur un graphe en étoile possédant un nombre impair de demi-droites, ($\mathrm{NLS}_{\mathcal{G}}$) ne possède que deux solutions non-nulles. Ces solutions sont opposées, l'une d'entre elles est positive et s'obtient en recollant des demi-solitons en leur point de maximum, comme l'illustre la figure I.12.

Remarquons que la condition de Kirchhoff est satisfaite pour ces solutions car toutes les dérivées en le nœud sont nulles.

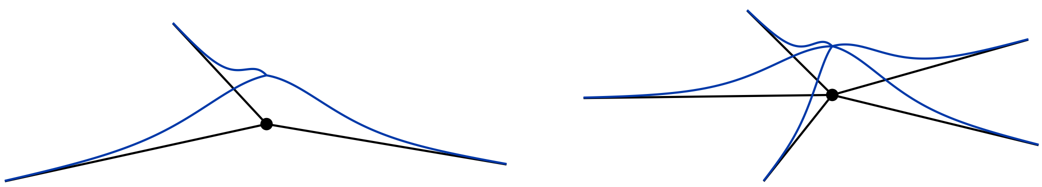

FIGURE I.12 : Solutions positives sur les étoiles à 3 et 5 branches

[9]Ce qui signifie que si u est une solution, alors $-u$ l'est également.

Les graphes en étoile avec un nombre pair de demi-droites

Lorsque le nombre de demi-droites est pair, il est possible de les grouper deux par deux et d'y placer un soliton entier, comme on peut le voir sur la figure I.13. Ainsi, on observe à nouveau la présence de familles continues de solutions.

La condition de Kirchhoff est satisfaite pour ces solutions car les dérivées en le nœud se compensent deux à deux.

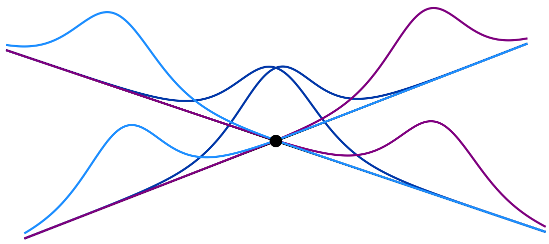

FIGURE I.13 : Une famille continue de solutions sur l'étoile à 4 branches

I.3.2 Des condensats de Bose-Einstein à l'atomtronique[10]

Deux situations physiques, a priori assez différentes, sont décrites par l'équation de Schrödinger non-linéaire sur les graphes : la transmission de signaux dans les fibres optiques et l'étude des condensats de Bose-Einstein. Citons Y.S. Kivshar et G.P. Agrawal dans [207, Preface, page xv] :

> In particular, the remarkable similarities between the matter-wave solitons and optical solitons emphasize the intimate connection between classical nonlinear optics and coherent atom optics and may lead to many discoveries in other, seemingly different fields.

On peut montrer que la propagation de signaux dans certaines fibres optiques amène à considérer une équation de Schrödinger non-linéaire focalisante (voir [207, Section 1.2.2]). Le terme *cubique* qui y apparaît provient de l'*effet Kerr*, qui désigne un changement non-linéaire dans l'indice de réfraction d'un matériau optique. La non-linéarité focalisante peut compenser les effets dispersifs. Cela mène à diverses notions de *solitons* (spatiaux, temporels, clairs, noirs[11], etc.).

[10] Pour une introduction aux aspects physiques de l'équation de Schrödinger non-linéaire sur les graphes métriques, on peut visionner le cours du Pr. Riccardo Adami lors des « Expository Quantum Lecture Series 8 » ayant lieu à l'Institute for Mathematical Research (INSPEM) de l'Université Putra Malaysia (UPM), voir `https://einspem.upm.edu.my/equals8/`. La première vidéo de cette série de leçons est disponible à l'adresse `https://www.youtube.com/watch?v=4ZsIV7iOwgI`.

[11] Ces solitons « noirs » convergent vers des constantes *non nulles* en l'infini et ne sont donc pas de carré intégrable. Nous renvoyons le lecteur intéressé vers [73, 325] et remercions André de Laire d'avoir présenté le sujet du point de vue mathématique lors d'un séminaire à Valenciennes et d'avoir mis en avant les deux références précitées.

Pour en savoir plus sur le rôle joué en optique par l'équation de Schrödinger non-linéaire, citons les livres de D.E. Pelinovsky [268], G. Fibich [152] et C. Sulem–P.L. Sulem [317], à l'interface entre les mathématiques et leurs applications. Citons également les articles [166, 291] qui expliquent comment l'équation de Schrödinger non-linéaire sur des graphes peut modéliser des *réseaux* de fibres.

À présent, intéressons-nous aux condensats de Bose-Einstein.

Lorsque des bosons[12] identiques sont refroidis à une température très proche du zéro absolu, ils occupent un unique état quantique de plus basse énergie.

En 1925, A. Einstein a publié [138] dans lequel il prédit ce phénomène (à présent connu sous le nom de *condensation de Bose-Einstein*), en se basant sur des travaux de S.N. Bose [83].[13]

Un phénomène quantique *macroscopique* ?

La condensation de Bose-Einstein est *macroscopique* et peut même survenir en présence de nombreuses particules, plus de 1000 lors des premières réalisations expérimentales décrites plus bas.

Cela semble contredire le principe de *décohérence* qui affirme qu'un système « suffisamment grand » perd ses caractéristiques quantiques. Ceci est illustré par le célèbre exemple du *chat de Schrödinger*. Comme les auteurs de [171, Section 12.5][14], nous recommandons la lecture de [296, Chapter 1, « Introducing Decoherence »].

Concernant les condensats de Bose-Einstein, soulignons que le phénomène a lieu dans un régime de *très basse énergie*, ce qui explique au moins partiellement pourquoi on peut s'attendre à observer des effets quantiques même pour un nombre de particules relativement élevé (voir aussi [296, Section 6.4.1]).

Aspects expérimentaux

Malgré ce qui a été prédit dans les années 1920, ce n'est qu'au milieu des années 1990 que les équipes de C. Wieman, d'E. Cornell et de W. Ketterle parviennent à réaliser expérimentalement des condensats de Bose-Einstein. Ces trois chercheurs obtiendront le prix Nobel de physique en 2001 pour ces travaux[15].

---

[12]Un *boson* est une particule de spin entier. En ce qui concerne la condensation de Bose-Einstein, on s'intéressera surtout à des bosons composites, tels que des atomes. Les particules qui ne sont pas des bosons ont un spin demi-entier et sont des *fermions*. Ceux-ci obéissent au *principe d'exclusion de Pauli* qui stipule que deux fermions identiques ne peuvent pas occuper le même état quantique. Il n'existe donc aucun équivalent de la condensation de Bose-Einstein pour les fermions. Pour plus d'informations sur les bosons, fermions et le principe d'exclusion, on pourra par exemple se référer à [171, Section 5.1.1].

[13]Signalons qu'une présentation audiovisuelle pédagogique peut être consultée à l'adresse `https://toutestquantique.fr/condensation-de-bose-einstein/` (produite par le groupe de recherche « La Physique Autrement », `https://vulgarisation.fr/`).

[14]Où on trouvera une discussion à propos du chat de Schrödinger.

[15]Voir le communiqué de presse [282] et la présentation [281] desquels proviennent la plupart des informations présentées dans la section consacrée aux aspects expérimentaux.

Sur le plan expérimental, un des challenges majeurs consiste à atteindre des températures extrêmement basses. À cette fin, Wieman, Cornell et Ketterle ont notamment utilisé les progrès obtenus dans le *refroidissement d'atomes par laser*.

Ainsi, Wiemann et Cornell ont réalisé la condensation dans un gaz d'atomes de rubidium refroidi à environ 20 nanokelvins[16]. Peu après, Ketterle a produit un condensat dans un gaz d'atomes de sodium.

Pour plus d'informations sur les aspects expérimentaux de la réalisation des condensats, on pourra se référer à l'article d'exposition [137] et aux « Conférences Nobel » [110, 203, 333].

Les condensats de Bose-Einstein, des systèmes quantiques à N corps[17]

Désignons par Ω la région de $\mathbb{R}^3$ dans laquelle le condensat de Bose-Einstein est confiné et par N le nombre de bosons. À chaque boson correspond une position $x_1, \dots, x_N \in \Omega$.

L'opérateur hamiltonien quantique associé au système s'écrit sous la forme

$$H_N = -\Delta + \sum_{1 \le j \le N} W(x_j) + \sum_{1 \le i < j \le N} V_N(x_i - x_j),$$

où $W : \Omega \to \mathbb{R}$ et $V_N : \mathbb{R}^3 \to \mathbb{R}$. L'hamiltonien représente l'énergie du système de bosons.

L'*état fondamental* du système quantique est la fonction propre $\psi_N(x_1, \dots, x_N)$ de H_N associée au niveau d'énergie E_N le plus bas. Autrement dit, on a

$$H_N \psi_N = E_N \psi_N, \tag{I.2}$$

où $E_N \in \mathbb{R}$ est la première valeur propre de H_N.

Si le système se trouve dans l'état fondamental, cela signifie[18] que pour tout ensemble mesurable $S \subseteq \Omega^N$, si l'on effectue une mesure des positions, alors la probabilité que les N bosons soient tels que le vecteur des positions $(x_1, \dots, x_N)$ appartienne à S vaut

$$\int_S |\psi_N(x_1, \dots, x_N)|^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Lorsque le nombre N de particules présentes dans le système devient grand, la condensation de Bose-Einstein implique que ψ_N s'approche d'un état *factorisé* :

$$\psi_N(x_1, \dots, x_N) \approx \varphi(x_1) \cdots \varphi(x_N). \tag{I.3}$$

D'une certaine façon, cela signifie que le système se comporte comme une unique particule quantique occupant l'état φ.

[16] À titre de comparaison, la température de l'espace intersidéral est d'environ 2.7 kelvins (sous l'effet du fond diffus cosmologique, voir [123, Page 3, Figure 1.2], [156]).

[17] Cette section et la suivante s'inspirent fortement de [20, Sections 1.1 and 1.2], que nous recommandons au lecteur intéressé par les aspects liés à la physique quantique.

[18] Voir [171, Section 1.2] pour plus d'explications quant à l'interprétation statistique (ou « règle de Born ») des *états d'un système quantique* (aussi appelés *fonctions d'onde*, « *wave functions* »).

Émergence d'un modèle quantique *non-linéaire* ?

Il s'avère (voir [20, Section 1.1]) que l'état quantique commun φ apparaissant dans (I.3) appartient à $H^1(\Omega)$ et est tel que le produit $N\varphi$ minimise la *fonctionnelle de Gross-Pitaevskii*

$$E_{GP}(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + 8\pi\alpha\int_\Omega |u(x)|^4\,\mathrm{d}x \tag{I.4}$$

sous la contrainte

$$\int_\Omega |u(x)|^2\,\mathrm{d}x = N. \tag{I.5}$$

La constante α apparaissant dans (I.4) dépend de l'intensité de l'interaction entre les particules[19]. Nous nous intéresserons uniquement au cas[20] $\alpha < 0$, dit *focalisant.*

On peut montrer que si $u \in H^1(\mathcal{G};\mathbb{R})$ est un point critique de E_{GP} sous la contrainte $\|u\|^2_{L^2(\Omega)} = N$, alors u est une solution de l'équation de Schrödinger non-linéaire[21]

$$-\Delta u + \lambda u = -32\pi\alpha|u|^2 u \tag{I.6}$$

pour un certain $\lambda \in \mathbb{R}$. Ainsi, on cherche à déterminer des *solutions normalisées* (pour la norme L^2) d'équations de Schrödinger non-linéaires. Nous reviendrons sur cette notion dans la section I.5 de cette introduction.

La présence du terme de degré quatre dans la fonctionnelle (I.4) rend le modèle de Gross-Pitaevskii *non-linéaire*, comme on peut le voir dans l'équation (I.6). Cela a de quoi surprendre. En effet, la mécanique quantique[22] se formule en termes d'équations *linéaires* (voir [171, Sections 2.1 et 3.1]). Ainsi, nous avons décrit dans la section précédente le système quantique par une fonction d'onde ψ_N, *fonction propre* de l'*opérateur linéaire* H_N (voir l'équation (I.2)).

Néanmoins, on observe une non-linéarité dans l'écriture (I.3), où ψ_N n'est pas linéaire en φ. L'apparition de la fonctionnelle (I.4) et de la non-linéarité cubique dans l'équation (I.6) résulte de plusieurs approximations réalisées sur le système quantique à N corps, notamment un passage à la *limite de champ moyen* (dans le régime asymptotique $N \to \infty$). Nous n'en présenterons pas les détails techniques ici et renvoyons vers [114, 279, 293] pour des références du point de vue physique et, entre autres, vers[23] [2, 3, 15, 141, 220, 221, 222, 223, 270, 295] pour des références concernant les preuves de convergence (voir aussi [20, Section 1.2]).

[19] Seules les interactions entre paires de particules sont prises en compte dans l'analyse.

[20] On peut s'attendre à ce que le paramètre α soit positif dans les expériences, ce qui correspond à une interaction répulsive. C'est par exemple le cas pour le rubidium utilisé par Wieman et Cornell. Néanmoins, la valeur de la constante α peut être modulée expérimentalement grâce à un champ magnétique externe (phénomène de « résonance de Feshbach »). Ainsi, il est possible de réaliser des expériences correspondant au régime $\alpha < 0$. (voir par exemple [182]).

[21] Au moins en ce qui concerne les minima de E_{GP}, on peut remplacer u par $|u|$ sans changer la valeur de la fonctionnelle et ainsi supposer que u est à valeurs réelles.

[22] Du moins au sens de l'équation de Schrödinger. Nous n'évoquerons pas ici les théories quantiques plus modernes, notamment celles de mécanique quantique relativiste.

[23] Un grand merci au Pr. Adami d'avoir transmis la référence [3].

Structures ramifiées et atomtronique

Dans ce qui précède, nous avons décrit la condensation de Bose-Einstein dans un domaine Ω arbitraire. Il est pertinent de considérer le cas d'un domaine Ω « quasi unidimensionnel », formé de « câbles quantiques » (dans l'esprit de [208]).

De tels dispositifs peuvent être réalisés expérimentalement, notamment grâce à des *jonctions de Josephson* (voir[24] par exemple [94, 204, 228, 328]). Cela nous amène à l'*atomtronique*, récent domaine de recherche dédié à l'étude des circuits guidant la propagation d'atomes ultra-froids (voir [29]). L'idée générale est de produire des composants tels que ceux qu'on trouve dans les circuits électroniques en utilisant des *circuits de matière* (froide) et, si possible, de réaliser des effets « purement quantiques » dans ces circuits.

Ainsi, la perspective de l'atomtronique amène à étudier l'équation ($\mathrm{NLS}_\mathcal{G}$) sur les graphes métriques. En particulier, il sera important de comprendre le rôle joué par les propriétés topologiques et métriques des graphes, ce qui a attiré l'attention de plusieurs chercheurs ces dernières années. Pour plus d'informations sur le rôle de ($\mathrm{NLS}_\mathcal{G}$) en physique, on peut consulter les articles d'exposition [20, 255].

I.3.3 Genèse de l'équation de Schrödinger non-linéaire sur les graphes

Mentionnons tout d'abord les travaux pionniers de F. Ali Mehmeti (par exemple son livre [25]) et de J. von Below (voir par exemple[25] [58]), dans lesquels ces auteurs étudient des équations d'évolution *semi-linéaires* sur des graphes.

Concernant ($\mathrm{NLS}_\mathcal{G}$), bien qu'il s'agisse d'un sujet assez récent, la littérature est vaste. On pourra consulter avec intérêt les articles d'exposition [20, 199, 255].

Comme vu dans la section précédente, le cas $p = 4$ (pour lequel la non-linéarité dans ($\mathrm{NLS}_\mathcal{G}$) est *cubique*) est particulièrement important du point de vue de la physique. Les premiers travaux ont donc naturellement étudié ce cas[26].

Un des premiers articles à s'intéresser à l'équation de Schrödinger non-linéaire (d'évolution) sur des graphes métriques est [56], où sont étudiés plus généralement des « champs quantiques » sur les graphes en étoile. La modélisation des « câbles quantiques » est une raison de se pencher sur la question (voir la discussion sur l'atomtronique). Par la suite, les trois travaux [92], [305] et [7] sont publiés. Ils portent sur l'équation de Schrödinger (d'évolution) cubique sur des graphes en étoile[27]. Les auteurs y étudient notamment la dynamique des solitons.

[24] Nous avons découvert les références [204, 328] grâce à l'introduction de [19], qui contient également des éléments de discussion sur la physique des condensats.

[25] Un grand merci au Pr. Serge Nicaise de nous avoir transmis cet article.

[26] Signalons que le cas $p = 4$ est commode en dimension 1 grâce aux propriétés d'*intégrabilité* de l'équation d'évolution (voir la section F.8).

[27] Et quelques autres graphes, voir [305, Section IV].

Les articles précédents mettent en évidence une richesse de phénomènes ainsi que l'*importance des conditions de transmission en le nœud.* Notons que dans [7], une « condition de type δ » (ou simplement δ-condition) est employée en le sommet. Cette condition est plus générale que la condition de Kirchhoff et s'écrit

$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = \alpha u(v), \tag{I.7}$$

où α est un paramètre[28]. Dans le langage des équations aux dérivées partielles, il s'agit essentiellement d'une *condition de Robin* en le sommet.

Remarque. La présence d'un terme $\alpha \neq 0$ dans (I.7) correspond à un terme d'*interaction ponctuelle* (ou *défaut ponctuel*) en le sommet, autrement dit à un δ de Dirac.[29] C'est la raison pour laquelle la « δ-condition » porte ce nom.

Signalons que la δ-condition se généralise à d'autres graphes que ceux en étoile, auquel cas le coefficient α peut varier d'un sommet à l'autre (voir par exemple [199, Equation (1.5)] et [255, Section 1 (a)]).

I.3.4 Étude de (NLS$_\mathcal{G}$) sur des graphes spécifiques

Dans cette section, on s'intéresse à l'étude de (NLS$_\mathcal{G}$) sur des graphes spécifiques. Les résultats présentés ci-dessous se basent, pour la plupart, sur la théorie des équations différentielles ordinaires (EDOs).

Les intervalles, la demi-droite et le cercle

Si on considère des graphes contenant une seule arête, on retrouve les intervalles bornés de $\mathbb{R}$, la demi-droite, ou encore le cercle. Ainsi, les problèmes au bord couplant l'EDO $-u'' + \lambda u = |u|^{p-2}u$ avec des conditions au bord de Neumann ou des conditions périodiques[30] sont des instances (simples) de (NLS$_\mathcal{G}$).

Graphes compacts[31]

Citons [292], un des premiers travaux qui étudie le problème elliptique (NLS$_\mathcal{G}$) (et non l'équation d'évolution). Dans cet article, les graphes considérés sont des « étoiles compactes », formées d'un nœud central joint à plusieurs nœuds de degré un, en lesquels la condition de Dirichlet est imposée et pas celle de Kirchhoff[32].

Passons à présent à l'étude d'exemples de graphes *non-compacts.* Nous en avons déjà rencontré dans la section I.3.1 où nous avons déterminé les solutions de (NLS$_\mathcal{G}$) sur les graphes en étoile.

[28]Notons qu'on retrouve la condition de Kirchhoff si on prend $\alpha = 0$.

[29]Le terme α traduit alors l'intensité de cette interaction. Signalons que les défauts ponctuels peuvent être modélisés également sur la droite réelle. C'est le cas dans [179], article ayant servi de source d'inspiration aux auteurs de [7].

[30]Ou encore la condition de Dirichlet, comme nous le ferons dans le chapitre 2.

[31]Rappelons qu'il s'agit des graphes formés d'un nombre fini d'arêtes de longueurs finies.

[32]Autrement dit, la solution s'annule en ces nœuds et non sa dérivée.

Le « graphe têtard » et les graphes en fleur

Le[33] « *graphe têtard* » (*tadpole graph*) possède un unique sommet auquel sont reliées une boucle de longueur ℓ et une demi-droite. Il est représenté par la figure I.14.

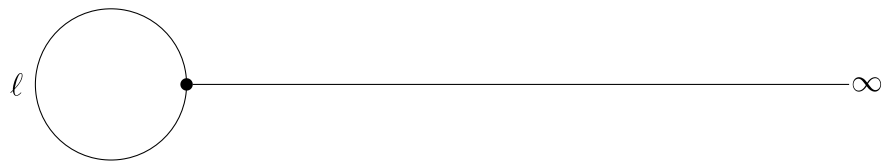

FIGURE I.14 : Le têtard

Le têtard est l'exemple le plus simple de graphe non-compact qui a une boucle.

Néanmoins, il peut réserver quelques surprises. Par exemple, considérons une solution[34] $v_0 : \mathbb{R} \to \mathbb{R}$ de période ℓ de l'EDO

$$-v_0'' = |v_0|^{p-2}v_0$$

telle que $v_0(0) = 0$. Définissons une fonction u sur le graphe têtard, égale à v_0 sur la boucle de longueur ℓ (dont on identifie le sommet avec 0) et nulle sur la demi-droite. Alors, par construction, u est une solution *à support compact* dans le graphe de l'équation (NLS$_\mathcal{G}$).

Ceci anéantit de façon spectaculaire tout espoir de *principe de continuation unique* pour les solutions de (NLS$_\mathcal{G}$). Nous y reviendrons dans la section I.9.

Quitte à considérer des solutions suffisamment oscillantes, on peut effectuer la même construction pour tout $\lambda \in \mathbb{R}$ (y compris pour $\lambda < 0$). On en déduit que, contrairement à la droite réelle pour laquelle l'EDO $-u'' + \lambda u = |u|^{p-2}u$ n'admet aucune solution telle que $u(x) \xrightarrow[x\to\pm\infty]{} 0$ lorsque $\lambda \leq 0$ (voir la proposition C.2), le problème (NLS$_\mathcal{G}$) admet des solutions pour tout $\lambda \in \mathbb{R}$. Signalons néanmoins que ces solutions sont nécessairement nulles sur la demi-droite lorsque $\lambda \leq 0$.

Le graphe têtard a été utilisé afin de montrer que les solutions de (NLS$_\mathcal{G}$) peuvent s'annuler sur des arêtes. Des études détaillées du problème (NLS$_\mathcal{G}$) sur ce graphe existent. Par exemple, C. Cacciapuoti, D. Finco et D. Noja classifient toutes les solutions lorsque $p = 4$ dans [90]. De plus, D. Noja, D.E. Pelinovsky et G. Shaikhova déterminent des branches de solutions et étudient leur stabilité dans [256]. Le cas $p = 6$ a, quant à lui, été étudié par Noja et Pelinovsky dans [257].

[33] Nous utiliserons le singulier pour désigner le têtard car, du point de vue topologique, c'est le seul graphe formé d'un unique sommet auquel sont reliées une boucle et une demi-droite. Néanmoins, rappelons que nos graphes sont métriques et qu'il existe une infinité de « têtards métriques » selon la longueur de la boucle. En pratique, aucune confusion ne devrait survenir.

[34] Il est standard de montrer que de telles solutions existent, voir le lemme 4.21.

Plus généralement, intéressons-nous aux « *graphes en fleur* » (*flower graphs*) à $N \geq 2$ pétales[35], constitués d'un unique sommet duquel partent une demi-droite ainsi que N boucles de même longueur, comme on peut le voir dans la figure I.15.

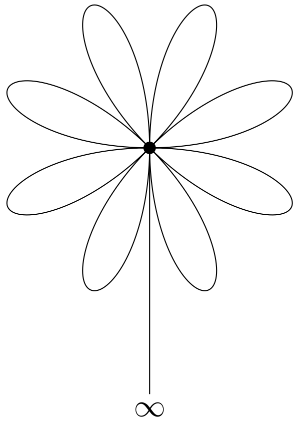

FIGURE I.15 : Une fleur à huit pétales

Dans [198], A. Kairzhan, R. Marangell, D.E. Pelinovsky et K. Xiao analysent sur ces graphes des *phénomènes de bifurcation* (lorsque $p = 4$), notamment grâce à des arguments de la théorie des EDO (analyse de la *fonction période*). On peut également se référer à [199, Section 6.3] pour une présentation de leurs résultats.

Le double-pont

Le « double-pont » (*double-bridge*) est représenté par la figure I.16.

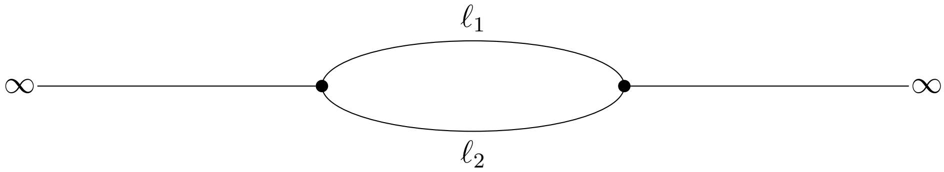

FIGURE I.16 : Le double-pont

Dans [258], D. Noja, S. Rolando et S. Secchi étudient les solutions de ($NLS_{\mathcal{G}}$) sur le double-pont pour une non-linéarité cubique. Dans ce cas, les solutions des équations différentielles sur les arêtes s'expriment grâce aux *fonctions elliptiques de Jacobi*[36], ce qui est utilisé de façon importante dans les raisonnements.

[35] Dans le cas $N = 1$, on retrouve le graphe têtard.

[36] Le cours du Pr. Diego Noja lors de l'école d'été *Nonlinear Quantum Graphs* à Valenciennes nous a beaucoup appris sur les travaux concernant ($NLS_{\mathcal{G}}$) dans le cas $p = 4$. C'est en particulier à cette occasion que nous avons découvert les fonctions elliptiques de Jacobi. Plusieurs références présentées dans cette introduction proviennent de ce cours. Pour en savoir plus sur l'emploi des fonctions elliptiques de Jacobi, on pourra par exemple se référer à [209] et à [199, Section 5].

L'étude se révèle être très riche. Il s'avère que la description de l'ensemble des solutions dépend de façon importante du rapport $r := \frac{\ell_1}{\ell_2}$ entre les longueurs des arêtes bornées. En particulier, le fait que r soit rationnel ou non a son importance.

Même si ce n'est pas flagrant « visuellement », il y a une grande différence entre l'étude de $(\mathrm{NLS}_{\mathcal{G}})$ sur la droite réelle et sur le double-pont !

Le $\mathcal{T}$-graphe

Le $\mathcal{T}$-graphe est représenté par la figure I.17.

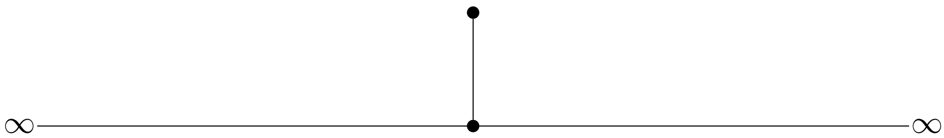

FIGURE I.17 : Le $\mathcal{T}$-graphe

Dans [22], les auteurs classifient complètement les solutions de $(\mathrm{NLS}_{\mathcal{G}})$ sur le $\mathcal{T}$-graphe et y donnent des informations sur leur stabilité et leurs caractérisations variationnelles. Nous y reviendrons dans la section I.6.4, où le $\mathcal{T}$-graphe est un exemple emblématique de graphe non-compact admettant des *ground states*.

I.3.5 Une formulation variationnelle de $(\mathrm{NLS}_{\mathcal{G}})$

Dans la section I.3.1, nous avons montré qu'on peut déterminer *toutes* les solutions de l'équation de Schrödinger non-linéaire sur les graphes en étoile. Pour ce faire, il faut combiner l'analyse de l'EDO sur les arêtes et un argument de « recollement » en le sommet.

Nous venons de voir qu'une telle analyse est parfois possible sur des graphes spécifiques[37]. Néanmoins, elle devient rapidement très complexe dès que le nombre de sommets augmente ou que des arêtes bornées sont présentes.

Afin d'étudier le problème $(\mathrm{NLS}_{\mathcal{G}})$ de façon plus générale, il convient d'adopter une formulation plus adaptée et moins sensible aux spécificités du graphe ou à la valeur de l'exposant.

C'est ainsi que nous adopterons une *approche variationnelle*, comme celle déjà rencontrée dans la section I.3.2 consacrée aux condensats de Bose-Einstein où nous avons vu qu'il est important de minimiser la fonctionnelle de Gross-Pitaveskii sur la contrainte de norme L^2.

À présent, introduisons un espace de Hilbert et une fonctionnelle utilisés dans l'approche variationnelle.

[37]Quitte à supposer que $p = 4$, voir la section I.3.4.

L'espace $H^1(\mathcal{G})$ et la fonctionnelle d'action J_λ

Sur l'espace de Sobolev[38]

$$H^1(\mathcal{G}) := \Big\{u : \mathcal{G} \to \mathbb{R} \mid u \text{ est continue}, u, u' \in L^2(\mathcal{G})\Big\},$$

on définit la *fonctionnelle d'action* $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$ par

$$J_\lambda(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \,\mathrm{d}x.$$

Remarquons que les dérivées directionnelles de J_λ sont données par

$$J'_\lambda(u)[v] = \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x + \lambda\int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

pour tout couple de fonctions u et v dans $H^1(\mathcal{G})$.

À présent, montrons que les points critiques de J_λ sur $H^1(\mathcal{G})$ sont des solutions[39] de ($\mathrm{NLS}_{\mathcal{G}}$). Le raisonnement est classique et analogue à celui reliant une équation aux dérivées partielles à sa formulation faible. Nous en présentons néanmoins les détails pour mettre en évidence l'apparition naturelle de la condition de Kirchhoff.

L'équation différentielle arête par arête

Si φ est une fonction $\mathcal{C}^\infty$ à support compact à l'intérieur d'une arête e reliant les sommets a et b (voir la figure I.18), on a[40]

$$\begin{aligned} 0 &= J'_\lambda(u)[\varphi] \\ &= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda\int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}u}{\mathrm{d}x_e}(b)\underbrace{\varphi(b)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(a)\underbrace{\varphi(a)}_{=0} \\ &\quad + \int_e \Big(-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x)\Big)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

Dès lors, on a

$$\int_e \Big(-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x)\Big)\varphi(x)\,\mathrm{d}x = 0$$

pour toute fonction test φ de classe $\mathcal{C}^\infty$ à support compact dans e. On en déduit que l'équation $-u'' + \lambda u = |u|^{p-2}u$ est satisfaite à l'intérieur de l'arête e.

[38] Voir la section A.4 pour davantage de détails sur la structure de l'espace $H^1(\mathcal{G})$. Puisque nous travaillons en dimension un, *toutes les fonctions de* $H^1(\mathcal{G})$ *sont continues*, comme prouvé dans la proposition A.7.

[39] La réciproque de cette affirmation est vraie et se prouve de façon semblable.

[40] Ici, on suppose que u est assez régulière que pour justifier les intégrations par parties. Plus rigoureusement, il faut comprendre l'équation comme étant satisfaite au sens faible et prouver un résultat de régularité elliptique. Les détails sont classiques et nous les omettons.

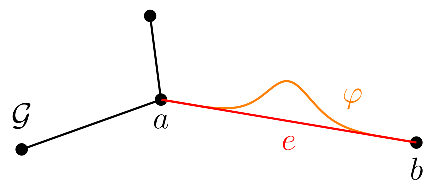

FIGURE I.18 : Fonction test dont le support compact est inclus dans l'intérieur d'une arête.

La condition de Kirchhoff

Soit a un sommet de degré D de $\mathcal{G}$ et soient $b_1, \ldots, b_D$ les sommets adjacents à a.

On définit une fonction φ, affine sur les arêtes de $\mathcal{G}$ et telle que $\varphi(a) = 1$ et $\varphi(v) = 0$ pour tout sommet $v \neq a$ (voir la figure I.19). En notant e_i l'arête joignant a et b_i, on obtient

$$\begin{aligned} 0 &= J'_\lambda(u)[\varphi] \\ &= \sum_{1\le i\le D} \left(\int_{e_i} u'\varphi' \,\mathrm{d}x + \lambda \int_{e_i} u\varphi \,\mathrm{d}x - \int_{e_i} |u|^{p-2}u\varphi \,\mathrm{d}x \right) \\ &= \sum_{1\le i\le D} \left(\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(b_i) \underbrace{\varphi(b_i)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a) \underbrace{\varphi(a)}_{=1} \right) \\ &\quad + \sum_{1\le i\le D} \int_{e_i} \underbrace{\left(-u'' + \lambda u - |u|^{p-2}u \right)}_{=0} \varphi(x) \,\mathrm{d}x \end{aligned}$$

donc $\sum_{1\le i\le D} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a_i) = 0$, ce qui est la condition de Kirchhoff.

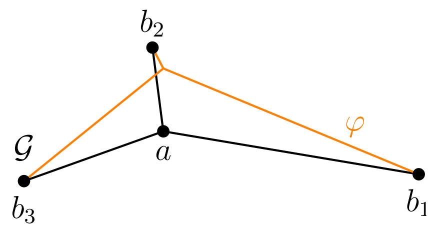

FIGURE I.19 : Fonction affine utilisée pour retrouver la condition de Kirchhoff.

Et maintenant ?

Nous venons de voir que les solutions de $(\mathrm{NLS}_\mathcal{G})$ correspondent aux points critiques de la *fonctionnelle d'action* $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$, définie par

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},$$

où

$$H^1(\mathcal{G}) = \left\{ u : \mathcal{G} \to \mathbb{R} \;\middle|\; u \text{ est continue et } u, u' \in L^2(\mathcal{G}) \right\}.$$

Remarquons que la fonctionnelle d'action J_λ n'est pas bornée inférieurement sur $H^1(\mathcal{G})$. En effet, si $u \neq 0$, alors

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t\to\infty]{} -\infty.$$

Une stratégie possible pour trouver des points critiques de J_λ consiste à étudier des *problèmes de minimisation avec contraintes* afin d'obtenir des solutions de ($\mathrm{NLS}_\mathcal{G}$) qui sont des minima de J_λ sous la contrainte. Nous allons présenter une façon de mener à bien cette démarche dans la section I.4 suivante.

I.3.6 Quelques mots sur les simulations numériques

Lorsqu'on veut étudier en détail un exemple spécifique de graphe, les calculs peuvent s'avérer très complexes. Dès lors, les simulations numériques sont un outil précieux.

À ce sujet, citons la librairie Python « GraFiDi[41] » développée par C. Besse, R. Duboscq et S. Le Coz (voir [71, 72]) ainsi que le package MATLAB « QGLAB[42] » développé par R. Goodman[43], G. Conte et J. Marzuola (voir [169]).

Nous emploierons aussi l'outil numérique dans le chapitre 5 dans lequel nous développerons une *preuve assistée par ordinateur* (voir la section I.11.5).

I.4 (Nodal) action ground states

I.4.1 Variété de Nehari et action ground states

Définitions et premières propriétés

Considérons la *variété de Nehari* associée à ($\mathrm{NLS}_\mathcal{G}$) et définie par

$$\begin{aligned}\mathcal{N}_\lambda(\mathcal{G}) &:= \Big\{u \in H^1(\mathcal{G}) \setminus \{0\} \;\Big|\; J'_\lambda(u)u = 0\Big\}\\ &= \Big\{u \in H^1(\mathcal{G}) \setminus \{0\} \;\Big|\; \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})}\Big\}.\end{aligned}$$

La variété[44] de Nehari contient tous les points critiques non nuls de J_λ et on peut montrer qu'un point critique de J_λ sous la contrainte d'appartenance à $\mathcal{N}_\lambda(\mathcal{G})$ est une solution de ($\mathrm{NLS}_\mathcal{G}$) (voir par exemple [35, Remark 2.3.13]).

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[41] Disponible à l'adresse `https://plmlab.math.cnrs.fr/cbesse/grafidi`.

[42] Disponible à l'adresse `https://github.com/manroygood/Quantum-Graphs/tree/master`.

[43] Qu'on remercie encore pour son cours donné lors de l'école d'été *Nonlinear Quantum Graphs*. Le lecteur pourra consulter le support utilisé durant les leçons à l'adresse `https://roygoodman.net/course/nqg_valenciennes/`.

[44] Il est possible de montrer que la variété de Nehari est une variété hilbertienne $\mathcal{C}^2$ modelée sur $H^1(\mathcal{G})$. Nous n'aurons en général pas besoin d'utiliser de telles considérations. Nous renvoyons vers [28, Chapitre 6] pour en savoir plus sur le point de vue des contraintes en tant que variétés.

D'après la définition de $\mathcal{N}_\lambda(\mathcal{G})$, on remarque que

$$u \in \mathcal{N}_\lambda(\mathcal{G}) \quad \Longrightarrow \quad J_\lambda(u) = \left(\tfrac{1}{2} - \tfrac{1}{p}\right)\|u\|^p_{L^p(\mathcal{G})}\,.$$

En particulier, J_λ est strictement positive sur $\mathcal{N}_\lambda$, donc bornée inférieurement. On définit

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u\in\mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u).$$

Un *action ground state* pour ($\mathrm{NLS}_{\mathcal{G}}$) est une fonction $u \in \mathcal{N}_\lambda(\mathcal{G})$ telle que

$$J_\lambda(u) = \mathcal{J}_{\mathcal{G}}(\lambda).$$

Si un action ground state existe, on peut montrer qu'il s'agit d'une solution de signe constant de ($\mathrm{NLS}_{\mathcal{G}}$), voir par exemple [35, Remark 2.3.13] ou [318, Corollary 10, (c)] pour une preuve. Remarquons que lorsque $\mathcal{G}$ n'est pas compact, l'existence d'action ground states n'est en général pas garantie.

Bien que nous venons d'introduire la variété de Nehari et la notion d'action ground state sur les graphes, on peut suivre la même démarche dans de nombreuses situations : équations différentielles, équations elliptiques sur des ouverts de $\mathbb{R}^N$, systèmes d'équations elliptiques, etc. L'article fondateur de Z. Nehari [250] est dédié à un problème d'EDO. Le lecteur désireux de s'introduire au sujet pourra notamment consulter [35, Section 2.3] ou l'article d'exposition [318].

Quelques travaux concernant les action ground states sur les graphes

Jusqu'à présent, les action ground states pour ($\mathrm{NLS}_{\mathcal{G}}$) ont été assez peu étudiés. En effet, la plupart des travaux s'intéressent aux solutions normalisées (voir la section I.5). Néanmoins, citons [10, 11, 126, 211, 218, 263, 302], qui considèrent la fonctionnelle d'action et la variété de Nehari sur les graphes.

Dans [10, 11], considérer des action ground states (sur des graphes en étoile, avec la δ-condition) permet d'employer la méthode d'étude de la stabilité orbitale des solutions développée par M.I. Weinstein et M. Grillakis–J. Shatah–W. Strauss (voir [172, 173, 330, 331] et la section F.7).

L'article [218] montre que quelques arguments développés pour les solutions normalisées sur les graphes (voir la section I.5) s'adaptent au cas de la variété de Nehari.

Le phénomène de *concentration* des action ground states est étudié dans les travaux [211, 302]. Nous y reviendrons dans la section I.7.

L'article [263] étudie ($\mathrm{NLS}_{\mathcal{G}}$) sur des graphes périodiques grâce à des techniques basées sur les action ground states. Nous considérerons des problèmes similaires dans la section 2.5.1.

Finalement, dans [126], S. Dovetta étudie le comportement asymptotique des action ground states sur une grille entière (voir la figure I.4 de la page 4) dont les longueurs convergent vers 0. L'auteur prouve des résultats de convergence entre l'action ground state sur les grilles (qui sont des graphes métriques) et le soliton de l'équation aux dérivées partielles (NLS). Il s'agit donc d'une manifestation du phénomène de « réduction dimensionnelle » par lequel nous avons introduit le modèle ($\mathrm{NLS}_\mathcal{G}$) dans la section I.3.

Les chapitres 1 et 2 de cette thèse étudieront en détail la notion d'action ground state sur les graphes (compacts et non-compacts), en fournissant notamment des théorèmes d'existence et de non-existence. Dans le chapitre 3, nous étudierons la masse L^2 des action ground states sur des domaines bornés de $\mathbb{R}^N$ afin d'y obtenir des solutions normalisées.

I.4.2 Ensemble de Nehari nodal et nodal ground states

Les action ground states présentés dans la section précédente sont des solutions *de signe constant* de ($\mathrm{NLS}_\mathcal{G}$). À présent, intéressons-nous aux solutions changeant de signe, également appelées *solutions nodales*.

Étant donné une fonction u à valeurs réelles, on pose

$$u^+ := \max(u, 0) \quad \text{et} \quad u^- := \min(u, 0).$$

Une *solution nodale* est, par définition, une solution u de ($\mathrm{NLS}_\mathcal{G}$) telle que $u^+ \neq 0$ et $u^- \neq 0$. Toutes les solutions nodales de ($\mathrm{NLS}_\mathcal{G}$) appartiennent à *l'ensemble*[45] *de Nehari nodal*

$$\mathcal{N}_\lambda^{nod}(\mathcal{G}) := \Big\{ u \in H^1(\mathcal{G}) \ \Big| \ u^\pm \in \mathcal{N}_\lambda(\mathcal{G}) \Big\} = \Big\{ u \in H^1(\mathcal{G}) \ \Big| \ u^\pm \neq 0,\ J_\lambda'(u)u^\pm = 0 \Big\}.$$

Une fonction $u \in \mathcal{N}_\lambda^{nod}(\mathcal{G})$ est un *nodal ground state* de ($\mathrm{NLS}_\mathcal{G}$) si

$$J_\lambda(u) = \inf_{v \in \mathcal{N}_\lambda^{nod}(\mathcal{G})} J_\lambda(v).$$

Lorsqu'ils existent, les nodal ground states sont des solutions changeant de signe de ($\mathrm{NLS}_\mathcal{G}$) (voir par exemple [318, Proof of Theorem 18]). Plus précisément, il s'agit de *solutions nodales d'action minimale* du problème.

L'article fondateur qui étudie la méthode de minimisation sur l'ensemble de Nehari nodal est celui de A. Castro, J. Cossio et J.M. Neuberger [93]. Pour un aperçu plus détaillé, nous renvoyons vers [53, 318] ainsi que vers les chapitres 2 et 3.

[45] Contrairement à la variété de Nehari, l'ensemble de Nehari nodal $\mathcal{N}_\lambda^{nod}$ n'est en général pas muni d'une structure de variété. Voir [53, Introduction et Lemma 3.2] pour une discussion et la présentation d'une stratégie pour y remédier, en utilisant la norme H^2 et non la norme H^1.

À notre connaissance, les nodal ground states n'avaient jamais été étudiés sur les graphes avant les travaux présentés dans cette thèse.

Les nodal ground states joueront un grand rôle dans le chapitre 2 (où l'on étudiera leur existence sur des graphes non-compacts) ainsi que dans le chapitre 3 (où l'on étudiera leurs masses L^2 sur des domaines ouverts de $\mathbb{R}^N$ afin d'obtenir des solutions nodales normalisées). Ils seront également considérés dans le chapitre 5 où l'on déterminera leurs propriétés qualitatives dans le cas des graphes compacts.

I.5 Solutions normalisées[46]

I.5.1 Que sont les solutions normalisées ?

Une solution *normalisée* de ($\mathrm{NLS}_{\mathcal{G}}$) est une solution dont la norme L^2 est imposée mais où la valeur du paramètre λ ne l'est pas. Cette notion est importante lorsqu'on étudie certains modèles physiques (voir la section I.3.2). Par exemple, lorsqu'on modélise un condensat de Bose-Einstein, nous avons vu dans la section I.3.2 que la norme L^2 correspond au nombre de particules dans le système décrit (voir la contrainte (I.5)).

Rechercher des solutions normalisées est également utile lorsqu'on étudie des équations d'évolution (voir l'annexe F et en particulier la section F.7).

Du point de vue variationnel, les solutions normalisées correspondent à des points critiques de la *fonctionnelle d'énergie*[47]

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},$$

sur la *contrainte de masse* $\|u\|^2_{L^2} = \mu$, μ étant la masse[48]. Dans ce cas, le paramètre λ apparaît en tant que multiplicateur de Lagrange provenant de la contrainte.

Sur un graphe métrique $\mathcal{G}$, étant donné $\mu > 0$, on peut montrer que l'infimum

$$\inf_{\substack{u\in H^1(\mathcal{G})\\ \|u\|^2_{L^2}=\mu}} E(u)$$

est fini lorsque $2 < p < 6$ et vaut $-\infty$ lorsque $p > 6$ (voir par exemple [20, Section 2.1])[49]. Apparaît ici l'*exposant masse-critique* du problème : il vaut $p = 6$.

[46] Un grand merci au Pr. Louis Jeanjean pour la discussion à propos de l'historique de l'étude des solutions normalisées.

[47] Déjà rencontrée dans le cas $p = 4$ de la section I.3.2 sous le nom de *fonctionnelle de Gross-Pitaevskii* E_{GP} définie par (I.4). Rappelons que le paramètre α dans (I.4) est strictement négatif.

[48] Nous utiliserons parfois d'autres conventions de normalisation « à constante multiplicative près », par exemple dans la section I.10.3 ainsi que dans le chapitre 3.

[49] Lorsque $p = 6$, la fonctionnelle E est bornée inférieurement sur la contrainte de masse si et seulement si μ est suffisamment petit.

Plus généralement, si Ω est un ouvert borné de $\mathbb{R}^N$, l'inégalité de Gagliardo–Nirenberg (voir par exemple [332])

$$\|u\|_{L^p}^p \leq K_p \|u\|_{L^2}^{p-N\left(\frac{p}{2}-1\right)} \|\nabla u\|_{L^2}^{N\left(\frac{p}{2}-1\right)}, \qquad \forall u \in H^1(\mathbb{R}^N),$$

valable pour tout[50] $p \in (2, 2^*)$, implique que la fonctionnelle d'énergie

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x$$

est bornée inférieurement sur la contrainte de masse lorsque $p < 2 + \frac{4}{N}$. Ce n'est pas le cas lorsque $p > 2 + \frac{4}{N}$ (voir [183]). Ainsi, la valeur $2 + \frac{4}{N}$ est l'*exposant masse-critique* en dimension N. Notons qu'on retrouve $p = 6$ en dimension $N = 1$.

L'exposant masse-critique joue aussi un grand rôle dans l'étude de l'équation d'évolution de Schrödinger non-linéaire (voir les sections F.3 à F.5).

Lorsque l'énergie est bornée inférieurement sur la contrainte, on appelle *energy ground state* un minimum de l'énergie sous la contrainte de masse. On montre alors qu'il s'agit nécessairement d'une solution de signe constant de (NLS). Dans tous les cas, on dispose également de la notion de *solution d'énergie minimale*, c'est-à-dire d'une solution minimisant la fonctionnelle d'énergie parmi l'ensemble des solutions de masse μ de (NLS) (pour un certain λ pouvant différer d'une solution à l'autre). Nous reviendrons sur ces notions dans la section I.10.

I.5.2 Genèse des solutions normalisées[51]

Bifurcation depuis le spectre essentiel

Dans [316], C.A. Stuart étudie des phénomènes de bifurcation. Illustrons ce concept sur un exemple, en considérant l'équation différentielle

$$-u''(x) - \frac{|u(x)|^{p-2}u(x)}{x} = \lambda u(x) \tag{I.8}$$

où $x \in (0, +\infty)$, $p \in (2, +\infty)$ et $\lambda \in \mathbb{R}$. Définissons l'espace de Hilbert

$$\tilde{H} := H^2(0, +\infty) \cap H_0^1(0, +\infty),$$

muni de la norme H^2 ainsi que l'ensemble

$$E := \left\{(u, \lambda) \in \tilde{H} \times \mathbb{R} \;\middle|\; u \text{ est une solution de (I.8)}, u \neq 0\right\}.$$

Suivant Stuart, on dit que $\lambda \in \mathbb{R}$ est *un point de bifurcation* si $(0, \lambda)$ appartient à l'adhérence de E dans $\tilde{H} \times \mathbb{R}$.

[50]Rappelons que 2^* est l'exposant critique de Sobolev, qui vaut $+\infty$ lorsque la dimension N vaut 1 ou 2 et vaut $2N/(N-2)$ lorsque $N \geq 3$.

[51]Un grand merci au Pr. Charles A. Stuart d'avoir transmis ses travaux sur le sujet et d'en avoir clarifié la chronologie. La lecture de [185, Section 4, Solutions de normes prescrites] nous a également été utile pour écrire cette section.

Lors de la publication de [316] dans les Comptes Rendus de l'Académie des Sciences de Paris en 1977, de tels phénomènes de bifurcation sont étudiés dans la littérature, mais uniquement à partir de valeurs propres (voir notamment [285] et les références dans [316] pour plus d'informations).

Celles-ci n'existent pas toujours, notamment sur les domaines non-compacts. Par exemple, sur $[0, +\infty)$, l'opérateur $v \mapsto -v''$ avec la condition de Dirichlet en l'origine admet $[0, +\infty)$ comme spectre mais ne possède aucune valeur propre.

Ainsi, l'originalité de la démarche de Stuart consiste à étudier le phénomène de *bifurcation depuis le spectre essentiel*. Les premiers résultats obtenus de la sorte sont présentés dans l'article [104] de R. Chiappinelli et C.A. Stuart.

À présent, voyons quel est le lien entre le phénomène de bifurcation depuis le spectre essentiel et la recherche de solutions normalisées.

Des problèmes de bifurcation aux solutions normalisées

Les travaux [311, 312, 313, 314, 315] de Stuart mettent en évidence un lien entre le phénomène de bifurcation et les solutions normalisées : *si un problème admet des energy ground states d'énergie négative pour toute masse L^2 suffisamment petite, alors* 0 *est un point de bifurcation*. Le lecteur pourra se référer aux références mentionnées ci-avant (voir notamment [313, Theorem 2.1] et [312, Theorem 3.3]).

Dans ces articles, des techniques pour prouver l'existence d'energy ground states sont développées.

P.L. Lions obtiendra ensuite dans [224, 226, 227] ses résultats de concentration-compacité, particulièrement bien adaptés pour étudier le problème de minimisation de l'énergie sous la contrainte de masse.

I.5.3 Solutions normalisées sur les graphes en étoile dans le régime masse-sous-critique ($2 < p < 6$)

Après leurs premiers travaux [7, 10], R. Adami, C. Cacciapuoti, D. Finco et D. Noja poursuivent leur étude des solutions sur les graphes en étoile.

La caractérisation variationnelle de la solution stationnaire (voir la figure I.12 à la page 13) sur l'étoile à trois branches, avec la condition de Kirchhoff et une non-linéarité cubique, est clarifiée dans [8]. Il apparaît que la solution est un *point de selle* de l'énergie sur la contrainte de masse et non un minimum.

Mentionnons [6], dédié à l'existence de solutions normalisées et à l'étude de leur stabilité orbitale sur des graphes en étoile. Ce travail améliore les résultats de [10] et amène Adami, Cacciapuoti, Finco et Noja à adapter les méthodes de concentration-compacité de Lions [224, 226, 227] au cas des graphes en étoile. Signalons que par la suite, A. Kairzhan a décrit précisément la stabilité de ces solutions dans [197] en utilisant des techniques basées sur la *théorie de Sturm*.

Pour conclure, soulignons que dans [9], les auteurs prouvent que les solutions stationnaires sur les graphes en étoile, *en présence d'une δ-interaction attractive en le sommet*, sont toujours des minima *locaux*[52] de l'énergie sur la contrainte de masse. Ce résultat remarquable implique la stabilité orbitale de ces solutions.

Les travaux mentionnés précédemment mettent en évidence une profondeur mathématique peut-être insoupçonnée. Et ce, *rien que sur les graphes en étoile* !

Des questions se posent naturellement. *Que se passe-t-il sur des graphes dont la structure est plus complexe que ceux en étoile ? Y a-t-il une richesse de phénomènes dans ce cas, même en l'absence de la δ-interaction en les sommets ?*

Nous avons déjà partiellement répondu à ces questions lors de nos études de ($\mathrm{NLS}_{\mathcal{G}}$) sur des graphes spécifiques dans la section I.3.4. Nous y reviendrons dans la section I.6 dédiée aux domaines *non-compacts*.

Il existe aussi des questions pertinentes sur des graphes compacts, comme nous allons le voir dans la section suivante.

I.5.4 Solutions normalisées sur les graphes compacts

Si $\mathcal{G}$ est un graphe compact de longueur totale $|\mathcal{G}|$ et si μ est un réel positif, alors la fonction constante $u : \mathcal{G} \to \mathbb{R}$ valant $c := \sqrt{\frac{\mu}{\mathcal{G}}}$ en tout point du graphe est telle que $\|u\|_{L^2}^2 = \mu$ et est une solution de ($\mathrm{NLS}_{\mathcal{G}}$) si on prend $\lambda = c^{p-2}$. Au signe près, il s'agit de l'unique solution constante du problème.

Deux questions se posent. *Y a-t-il d'autres solutions que la solution constante ? La solution constante est-elle un energy ground state ?*

La première question admet une réponse positive dans tous les cas. En effet, il existe toujours une infinité de solutions lorsque $2 < p < 6$, comme prouvé dans [124] (l'auteur y étudie également le cas critique $p = 6$).

La réponse à la deuxième question dépend en général des paramètres. Une analyse des propriétés variationnelles de la solution constante et de sa stabilité orbitale est réalisée dans [89] pour les cas $p < 6$ et $p = 6$. Une étude plus précise de cette seconde question a été réalisée par J.L. Marzuola et D.E. Pelinovsky dans [231] sur « l'*haltère* » (*dumbbell graph*) représenté ci-dessous par la figure I.20.

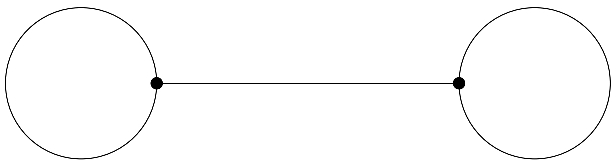

FIGURE I.20 : L'haltère

[52]Rappelons qu'il n'existe pas d'energy ground states sur les étoiles à $N \geq 3$ branches, c'est-à-dire que l'énergie n'admet pas de minima globaux sous la contrainte de masse.

Ces deux auteurs considèrent des *phénomènes de bifurcation* pour les solutions sur l'haltère grâce à des méthodes plus proches des travaux mentionnés dans la section I.3.4. Cela permet de déterminer dans quels cas la solution constante de ($\text{NLS}_\mathcal{G}$) est un ground state. Cette démarche sur l'haltère a été poursuivie par R. Goodman dans [168]. Le lecteur pourra se référer à [199, Section 6.3] pour une présentation des résultats obtenus dans les travaux mentionnés précédemment.

Remarque. Il n'est pas étonnant qu'une forme d'haltère soit considérée pour étudier la brisure de symétrie des ground states. En effet, des versions « épaissies » (comme dans la figure I.8 de la page 7) du graphe en haltère sont des exemples classiques de domaines ouverts de $\mathbb{R}^N$ illustrant ce phénomène pour des équations aux dérivées partielles superlinéaires, voir par exemple [118] et [175, Page 18, Figure 3].

Remarque. Sur les graphes compacts, la solution constante est présente dans les trois régimes $p < 6$, $p = 6$ et $p > 6$. Si l'on développe un résultat d'existence de solutions, il faudra donc prendre soin de trouver des solutions non-constantes, sous peine d'avoir prouvé un théorème sans intérêt.

Pour conclure, signalons que les graphes compacts sont des cadres commodes dans lesquels on peut étudier les *propriétés qualitatives* ou *l'unicité* de certains types de solutions de ($\text{NLS}_\mathcal{G}$). On peut aussi y voir le rôle joué par les sommets de Dirichlet, ou encore le phénomène des solutions s'annulant identiquement sur des arêtes (voir la section I.9). Ces questions seront au cœur du chapitre 5.

I.5.5 Quelques concepts importants

Dans la suite de la section I.5, nous allons considérer le régime masse-supercritique, pour lequel l'énergie n'est plus bornée inférieurement sur la contrainte de masse.[53] Nous présenterons les travaux sur $\mathbb{R}^N$ et plusieurs méthodes « abstraites ». Nous reviendrons ensuite sur le cas des domaines bornés et des graphes métriques, qui seront étudiés dans les chapitres 3 et 4.

Avant de poursuivre, il convient d'introduire plusieurs concepts classiques dans la théorie des équations aux dérivées partielles elliptiques semi-linéaires. Nous avons suivi [136, Chapter 1] et [115] pour les présenter.

Stabilité, indices de Morse, non-dégénérescence

Étant donné une solution[54] $u \in H_0^1(\Omega)$ de (NLS), c'est-à-dire un point critique de J_λ sur $H_0^1(\Omega)$, on considère la forme quadratique Q_u définie par

$$Q_u(\varphi) := J_\lambda''(u)[\varphi, \varphi] = \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x + \lambda \int_\Omega \varphi^2 \,\mathrm{d}x - (p-1) \int_\Omega |u|^{p-2}\varphi^2 \,\mathrm{d}x.$$

[53] Dans ce cas, on ne peut plus étudier un problème de minimisation sous contraintes pour trouver des solutions normalisées.

[54] Nous avons choisi de présenter les concepts pour le problème posé sur un domaine $\Omega \subseteq \mathbb{R}^N$ avec la condition au bord de Dirichlet. Les définitions s'adaptent, *mutatis mutandis*, aux autres cadres : problèmes posés sur $\mathbb{R}^N$, sur des graphes métriques, etc.

On dit alors que la solution u est :

- *stable* si $Q_u(\varphi) \geq 0$ pour tout[55] $\varphi \in \mathcal{C}^1_c(\Omega)$ ([136, Definition 1.1.2]) ;
- *stable en dehors du compact* $K \subset \Omega$ si $Q_u(\varphi) \geq 0$ pour tout $\varphi \in \mathcal{C}^1_c(\Omega \setminus K)$ ([136, Definition 1.5.1]) ;
- d'*indice de Morse* k si k est la dimension maximale d'un sous-espace vectoriel V de $\mathcal{C}^1_c(\Omega)$ tel que $Q_u(\varphi) < 0$ pour tout $\varphi \in V \setminus \{0\}$ ([136, Definition 1.5.2]) ;
- *non-dégénérée* si 0 n'est pas une valeur propre de l'opérateur linéarisé autour de u. Cet opérateur[56] $L_u : H^1_0(\Omega) \to H^{-1}(\Omega)$ est associé à la forme quadratique Q_u au sens où
$$\Big\langle L_u[\varphi] \mid \psi \Big\rangle := \frac{1}{2} Q'_u(\varphi)[\psi]$$
pour toute fonction $\psi \in H^1_0(\Omega)$. Plus concrètement, cela signifie que
$$L_u[\varphi] = -\Delta\varphi + \lambda\varphi - (p-1)|u|^{p-2}\varphi,$$
voir par exemple [115, (3.21), page 94].

Les notions de stabilité et d'indice de Morse (ainsi que celle d'*indice de Morse approché*) joueront un grand rôle dans le chapitre 4. Celle de solution non-dégénérée sera très importante dans le chapitre 5.

Pour en savoir plus sur le rôle joué par la stabilité et l'indice de Morse dans l'étude des équations elliptiques semi-linéaires, on pourra par exemple consulter [41, 115, 136, 147].

Suites et condition de Palais-Smale

En dimension infinie, la compacité fait souvent défaut. Ainsi, avant d'obtenir des points critiques des fonctionnelles, on devra souvent considérer des *suites de points critiques approchés*.

Ainsi, on dit[57]qu'une suite $(u_n)_n \subseteq H^1_0(\Omega)$ est une *suite de Palais-Smale au niveau* $c \in \mathbb{R}$ pour la fonctionnelle J_λ si
$$J_\lambda(u_n) \xrightarrow[n\to\infty]{} c \quad \text{et} \quad J'_\lambda(u_n) \xrightarrow[n\to\infty]{H^{-1}(\Omega)} 0.$$

De plus, on dit[58] que J_λ vérifie la *condition de Palais-Smale au niveau* c si toutes les suites de Palais-Smale au niveau c admettent une sous-suite convergente.

Cette notion s'est avérée très fructueuse en analyse non-linéaire. Elle admet plusieurs généralisations, nous renvoyons vers [234] pour plus d'informations.

[55] Il y a une certaine latitude dans le choix de la régularité à imposer aux fonctions φ, qu'on pourrait par exemple supposer $\mathcal{C}^\infty_c$.

[56] De même que dans [85, page 291], nous noterons $H^{-1}(\Omega)$ le dual de $H^1_0(\Omega)$. Même si $H^1_0(\Omega)$ est un espace de Hilbert, on ne l'identifiera pas à son dual, voir [85, Chapter 5, Remark 3]. Nous noterons $\langle T \mid \psi\rangle := T[\psi]$ le *crochet de dualité* entre ces espaces (voir [85, Notation, page 3]).

[57] Définition issue de [335, Introduction].

[58] À nouveau, définition extraite de [335, Introduction].

Géométrie de col

Dans la section I.4, nous avons vu que la fonctionnelle d'action J_λ n'est pas bornée inférieurement sur H^1. Nous avons également présenté une façon de traiter cette difficulté en introduisant la variété de Nehari $\mathcal{N}_\lambda$ et en considérant le problème de minimisation de J_λ sur $\mathcal{N}_\lambda$.

Illustrons ici une méthode alternative. Celle-ci montrera comment exploiter la *géométrie de col* de la fonctionnelle. Une démarche similaire aura un rôle important dans les sections suivantes où nous étudierons le régime masse-supercritique.

Revenons un instant sur la géométrie de la fonctionnelle d'action. Tout d'abord, regardons « direction par direction ». Si $u \in H^1 \setminus \{0\}$, alors il existe un unique $t_u > 0$ tel que $t_u u$ appartient à $\mathcal{N}_\lambda$. Le nombre t_u est caractérisé par

$$J_\lambda(t_u u) = \max_{t>0} J_\lambda(tu).$$

Ainsi, la situation ressemble à celle illustrée par la figure I.21. Dans celle-ci, l'espace H^1 correspond au plan horizontal et les niveaux d'action correspondent à l'axe vertical.

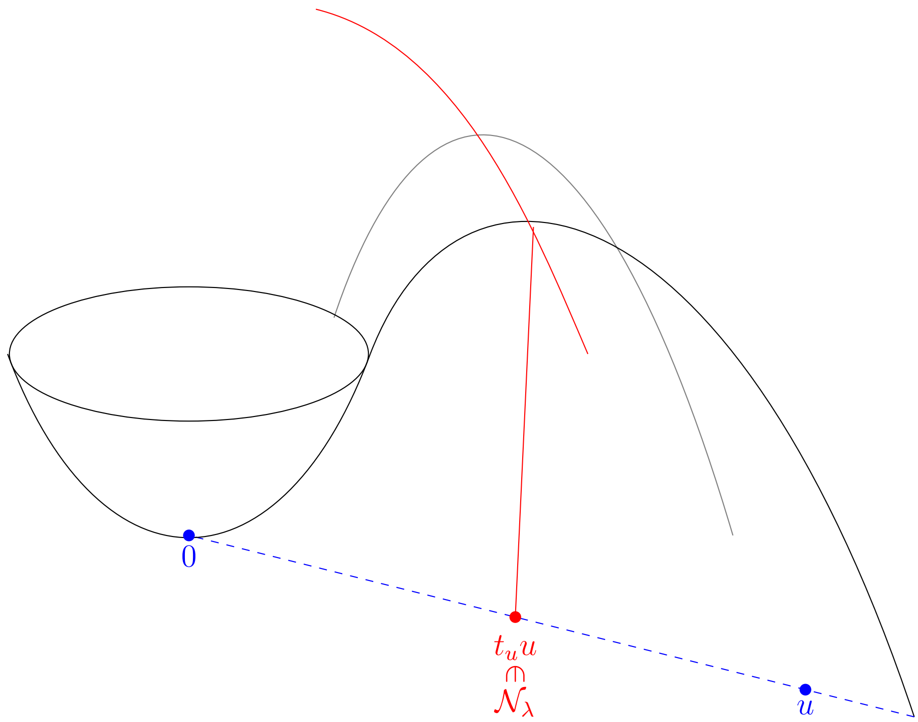

FIGURE I.21 : Représentation géométrique de la variété de Nehari[59]

Sur cette figure, on peut observer que minimiser la fonctionnelle J_λ sur la variété de Nehari $\mathcal{N}_\lambda$ revient à chercher le « col » de la « montagne » $\mathcal{N}_\lambda$.

[59] Un grand merci au Pr. Christophe Troestler qui m'a fourni cette figure.

De manière intuitive, rechercher le col revient à comprendre comment joindre l'origine à « l'autre côté de la montagne » (ce qu'on caractérisera comme étant la région où l'action est strictement négative), tout en « montant aussi peu que possible dans les niveaux d'action ».

Cela nous amène à considérer la *classe des chemins*

$$\Gamma := \Big\{\gamma \in \mathcal{C}([0,1], H^1) \;\Big|\; \gamma(0) = 0, J_\lambda(\gamma(1)) < 0\Big\}$$

ainsi que le *niveau d'action du col*

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\lambda(\gamma(t)).$$

La présence d'une géométrie de col se traduit par l'inégalité $c > 0$: « pour traverser la montagne, il faut nécessairement monter dans les niveaux d'action ». Elle implique l'existence d'une suite de Palais-Smale au niveau c pour J_λ (qui correspond intuitivement au point de selle à hauteur du col). Nous renvoyons vers [214, Section 3], [35, Chapter 4] et [335, Section 1.3] pour plus d'informations.

Pour la fonctionnelle d'action J_λ, cette approche équivaut à la minimisation sur la variété de Nehari (voir [214, Proposition 3.12]).

L'utilisation de la géométrie de col peut s'adapter à de nombreuses situations dans lesquelles on ne dispose pas forcément d'un équivalent de la variété de Nehari. Nous allons le constater dans les sections suivantes.

I.5.6 Régime L^2-supercritique sur $\mathbb{R}^N$

Existence d'une solution normalisée de (NLS)

Le premier article consacré aux solutions normalisées de (NLS) dans le régime masse-supercritique ($p > 2 + \frac{4}{N}$) est celui de L. Jeanjean [183].

La situation y est très différente du régime masse-sous-critique. En effet, dans ce cas, l'énergie n'est plus bornée inférieurement sur la contrainte de masse L^2. Il faut alors chercher le point critique en exploitant la géométrie de col que possède l'énergie sur la contrainte de masse.

Comme nous l'avons vu dans la section précédente, la présence d'une géométrie de col permet d'obtenir l'existence d'une suite de Palais-Smale. Néanmoins, un problème se pose[60] : *il n'est pas clair qu'une telle suite soit bornée dans* $H^1(\mathbb{R}^N)$. Dès lors que la suite est bornée, on peut supposer qu'elle converge faiblement[61] et essayer de montrer que la limite faible est une solution normalisée.

[60]Ce problème n'apparaît pas lorsqu'on recherche des points critiques sur la variété de Nehari ou des energy ground states dans le régime masse-sous-critique.

[61]En passant à une sous-suite.

La stratégie employée pour montrer que la suite de Palais-Smale est bornée se base sur l'*identité de Pohožaev*[62] que satisfont les solutions de (NLS) sur $\mathbb{R}^N$:

$$(N-2)\|\nabla u\|_2^2 + \lambda N \|u\|_2^2 = \frac{2N}{p}\|u\|_p^p. \tag{I.9}$$

On peut par exemple se référer à [183, Section 2.2] qui détaille comment utiliser l'identité de Pohožaev afin de prouver que les suites de Palais-Smale sont bornées.

Après avoir résolu cette difficulté technique, Jeanjean prouve que l'équation (NLS) admet une solution normalisée de masse μ sur $\mathbb{R}^N$ pour toute masse $\mu > 0$, dans le régime masse-supercritique.

Une infinité de solutions normalisées à (NLS)

Dans [51], T. Bartsch et S. de Valeriola montrent l'existence d'une *infinité de solutions* radiales normalisées dans le cas L^2-supercritique[63].

À cette fin, ils utilisent le lemme du col (*Mountain pass theorem*) dans une version assez générale pour des problèmes variationnels invariants par l'application $u \mapsto -u$. Notons que dans [51], la preuve que les suites de Palais-Smale sont bornées utilise l'identité de Pohožaev de la même façon que dans [183, Section 2.2].

Extensions : équations autonomes plus générales, systèmes, problèmes avec potentiel

L'article [183] traite d'équations *autonomes*[64] plus générales que (NLS)[65]. Cela est possible car l'identité de Pohožaev se prête bien à ces cas sur $\mathbb{R}^N$.

On pourra se référer à [189, 191, 192] pour des travaux ultérieurs et à [27, 187, 188] pour des extensions où l'exposant critique de Sobolev 2^* intervient.

D'autres approches que celle de [183] sont possibles pour prouver l'existence de solutions normalisées. Citons par exemple l'article [194], où les auteurs montrent l'existence et étudient les masses L^2 de branches de solutions dans l'espace des fonctions H^1 radiales.

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[62]Prouvée originellement par S. Pohožaev dans [280]. Pour une preuve dans le cas de $\mathbb{R}^N$, se référer par exemple à [335, Theorem B.3].

[63]Lorsque $\lambda > 0$ est fixé, la multiplicité infinie des solutions radiales est prouvée dans [66].

[64]C'est-à-dire de la forme $-\Delta u = f(u)$ et non pas $-\Delta u = f(x, u)$.

[65]Sur $\mathbb{R}^N$, si on utilise une non-linéarité « de type puissance » comme dans (NLS), on peut exprimer explicitement la masse L^2 des action ground states en fonction de λ et obtenir les mêmes résultats d'existence de solutions normalisées. Dans l'annexe C, nous présenterons des calculs semblables pour le soliton sur la droite réelle, voir la proposition C.4. Si la non-linéarité est modifiée ou si le domaine change, cet argument n'est plus applicable et la méthode de Jeanjean est nécessaire.

Il est également possible d'étudier les solutions L^2-normalisées de *systèmes* d'équations elliptiques avec des techniques variationnelles, voir par exemple [42, 43, 44, 48, 49, 50, 170, 195]. Tous ces travaux utilisent de façon importante des identités de Pohožaev.

Dans [46, 245], les auteurs étudient des généralisations de (NLS) où est ajouté un terme de *potentiel* $V(x)$, ce qui rend l'équation

$$-\Delta u + V(x)u + \lambda u = |u|^{p-2}u$$

non-autonome. L'identité de Pohožaev associée est[66]

$$\begin{aligned}(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \lambda N \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x \\ = \frac{2N}{p}\int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x + 2\int_{\mathbb{R}^N} (x\cdot\nabla u)V(x)u \, \mathrm{d}x.\end{aligned} \tag{I.10}$$

Afin de pouvoir montrer l'existence de solutions normalisées, il faut imposer des hypothèses sur le potentiel, dont suffisamment d'intégrabilité de $V(x)$ et de $x \cdot V(x)$ afin de traiter le dernier terme dans (I.10).

Les travaux mentionnés jusqu'à présent utilisent une identité de Pohožaev afin d'établir le caractère borné des suites de Palais-Smale ainsi que leur convergence. Cela peut s'avérer parfois contraignant et impose, par exemple, des restrictions sur le potentiel dans [46, 245].

Afin de poursuivre l'étude des solutions normalisées dans le régime $p > 2 + \frac{4}{N}$, nous sommes amenés à rechercher des méthodes qui ne se basent pas sur une identité de Pohožaev. Dans la section suivante, nous allons présenter de telles méthodes consacrées à l'existence de suites de Palais-Smale bornées.

I.5.7 « Monotonicity trick », existence de suites de Palais-Smale bornées et indices de Morse

Le monotonicity trick[67]

Dans [308, 309], M. Struwe se retrouve aussi confronté au problème d'existence de suites de Palais-Smale bornées, dans des problèmes relevant de la géométrie et de l'étude des systèmes hamiltoniens. Il y développe un argument désormais connu sous le nom de « *monotonicity trick* » (voir aussi [310, Chapter II, Section 9]).

[66] Du moins formellement : se référer à [155, Lemma 1.1] où l'on trouvera l'expression que prend l'identité de Pohožaev d'une équation non-autonome sur un domaine borné et à [335, Appendix B, Section 3] pour passer du cas des domaines bornés à $\mathbb{R}^N$. Nous ne tenterons pas de justifier rigoureusement l'identité (I.10) et renvoyons vers [46, 245] où l'on explique comment des identités telles que (I.10) interviennent dans les preuves.

[67] La lecture de [185, Section 5, Suites de Palais-Smale bornées] nous a été très utile pour écrire cette section. Nous la recommandons au lecteur souhaitant approfondir ses connaissances sur la problématique du caractère borné des suites de Palais-Smale.

Dans [186], Jeanjean prouve que les techniques développées auparavant par Struwe peuvent être généralisées en une version « abstraite ». Citons son résultat.

Théorème ([186, Theorem 1.1])**.** *Soit* $(X, \|\cdot\|)$ *un espace de Banach.*

Considérons un intervalle $I \subset (0, +\infty)$ *et une famille* $\Phi_\rho \colon X \to \mathbb{R}$ *de fonctionnelles* $\mathcal{C}^1$ *de la forme*

$$\Phi_\rho(u) := A(u) - \rho B(u) \quad \textit{où } \rho \in I.$$

On suppose que $B(u) \geq 0$ *pour tout* $u \in X$ *et que*

$$A(u) \xrightarrow[\|u\|\to+\infty]{} +\infty \quad \textit{ou} \quad B(u) \xrightarrow[\|u\|\to+\infty]{} +\infty.$$

Étant donné $v_0, v_1 \in X$*, on définit*

$$\Gamma := \Big\{\gamma \in \mathcal{C}([0,1], X) \;\Big|\; \gamma(0) = v_0, \gamma(1) = v_1\Big\}.$$

Supposons que, pour tout $\rho \in I$*, le nombre*

$$c_\rho := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\rho(\gamma(t)),$$

satisfait l'inégalité $c_\rho > \max\{\Phi_\rho(v_0), \Phi_\rho(v_1)\}$.

Alors, pour presque tout $\rho \in I$*, il existe une suite* $(u_n)_{n\geq 1} \subseteq X$ *de Palais-Smale bornée au niveau* c_ρ *pour* Φ_ρ*. Autrement dit :*

(i) $(u_n)_n$ *est bornée ;*

(ii) $\Phi_\rho(u_n) \to c_\rho$ *lorsque* $n \to \infty$ *;*

(iii) $\Phi'_\rho(u_n) \to 0$ *dans le dual de* X *lorsque* $n \to \infty$.

La preuve de ce résultat utilise le « monotonicity trick » : puisque $I \subset (0, +\infty)$ et que $B(u) \geq 0$ pour tout $u \in X$, l'application $I \to \mathbb{R} : \rho \mapsto c_\rho$ est croissante. Elle est donc dérivable presque partout. Jeanjean montre alors que, lorsque c'_ρ existe, alors la fonctionnelle Φ_ρ possède une suite de Palais-Smale bornée au niveau c_ρ (voir [186, Section 2]).

Dans [193], L. Jeanjean et J.F. Toland généralisent ce résultat à des situations où l'application $\rho \mapsto c_\rho$ n'est pas monotone.

Le résultat d'existence de suites de Palais-Smale bornées est particulièrement utile et a trouvé beaucoup d'applications : existence de solutions à un problème « du type Landesman-Lazer » dans [186], extension pour le régime L^2-supercritique des résultats de bifurcation étudiés par Stuart dans[68] [160, 184], etc.

[68]Voir aussi [185, Section 5, Retour sur les problèmes de bifurcation].

Plusieurs questions s'offrent à nous.

Étant donné une suite de Palais-Smale bornée, comment démontrer qu'elle converge ? Peut-on trouver des suites de Palais-Smale bornées pour des problèmes avec contraintes[69] *? Peut-on démontrer l'existence de telles suites à des niveaux différents afin d'obtenir des résultats de multiplicité ?*

Comme nous allons le constater, posséder une information de type « indice de Morse » sur des éléments d'une suite de Palais-Smale est un outil précieux pour montrer la convergence de celle-ci.

Informations sur les indices de Morse (approchés)

Une suite de Palais-Smale est constituée de points critiques *approchés* qui ne sont a priori pas associés à une notion d'indice de Morse. Néanmoins, G. Fang et N. Ghoussoub ont montré dans [145, 146] qu'il est possible de définir une notion d'*indice de Morse approché* et prouver l'existence de suites de Palais-Smale avec informations sur les indices de Morse approchés des éléments.

Ceci s'avère utile pour étudier les questions de convergence, comme cela avait été observé dans [39, 41, 225] par A. Bahri et P.L. Lions. Dans ces articles, les auteurs mettent en évidence le rôle que peuvent jouer des informations sur les indices de Morse dans des équations elliptiques non-linéaires, notamment lorsqu'il manque de la compacité.

Notons que les travaux de Fang et Ghoussoub concernent des suites de Palais-Smale pour des problèmes variationnels *sans contraintes*.

Dans [81], J. Borthwick, X. Chang, L. Jeanjean et N. Soave ont montré qu'il existe des suites de Palais-Smale *bornées, avec informations sur l'indice de Morse approché*, dans le cas de fonctionnelles perturbées *contraintes sur une sphère* (voir [81, Theorem 1.5]) possédant une géométrie de col. Ainsi, les auteurs affinent le monotonicity trick afin d'obtenir également des informations sur les indices de Morse approchés des éléments de la suite de Palais-Smale, comme le font Fang et Ghoussoub.

Dans [81, Theorem 1.12], les auteurs prouvent aussi un théorème « généralisé » permettant d'obtenir des résultats de multiplicité pour des fonctionnelles paires[70] (dans l'esprit de la méthode employée sur $\mathbb{R}^N$ par Bartsch et de Valeriola [51]). Ce théorème sera crucial dans le chapitre 4.

Nous renvoyons vers l'introduction de [81] et également vers [139] pour en apprendre davantage sur l'existence de suites de Palais-Smale avec informations sur les indices de Morse des éléments et l'utilisation de tels résultats.

[69] Le théorème présenté ci-dessus porte sur des problèmes variationnels *libres*, ce qui signifie que les chemins dans Γ sont à valeurs dans X et ne sont pas contraints à une sous-variété de X.

[70] C'est le cas de la fonctionnelle d'énergie.

I.5.8 Solutions normalisées sur les graphes lorsque $6 < p$

Comme nous l'avons vu, on ne dispose pas sur les graphes d'un équivalent des arguments utilisés sur $\mathbb{R}^N$ qui se basent sur l'identité de Pohožaev. Il en sera généralement de même dans la section suivante, dans laquelle nous étudierons des solutions normalisées sur des domaines bornés.

Remarque. Néanmoins, signalons que l'identité de Pohožaev est commode à utiliser sur les domaines *étoilés* (nous le verrons dans le chapitre 3). Par exemple, en dimension $N \geq 3$, elle implique qu'il n'existe aucune solution non-nulle de (NLS) lorsque p est supérieur ou égal à $2^* := \frac{2N}{N-2}$ sur des domaines étoilés (voir par exemple [310, Chapter III, Section 1]).

Dès lors, les résultats sur l'existence de suites de Palais-Smale bornées avec informations sur les indices de Morse approchés prennent tout leur intérêt sur les graphes et les domaines bornés. C'est d'ailleurs l'étude des solutions normalisées sur les graphes qui a mené Borthwick, Chang, Jeanjean et Soave à développer [81].

Lorsque $p > 6$, ces résultats permettent d'obtenir une solution normalisée non-constante lorsque la masse est suffisamment petite sur les graphes compacts, voir [102]. Pour le problème à non-linéarité localisée, on obtient une solution normalisée pour toutes les masses, voir [82].

Dans [82, 102], après avoir démontré l'existence d'une suite de Palais-Smale bornée, il faut également montrer que la suite de « presque multiplicateurs de Lagrange » (*almost Lagrange multipliers*) associée converge vers un réel strictement positif. Cela nécessite d'exclure[71] la convergence des presque multiplicateurs vers l'infini ainsi que leur convergence vers zéro[72].

Comme annoncé dans la section précédente, nous étendrons le résultat de [82] en démontrant que le problème à non-linéarité localisée possède *une infinité* de solutions normalisées. À cette fin, nous devrons développer un argument excluant le cas $\lambda = 0$. Celui-ci est plus délicat à traiter lorsque les solutions peuvent changer de signe à cause de la présence possible de solutions s'annulant identiquement sur des arêtes du graphe (voir la section I.9). Cet argument est l'objet de la section 4.3 (voir en particulier la proposition 4.23).

Concernant les solutions normalisées sur les graphes dans le régime $p > 6$, mentionnons également [31]. Dans cet article, A.H. Ardila étudie des solutions normalisées sur des graphes dans le régime masse-supercritique, mais où la présence d'un potentiel modifie la géométrie de la fonctionnelle d'énergie, ce qui permet à l'auteur de trouver des solutions par minimisation sous la contrainte de masse.

À notre connaissance, il n'existe à ce jour qu'un seul article traitant du cas masse-supercritique sur les graphes *non-compacts* : celui de S. Dovetta, L. Jeanjean et E. Serra [128]. Nous y reviendrons dans la section I.6.

[71] Grâce à une *analyse de blow-up* utilisant les informations sur les indices de Morse.

[72] Ce qui est plutôt aisé lorsqu'on cherche des solutions *positives* comme dans [82, 102].

I.5.9 Solutions normalisées sur des domaines bornés dans le régime masse-supercritique

Dans [259], B. Noris, H. Tavares et G. Verzini étudient l'existence de solutions normalisées et la stabilité orbitale de celles-ci pour les solutions de (NLS) sur la boule unité avec la condition de Dirichlet. L'unicité de la solution positive de (NLS) (pour un λ donné) y joue un grand rôle, ce qui complique la généralisation des résultats à d'autres domaines où l'unicité n'est que rarement connue.

L'étude des solutions normalisées sur des domaines bornés est poursuivie dans[73] les articles [273, 274].

Citons aussi [269], qui prouve des résultats d'existence de solutions normalisées *concentrées*. Nous y reviendrons dans la section I.7.

Signalons dès à présent que dans la section I.10, nous présenterons une nouvelle méthode permettant d'étudier les solutions normalisées sur des domaines bornés, y compris dans le régime masse-supercritique. Celle-ci fait l'objet du chapitre 3.

I.6 Rôle des propriétés topologiques et métriques des domaines

Les deux cadres les plus classiques dans lesquels on peut étudier une équation elliptique telle que (NLS) sont les domaines bornés et $\mathbb{R}^N$ tout entier.

Sur un domaine borné régulier $\Omega \subseteq \mathbb{R}^N$, on dispose de compacité dans les espaces fonctionnels. Ainsi, l'injection de $H^1(\Omega)$ dans $L^2(\Omega)$ est compacte d'après le théorème de Rellich-Kondrachov (voir par exemple [85, Theorem 9.16]).

Au contraire, nous nous intéresserons dans ce qui suit au cas des domaines non-bornés et à l'étude de l'équation ($NLS_{\mathcal{G}}$) sur des graphes métriques non-compacts. Dans ces cas, le manque de compacité sera un des challenges à surmonter.

Ci-dessous, nous mentionnerons également d'autres sources de non-compacité que celle du domaine. Ainsi, si $\Omega \subseteq \mathbb{R}^N$ est un domaine en dimension $N \geq 3$, l'injection de $H^1(\Omega)$ dans $L^{2^*}(\Omega)$ n'est pas compacte, même si Ω est borné (voir [85, Corollary 9.14 et page 286, Remark 14]).

Comme nous l'avons vu au début de la section précédente, l'exposant 2^* n'a pas qu'un rôle « technique » et l'équation (NLS) ne possède pas de solutions non-nulles lorsque $p \geq 2^*$ sur des domaines étoilés.

[73]Signalons que la preuve du résultat d'existence présenté dans [273] ne montre pas comment vérifier que les suites de Palais-Smale obtenues sont bornées. Ceci a été commenté par les auteurs, voir [274, haut de la page 3] où cette étape est détaillée.

À présent, formulons quelques remarques.

- Sur les graphes, on ne rencontre pas de problèmes de compacité liés à un exposant critique de Sobolev car les graphes sont de dimension un.
- L'exposant critique de Sobolev apparaît naturellement dans certaines questions de géométrie riemannienne, par exemple dans le *problème de Yamabe* [336] où l'on se demande si toute variété peut être munie d'une métrique riemannienne dont la courbure scalaire est constante. L'article d'exposition [216] contient plus d'informations sur ce problème, son histoire et sa résolution.
- Le lecteur intéressé par une présentation plus exhaustive des diverses sources de non-compacité pouvant apparaître dans les problèmes variationnels pourra se référer à [310, Chapter III].

Tout d'abord, considérons le domaine non-compact le plus naturel : l'espace euclidien tout entier.

I.6.1 Solutions entières sur $\mathbb{R}^N$

On dit parfois que les solutions définies sur $\mathbb{R}^N$ sont *entières*. La littérature est très vaste (voir par exemple l'ouvrage [212]).

Si on étudie une équation *autonome* sur $\mathbb{R}^N$, c'est-à-dire de la forme

$$-\Delta u = f(u),$$

alors le problème est *invariant par translations*, ce qui permet de retrouver de la compacité comme l'illustre le lemme [219, Lemma 6] et son application[74] dans [86].

Nous utiliserons également l'invariance par translations lors de notre étude des *graphes périodiques*, qui sera présentée dans la section I.6.5.

Le rôle des actions de groupes non-compacts dans les pertes de compacité a été grandement clarifié par les travaux de P.L. Lions sur le principe de *concentration-compacité* [224, 226, 227]. Pour une introduction au sujet et à ses développements plus récents, nous renvoyons vers [217].

Pour les équations autonomes sur $\mathbb{R}^N$, on peut également chercher des solutions *radiales*, ce qui apporte aussi de la compacité, voir par exemple [65, 299, 307].

Les deux techniques mentionnées précédemment (usage des translations ou recherche de solutions radiales) ne s'appliquent pas à des équations *non-autonomes* de la forme $-\Delta u = f(x, u)$.

C'est par exemple le cas lorsqu'un *potentiel* est présent dans l'équation, comme pour les équations du type $-\Delta u + V(x)u + \lambda u = |u|^{p-2}u$.

Bien que la présence du potentiel $V(x)$ rende l'équation non-autonome, il peut parfois cependant y amener de la compacité, voir par exemple [47].

[74]Voir aussi [157, Section 3], dans lequel nous avons trouvé la référence [219].

Dans cette thèse, nous mettrons surtout l'accent sur le rôle du *domaine* (et de ses conditions au bord) et non sur *l'équation*.

À présent, considérons les cas « les plus simples » de domaines non-compacts autres que $\mathbb{R}^N$: les demi-espaces.

I.6.2 La demi-droite et les demi-espaces

En dimension un, un *demi-espace* est simplement une demi-droite, par exemple $[0, +\infty)$. Sur celle-ci, on étudie les solutions de l'EDO $-u'' + \lambda u = |u|^{p-2}u$ qui convergent vers 0 en l'infini.

Si l'on impose la condition de Neumann en $x = 0$ (c'est-à-dire $u'(0) = 0$), nous avons vu dans la section I.3.1 que l'EDO admet deux solutions non-nulles, qui sont opposées et données par des demi-solitons $\pm\phi_\lambda(x)_{|\mathbb{R}^+}$.

Si l'on impose la condition de Dirichlet en $x = 0$ (c'est-à-dire $u(0) = 0$), le problème n'admet aucune solution non-nulle (voir la proposition C.2).

Ceci montre que la condition au bord du domaine peut jouer un rôle crucial.

Comme nous le verrons dans le chapitre 2, la présence ou non de sommets de Dirichlet sera importante dans les résultats d'existence de (nodal) action ground states sur les graphes métriques.

Plus généralement, on peut considérer un demi-espace en dimension N, par exemple $\mathcal{H} := [0, +\infty) \times \mathbb{R}^{N-1}$.

Si l'on impose la condition de Dirichlet au bord de $\mathcal{H}$, alors le théorème [143, Theorem I.1] implique que (NLS) n'admet aucune solution dans $H_0^1(\mathcal{H})$.

La situation est très différente si l'on impose une condition de Dirichlet *non-homogène* $u = c$ sur le bord de $\mathcal{H}$. Dans ce cas, le nombre de solutions dépend de façon cruciale de la valeur de la constante c, voir [150].

I.6.3 Domaines possédant une géométrie plus riche

La géométrie du domaine joue un rôle dans les questions de compacité comme nous pourrons le constater dans les situations suivantes.

- Les travaux fondateurs de J.M. Coron et d'A. Bahri [36, 37, 111] montrent que, sur certains domaines dont la topologie est non-triviale (ceux qui, d'une manière intuitive, possèdent des « trous », voir la figure I.22), l'équation (NLS) peut admettre des solutions même lorsque p est égal à l'exposant critique[75] de Sobolev 2^*. Notons que dans le cas Sobolev-supercritique, la présence d'une topologie non-triviale ne suffit pas toujours à obtenir l'existence de solutions, comme prouvé par D. Passaseo [265].

[75] Aucun de ces domaines n'est étoilé suite à la présence des trous. Ceci explique pourquoi l'obstruction à l'existence de solutions de (NLS) obtenue par Pohožaev ne s'applique pas.

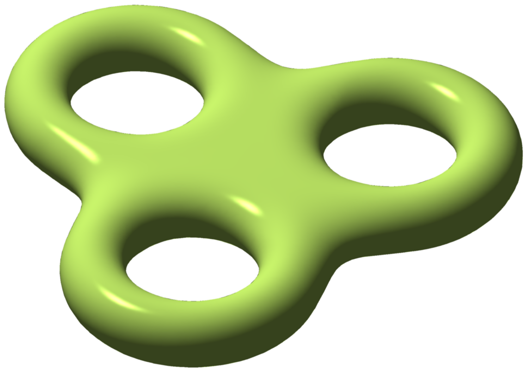


FIGURE I.22 : Le « tore triple[76] », un domaine à la topologie non-triviale dans $\mathbb{R}^3$

- L'étude de Bahri et Coron a mené à plusieurs développements ultérieurs pour des domaines possédant des « petits trous » (voir une illustration en dimension deux dans la figure I.23). Ainsi :
  - V. Benci et G. Cerami [61, 63] montrent des liens entre la topologie du domaine (en particulier, des notions liées au « nombre de trous ») et la multiplicité des solutions positives de (NLS) ;
  - O. Rey [287, 288] obtient des résultats de multiplicité en fonction du nombre de trous lorsque $p = 2^*$;
  - A. Bahri, Y. Li et O. Rey [38] étudient les points de concentration des solutions dans un régime presque Sobolev-critique ;
  - M. del Pino et J. Wei [276] prouvent l'existence de solutions pour certains $p > 2^*$ dans de tels domaines ;
  - O. Rey et A. Pistoia [278] construisent des solutions développant plusieurs pics dans le régime $p > 2^*$;
  - etc.

FIGURE I.23 : Un domaine borné avec des petits trous dans $\mathbb{R}^2$

[76] Image provenant de `https://upload.wikimedia.org/wikipedia/commons/f/f0/Triple_torus_illustration.png`, domaine public.

- Plusieurs auteurs ont étudié les solutions de (NLS) sur les *domaines extérieurs* (les ouverts de $\mathbb{R}^N$ dont le complémentaire est compact). Citons, entre autres, A. Bahri, V. Benci, G. Cerami, M. Clapp, P.L. Lions, R. Molle, D. Passaseo, etc. Leurs travaux révèlent que des équations du type (NLS) :
  - peuvent admettre des solutions si le « trou » est suffisamment petit ([62, Theorem B]) ou en présence de certains termes de potentiel ([40]) ;
  - admettent des solutions nodales en présence d'assez de symétries ([100]) ;
  - admettent de multiples solutions en fonction du nombre de « trous » ([101]) ;
  - etc.

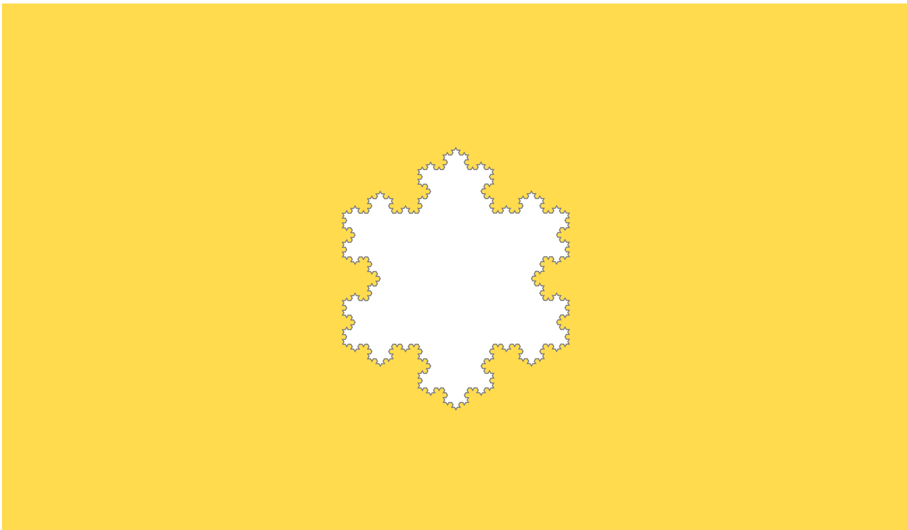


FIGURE I.24 : Un domaine extérieur[77]

- Dans [275], M.A. del Pino et P.L. Felmer considèrent des domaines de $\mathbb{R}^2$ de la forme
$$\Big\{(t,x) \in \mathbb{R}^2 \mid -f(t) < x < f(t)\Big\},$$
où f est une fonction strictement positive, de classe $\mathcal{C}^\infty$ sur $\mathbb{R}$ et convergeant vers 0 à l'infini.

  Bien que de tels domaines soient non-compacts, les auteurs montrent qu'il existe des solutions strictement positives de (NLS) sur ces domaines et sur des généralisations de ceux-ci, y compris en dimension $N \geq 3$.
- R. Molle a étudié dans [243, 244] des domaines $\Omega \subset \mathbb{R}^N$ tels que ni Ω ni $\mathbb{R}^N \setminus \Omega$ ne sont bornés.
- M.J. Esteban et P.L. Lions obtiennent, dans [143], des résultats de non-existence de solutions non-nulles pour certains domaines non-compacts.
- etc.

[77]Dessin réalisé grâce à la librairie TikZ `decorations.fractals`, à partir d'explications se trouvant sur `https://latex.org/forum/viewtopic.php?t=17902`.

Comme l'illustrent les travaux ci-dessus, certaines *propriétés géométriques* des domaines *peuvent, ou non, amener de la compacité.* Nous allons voir qu'il en est de même sur les graphes métriques, qui forment un excellent cadre pour étudier plus en détail ce phénomène.

I.6.4 (Non-)existence de ground states sur des graphes avec un nombre fini d'arêtes lorsque $2 < p < 6$

Première étude sur les graphes en pont

Tout d'abord, présentons l'article [16] dans lequel R. Adami, E. Serra et P. Tilli prouvent qu'il n'existe pas d'energy ground states lorsque $2 < p < 6$ sur des *graphes en pont* (*bridge-type graphs*), tels que celui représenté par la figure I.25 (un autre exemple étant le double-pont représenté par la figure I.16 de la page 21).

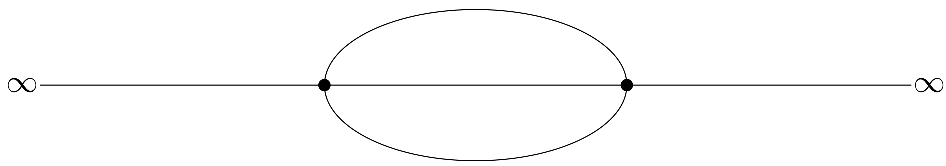


FIGURE I.25 : Le triple-pont

Dans le cas d'un nombre impair de « ponts » entre les deux sommets, le résultat de non-existence se base sur une technique consistant à « déplier » les fonctions définies sur les graphes afin d'obtenir des fonctions sur la droite réelle. Le cas général s'ensuit par comparaisons de niveaux entre le N-pont et le $(N-1)$-pont (voir la preuve de [16, Theorem 1.2]).

Ceci montre qu'ajouter des arêtes bornées à la droite réelle peut complètement changer la situation. En effet, sur $\mathbb{R}$, les energy ground states existent pour toute masse alors que sur les graphes en pont, ils n'existent jamais !

Les considérations mentionnées ci-dessus sont considérablement clarifiées et généralisées dans l'important article [19], « NLS ground states on graphs », des trois auteurs précités.

Dans cet article, le focus est mis sur *le graphe sur lequel l'équation est posée* et Adami, Serra et Tilli cherchent à *dégager des arguments s'appliquant à de larges classes de graphes.*

Présentons maintenant plusieurs résultats de l'article [19] et observons le lien entre ceux-ci et les travaux de cette thèse.

Réarrangement décroissant, nombre de préimages et hypothèse (H)

La droite réelle et la demi-droite sont deux cas « extrêmes », pour lesquels le niveau de l'energy ground state sur la contrainte de masse est maximisé ou minimisé, comme l'affirme le théorème ci-dessous (obtenu dans [19, Theorem 2.2] pour des graphes avec un nombre fini d'arêtes, mais vrai en toute généralité).

Théorème I.1 ([19, Theorem 2.2]). *Soient deux nombres réels* $p \in (2,6)$ *et* $\mu > 0$. *Si un graphe métrique* $\mathcal{G}$ *contient au moins une demi-droite, alors*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) \leq \inf_{u \in H^1_\mu(\mathbb{R})} E_{\mathbb{R}}(u) = E_{\mathbb{R}}(\widehat{\phi}_\mu). \tag{I.11}$$

où $\widehat{\phi}_\mu$ *est le soliton de masse* μ *(voir la définition C.6). De plus*[a], *on a*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) \geq \inf_{u \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(u) = \frac{1}{2} E_{\mathbb{R}}(\widehat{\phi}_{2\mu}). \tag{I.12}$$

[a]Notons qu'un energy ground state de masse μ sur $[0,+\infty)$ s'obtient en "coupant $\widehat{\phi}_{2\mu}$ en deux en son point de maximum" (voir la proposition C.9).

Vu l'importance, dans les deux premiers chapitres, des arguments utilisés dans la preuve de ce théorème, nous avons décidé d'en présenter les idées essentielles.

Éléments de la preuve du théorème I.1. L'inégalité (I.11) se prouve facilement. En effet, sur tout graphe possédant au moins une demi-droite, on peut considérer des « presque solitons $\widehat{\phi}_\mu$ ». Plus précisément, pour tout $\varepsilon > 0$, il existe $v_\varepsilon \in H^1_\mu(\mathcal{G})$ tel que $\|\widehat{\phi}_\mu - v_\varepsilon\|_{H^1} \leq \varepsilon$ et tel que v_ε a un support compact inclus dans une demi-droite de $\mathcal{G}$ (voir la figure I.26).

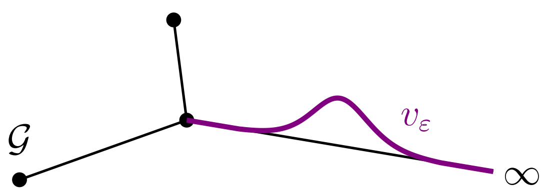


FIGURE I.26 : Un soliton tronqué sur une demi-droite

L'inégalité (I.12) est basée sur un argument de *réarrangement décroissant.* Étant donné $u \in H^1_\mu(\mathcal{G})$, on veut montrer que

$$E(u) \geq \inf_{v \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(v).$$

Quitte à remplacer u par $|u|$, nous pouvons supposer que u est à valeurs positives.

Considérons $u^* : [0, +\infty) \to \mathbb{R}$, le *réarrangement décroissant* de u. De manière intuitive[78], cela revient à « découper l'image de u en tranches verticales » et à les replacer sur une demi-droite, par ordre décroissant de hauteur. Ce procédé est représenté dans la figure ci-dessous.

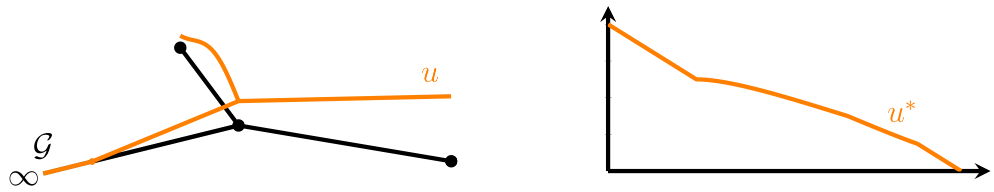


FIGURE I.27 : Une fonction $u : \mathcal{G} \to [0, +\infty)$ et son réarrangement décroissant u^*.

Dès lors, pour tout $t > 0$, on a[79]

$$\lambda_{\mathcal{G}}\Big(\{x \in \mathcal{G}, u(x) > t\}\Big) = \lambda_{\mathbb{R}^+}\Big(\{x \in (0, |\mathcal{G}|), u^*(x) > t\}\Big). \tag{I.13}$$

D'après (I.13), on a $\|u^*\|_{L^2(0,+\infty)} = \|u\|_{L^2(\mathcal{G})}$ et $\|u^*\|_{L^p(0,+\infty)} = \|u\|_{L^p(\mathcal{G})}$. De plus, l'*inégalité de Pólya-Szegő* (Lemme B.25) affirme que

$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \|u'\|_{L^2(\mathcal{G})}. \tag{I.14}$$

Ainsi, on obtient

$$\begin{aligned} E(u) &= \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} \\ &\geq \frac{1}{2}\|(u^*)'\|^2_{L^2(0,+\infty)} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} \\ &\geq \inf_{v \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(v), \end{aligned}$$

ce qui conclut la preuve de l'inégalité (I.12). □

En dimension un, il s'avère que l'on peut (sous conditions) affiner l'inégalité de Pólya-Szegő (I.14). Plus précisément, si une fonction $u : \mathcal{G} \to [0, +\infty)$ est telle que

$$\# u^{-1}(\{t\}) \geq N \tag{I.15}$$

pour presque tout t dans son image (où $N \geq 1$ est entier), alors son réarrangement décroissant u^* vérifie

$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \frac{1}{N}\|u'\|_{L^2(\mathcal{G})}. \tag{I.16}$$

Ainsi, *on a amélioré l'inégalité de Pólya-Szegő d'un facteur N grâce à l'hypothèse* (I.15) *sur le nombre de préimages !*

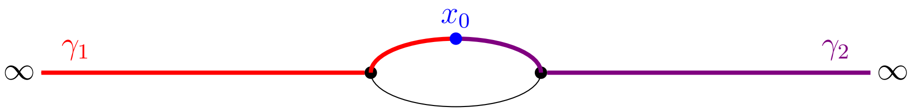
Certains graphes sont tels que *toutes les fonctions continues positives vérifient la condition* (I.15) *avec $N = 2$*. Cela nous mène à la définition suivante.

[78] Ces explications heuristiques se basent sur [20, Section 5.1]. Nous renvoyons vers l'annexe B pour les définitions rigoureuses et les preuves des propriétés annoncées.

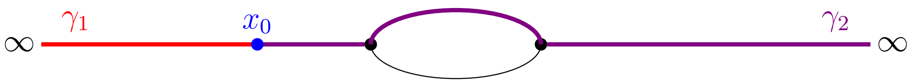
[79] Où $\lambda_{\mathcal{G}}$ et $\lambda_{\mathbb{R}}$ désignent respectivement les mesures de Lebesgue sur $\mathcal{G}$ et $\mathbb{R}$, voir la section A.3.

Définition. Un graphe métrique $\mathcal{G}$ *satisfait l'hypothèse*[80](H) si, pour tout point $x_0 \in \mathcal{G}$, il existe deux courbes injectives $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ qui sont paramétrisées par longueur d'arc, qui ont des images disjointes sauf pour un nombre au plus dénombrable de points et telles que $\gamma_1(0) = \gamma_2(0) = x_0$.

Exemple. Les graphes en pont satisfont l'hypothèse (H). Par exemple, si x_0 se trouve sur l'arête « du haut » du graphe, on peut considérer les deux courbes γ_1 et γ_2 représentées dans la figure ci-dessous :



De même, si x_0 se trouve sur la demi-droite « de gauche », on considère les courbes γ_1 et γ_2 comme suit :



Les graphes qui satisfont l'hypothèse (H) ne possèdent en général pas d'energy ground states, comme l'affirme l'énoncé suivant (voir [19, Theorems 2.2, 2.3, 2.5]).

Théorème I.2. *Soient deux réels $p \in (2, 6)$ et $\mu > 0$. Si un graphe métrique $\mathcal{G}$ satisfait* (H), *alors*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_\mathcal{G}(u) = \inf_{u \in H^1_\mu(\mathbb{R})} E_\mathbb{R}(u) = E_\mathbb{R}(\widehat{\phi}_\mu). \tag{I.17}$$

De plus, l'infimum

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_\mathcal{G}(u)$$

n'est pas atteint, c'est-à-dire qu'il n'existe pas d'energy ground states de masse μ, sauf si $\mathcal{G}$ est isométrique à la droite réelle ou à un des graphes en « tour de boucles », voir les figures I.28, I.29 et I.30.

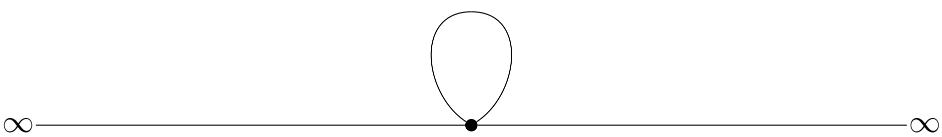


FIGURE I.28 : Une tour à une boucle

[80] Appelée (H') dans [19]. Il s'avère que l'hypothèse (H') est équivalente à l'hypothèse (H), voir [19, Lemma 5.1] (qui prouve une des implications de l'équivalence).

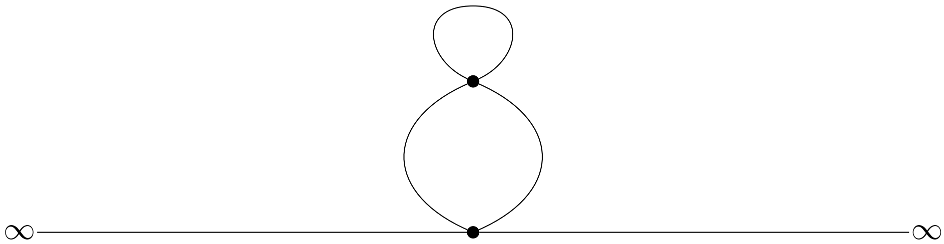


FIGURE I.29 : Une tour à deux boucles

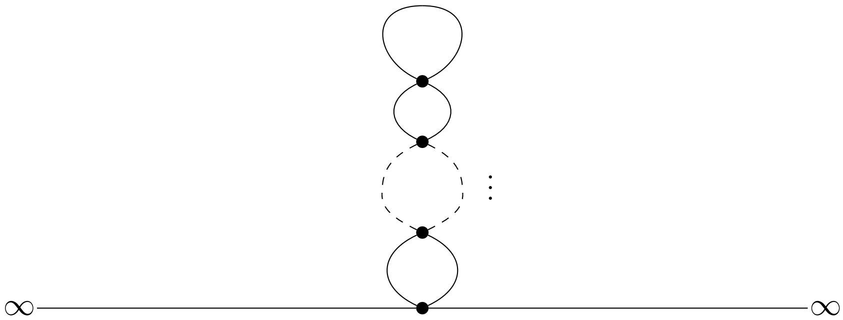


FIGURE I.30 : Une tour de boucles (cas général)

Puisque les graphes en pont vérifient l'hypothèse (H), le théorème précédent permet de retrouver les résultats de non-existence mentionnés auparavant.

Pour conclure la présentation de [19], signalons également qu'Adami, Serra et Tilli y étudient le $\mathcal{T}$-graphe (voir l'exemple (a) de la figure I.31 à la page suivante) et montrent que celui-ci admet des energy ground states pour toutes les valeurs de la masse (voir [19, Theorems 2.6 and 2.7]). Ce résultat découle de l'inégalité

$$\inf_{u \in H^1_\mu(\mathcal{T})} E_{\mathcal{T}}(u) < E_{\mathbb{R}}(\widehat{\phi}_\mu),$$

prouvée à l'aide de constructions basées sur des techniques de réarrangement et adaptées à la structure du $\mathcal{T}$-graphe (voir [19, Lemma 6.1]). Remarquons que le $\mathcal{T}$-graphe ne satisfait pas l'hypothèse (H).

L'étude se poursuit dans l'article [21], où l'on trouve notamment le résultat suivant (voir [21, Theorem 3.3]).

Théorème I.3. *Soit $\mathcal{G}$ un graphe métrique formé d'un nombre fini d'arêtes dont au moins une demi-droite. Soient deux nombres réels $p \in (2, 6)$ et $\mu > 0$. Si l'inégalité*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) < E_{\mathbb{R}}(\widehat{\phi}_\mu),$$

est satisfaite, alors il existe un energy ground state sur $\mathcal{G}$.

Les auteurs prouvent également, grâce à des constructions basées entre autres sur le procédé de réarrangement décroissant, que certains exemples de graphes satisfont l'hypothèse du théorème I.3 précédent (voir la figure I.31).

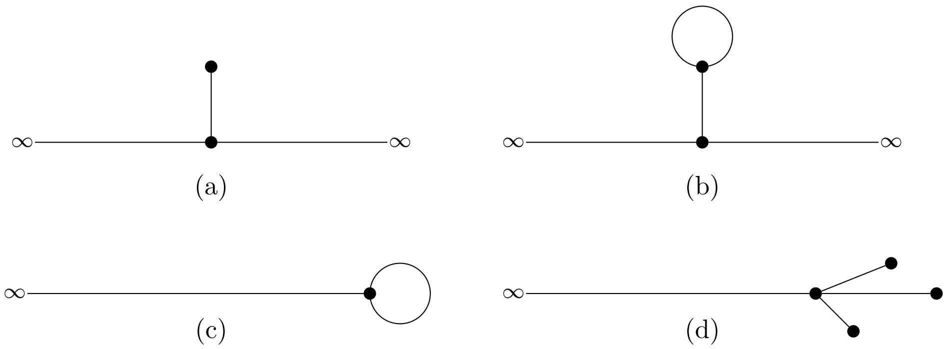


FIGURE I.31 : Exemples de graphes qui possèdent des energy ground states.
(a) le $\mathcal{T}$-graphe ; (b) le panneau de signalisation ; (c) le têtard ; (d) la 3-fourche

Pour une présentation des travaux d'Adami, Serra et Tilli, nous recommandons la lecture des articles d'exposition [1, 20].

Les différents thèmes rencontrés jusqu'ici (comparaisons avec la droite et la demi-droite, arguments de réarrangement, existence vs non-existence de ground states, etc.) seront également au centre des chapitres 1 et 2.

I.6.5 Graphes périodiques

Parmi les exemples les plus simples de graphes périodiques, on trouve les grilles infinies comme celle représentée par la figure I.4 de la page 4. De telles grilles (et certaines de leurs généralisations) ont été étudiées dans [12, 13, 14].

Ces travaux ont mis en évidence le phénomène de « *dimensional crossover* ». Celui-ci se traduit par la coexistence de phénomènes uni-dimensionnels liés à la structure *locale* des graphes et de phénomènes typiques de la dimension supérieure liés à la structure *globale* des graphes.

Par exemple, la grille dans la figure I.4 est, en un certain sens, de dimension 2. Lorsqu'on étudie l'existence d'energy ground states sur celle-ci, l'exposant critique

$$p_{\text{crit},\, N=1} := 2 + \frac{4}{1} = 6$$

joue un rôle comme on peut s'y attendre. De façon remarquable, il apparaît un *second* exposant particulier, donné par

$$p_{\text{crit},\, N=2} := 2 + \frac{4}{2} = 4.$$

Pour un énoncé précis à propos du *dimensional crossover*, on se référera par exemple à [14, Theorem 1.2]. La persistance de ce phénomène lors de perturbations de la grille (c'est-à-dire la suppression de certaines arêtes) est étudiée dans [134].

S. Dovetta a apporté plusieurs contributions remarquables à l'étude des graphes périodiques[81]. Ainsi, dans [125, Theorem 1.1], il prouve que tous les graphes $\mathbb{Z}$-périodiques[82] possèdent un energy ground state lorsque $2 < p < 6$. Le cas critique est plus complexe et nous renvoyons à [125, Theorems 1.2 and 1.3] pour plus d'informations à son sujet.

Remarque. L'invariance d'un graphe périodique sous l'action de son groupe de « translations » joue un grand rôle dans ses propriétés de compacité, comme nous le verrons dans la section 2.5.1. Nous y prouverons que tous les graphes périodiques possèdent un action ground state pour tout $p > 2$ et tout $\lambda > 0$, ce qui montre que les action ground states ne sont pas sujets au phénomène de dimensional crossover.

I.6.6 Autres travaux

De nombreux travaux étudient le problème ($\mathrm{NLS}_\mathcal{G}$) et ses variantes (pour un aperçu récent de la littérature, voir [199]). Ils portent sur :

- l'unicité et la multiplicité des ground states ([131]) ;
- des arbres infinis ([130]) ;
- des problèmes avec plusieurs sources de non-linéarité ([4, 77, 78, 271]) ;
- le cas masse-critique $p = 6$ ([18, 272]) ;
- etc.

I.6.7 Contributions au sujet développées dans cette thèse

Comme nous l'avons déjà signalé, les deux premiers chapitres s'inscrivent dans l'esprit des travaux d'Adami, Serra et Tilli. Bien que nous étudierons les notions d'action ground state et de nodal ground state et non d'energy ground state, de nombreux arguments s'inspireront des résultats mentionnés ci-avant. Ainsi,

- le théorème 2.3 adapte le théorème I.3 à l'étude des action ground states (et leurs équivalents nodaux), sous une forme « abstraite » permettant d'étudier de façon unifiée diverses familles de graphes. Nous appliquons cette méthode à plusieurs classes de graphes et obtenons en particulier l'existence d'action ground states pour les graphes représentés dans la figure I.31. À propos des *graphes périodiques* et des *arbres infinis*, les théorèmes 2.7 et 2.8 caractérisent complètement les cas où les ground states et les nodal ground states existent ;
- nous introduisons des *conditions topologiques* généralisant l'hypothèse (H) et nous montrons que celles-ci impliquent la *non-existence des action ground states et des nodal ground states* (théorème 2.6).

[81] Pour une présentation plus détaillée des graphes périodiques, voir le début de la section 2.5.1 dans le chapitre 2. On pourra se référer à [68, Definition 4.1.1] pour une définition précise.

[82] C'est à dire ceux formés de copies d'un graphe compact « disposées en ligne » (voir [125] pour plus de détails). Ainsi, ces graphes sont à la fois localement et globalement uni-dimensionnels, ce qui explique pourquoi on n'y observe pas de phénomène de dimensional crossover.

I.6.8 Solutions normalisées sur les graphes non-compacts dans le cas $p > 6$

À notre connaissance, le seul[83] travail qui étudie les solutions normalisées de ($NLS_{\mathcal{G}}$) sur les graphes non-compacts dans le régime masse-supercritique est celui de Dovetta, Jeanjean et Serra [128].

Les auteurs y prouvent l'existence de solutions *positives*, *d'énergie positive* pour les graphes ayant un nombre fini d'arêtes et auxquels sont attachés au moins une arête se terminant en un nœud de degré un (comme pour le $\mathcal{T}$-graphe, voir la figure I.31 (a)) ou un « panneau de signalisation » (voir la figure I.31 (b)), *en supposant que la masse est assez petite*.

Ils prouvent également l'existence de solutions d'énergie positive sur les graphes périodiques, en supposant que la masse est assez grande.

Ces preuves utilisent à la fois des constructions basées sur le lemme du col et le monotonicity trick (voir les sections I.5.6 et I.5.7), mais aussi des arguments spécifiques aux graphes métriques non-compacts.

Ainsi, pour les graphes ayant un nombre fini d'arêtes, les auteurs comparent les niveaux d'énergie des fonctions définies sur le graphe avec les niveaux d'énergie du soliton et du demi-soliton de même masse (voir [128, Section 3]), étendant d'une certaine façon la démarche du cas $2 < p < 6$. Néanmoins, le cas $p > 6$ est bien plus délicat.

I.7 Solutions concentrées

I.7.1 Concentration dans le régime $\lambda \to +\infty$

L'unique solution H^1 non-nulle de l'équation (NLS) sur la droite réelle, au signe près et à translations près, est le *soliton*

$$\phi_{\lambda,p}(x) := \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}}x\right)^{\frac{-2}{p-2}}, \tag{I.18}$$

comme prouvé dans l'annexe C.

L'expression ci-dessus implique que :

- la norme L^∞ du soliton converge vers l'infini lorsque $\lambda \to +\infty$;
- pour tout réel $x \neq 0$, on a $\phi_{\lambda,p}(x) \xrightarrow[\lambda\to+\infty]{} 0$.

Dans la figure I.32 ci-après, on constate que les solitons $\phi_{\lambda,3}$ « *se concentrent* » lorsque λ augmente.

[83]Signalons aussi l'article [31] (voir le bas de la page 40), mais dont les enjeux sont différents vu la présence d'un terme de potentiel.

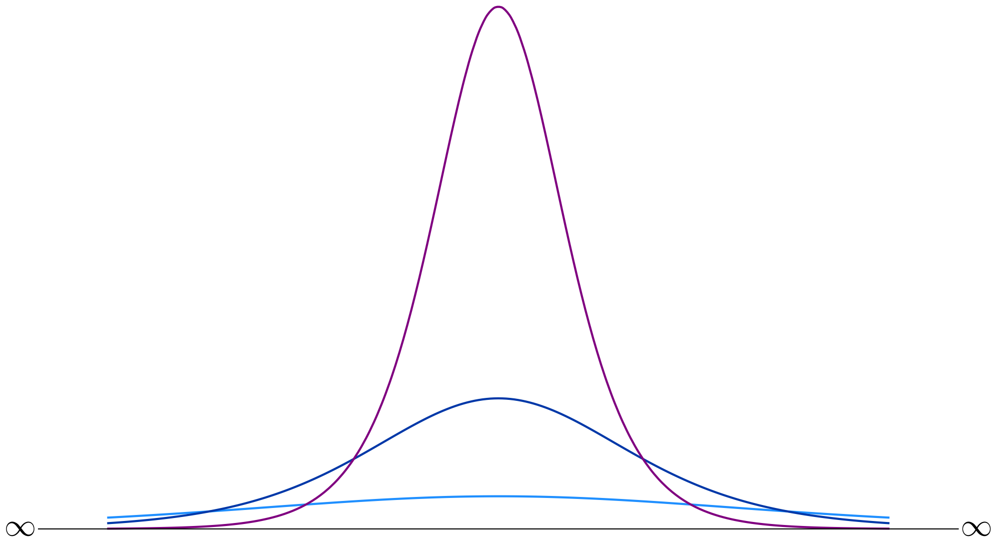


FIGURE I.32 : Une portion des solitons $\phi_{1/4,3}$, $\phi_{1,3}$ et $\phi_{4,3}$

Plus généralement, des équations elliptiques superlinéaires telles que (NLS) peuvent admettre des solutions *concentrées* au voisinage d'un point du domaine dans certains régimes asymptotiques.

Il s'agit d'un thème de recherche fort actif. Nous renvoyons le lecteur vers [277] (voir aussi [321, Section 2.4]) pour un plus large aperçu.

Remarque. De nombreux résultats d'existence de solutions concentrées utilisent la méthode de *réduction de Lyapunov-Schmidt*. Nous ne la présenterons pas ici et renvoyons vers l'article d'exposition [277].

Néanmoins, signalons que nous emploierons cette méthode afin d'étudier (NLS) dans le régime $p \approx 2$. Nous y reviendrons dans la section I.11.

À présent, présentons des résultats portant sur des solutions localisées sur les arêtes d'un graphe. Ils généraliseront les phénomènes observés sur la droite réelle.

I.7.2 Solutions localisées sur les arêtes d'un graphe

Un calcul basé sur (I.18) montre que la masse L^2 du soliton $\phi_{\lambda,p}$ est donnée par

$$\|\phi_{\lambda,p}\|_{L^2}^2 = \lambda^{\frac{6-p}{4(p-2)}} \|\phi_{1,p}\|_{L^2}^2.$$

Dès lors, *dans le régime masse-sous-critique* $2 < p < 6$, considérer un soliton associé à une grande valeur de λ revient à considérer un soliton de grande masse. De plus, une arête bornée « suffisamment longue » peut accueillir des fonctions qui sont « très proches » d'un soliton donné.

Cela mène à trois régimes différents où on peut trouver des solutions localisées :

($\lambda \to +\infty$) étant donné $p \in (2, +\infty)$, chercher les solutions (à λ fixé) de ($\mathrm{NLS}_{\mathcal{G}}$) et faire tendre λ vers $+\infty$;

($\mu \to +\infty$) étant donné $p \in (2, 6)$, chercher les solutions normalisées de masse μ (où, cette fois, λ n'est pas fixé) de ($\mathrm{NLS}_{\mathcal{G}}$) et faire tendre μ vers $+\infty$;

($\ell \to +\infty$) étant donné $p \in (2, +\infty)$, chercher les solutions (à λ fixé) de ($\mathrm{NLS}_{\mathcal{G}}$) « localisées sur une arête » de longueur ℓ et faire tendre ℓ vers $+\infty$.

Dans la littérature, il existe également trois approches assez différentes pour prouver l'existence de solutions localisées sur une arête d'un graphe :

(Var) une méthode variationnelle, consistant à chercher les solutions comme des minima locaux de la fonctionnelle d'énergie sur la contrainte de masse ou des minima locaux de la fonctionnelle d'action sur la variété de Nehari ;

(L-S) la méthode de Lyapunov-Schmidt, qui permet de prouver l'existence de solutions concentrées pour des équations aux dérivées partielles et qui peut être transposée aux graphes, en tenant compte de leur structure locale ;

(EDO) une méthode basée sur des techniques d'EDO, en particulier une analyse de la fonction période (dans le cas $p = 4$).

Résumons les articles qui étudient les solutions concentrées sur les graphes dans le tableau suivant (voir aussi [199, Sections 5 et 6]).

Méthode / Régime	(Var)	(L-S)	(EDO)
($\lambda \to +\infty$)	[211, 302]	[103, 127]	
($\mu \to +\infty$)	[17]		[70, 200]
($\ell \to +\infty$)	Chapitre 1		

Remarque. Dans [211, 302], K. Kurata et M. Shibata considèrent la limite $\varepsilon \to 0^+$ pour le problème

$$-\varepsilon^2 u'' + u = |u|^{p-2} u.$$

Néanmoins, le changement de variables $v := \varepsilon^{\frac{-2}{p-2}} u$ transforme cette équation en

$$-v'' + \varepsilon^{-2} v = |v|^{p-2} v,$$

pour laquelle la limite $\varepsilon \to 0^+$ est du type « $\lambda \to +\infty$ ».

Dans les articles [103, 127], il est manifeste que la structure locale des graphes joue un rôle. Ainsi, les *parités des degrés des nœuds* interviennent dans les résultats de [103]. C'est logique : si on « zoome » autour d'un nœud de degré D, on observe une étoile à D branches. La description des solutions sur une telle étoile dépend de la parité de D, comme nous l'avons vu dans la section I.3.1.

Pour conclure notre discussion à propos des solutions localisées sur une arête d'un graphe, présentons le résultat prouvé dans le chapitre 1.

Étant donné une arête bornée e d'un graphe $\mathcal{G}$, on définit l'ensemble

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}.$$

Pour tout réel $\lambda > 0$, on considère le niveau

$$\mathcal{J}_{\mathcal{G},e}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e} J_\lambda(u), \tag{I.19}$$

correspondant à un problème de minimisation doublement contraint.

Le résultat obtenu est le suivant (sous une forme légèrement simplifiée, voir les théorèmes 1.12 et 1.13 pour des énoncés plus précis).

Théorème I.4. *Soient deux nombres réels $p > 2$ et $\lambda > 2$. Soit $\mathcal{G}$ un graphe métrique possédant un nombre fini d'arêtes et satisfaisant l'hypothèse* (H). *Si e est une arête bornée de $\mathcal{G}$ suffisamment longue (où le seuil sur la longueur dépend de λ et de p), alors le problème de minimisation* (I.19) *possède un minimum $u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$. De plus, u est une solution de signe constant de* ($\mathrm{NLS}_{\mathcal{G}}$) *et on a*

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G}\setminus e)}. \tag{I.20}$$

Remarque. L'inégalité (I.20) traduit, dans ce cas, le fait que la solution u est « localisée » sur l'arête e : la valeur absolue de u atteint son maximum sur e et sur e uniquement.

Finalement, remarquons qu'un certain nombre de constructions de solutions localisées (que ce soit sur des graphes ou sur des domaines de dimension supérieure) s'appliquent y compris quand le domaine n'est pas compact. Il n'est pas surprenant que ces résultats « supportent plutôt bien la non-compacité du domaine » car leurs constructions sont *locales* par nature.

Ces considérations nous aideront grandement dans le chapitre 1. En effet, des constructions de solutions concentrées nous permettront de prouver l'existence de suites de solutions sur des domaines non-compacts.

I.8 Problèmes avec non-linéarité localisée

I.8.1 Définition et motivations

Intéressons-nous aux graphes métriques possédant un nombre fini d'arêtes et de sommets, comprenant au moins une arête bornée ainsi qu'au moins une demi-droite. La classe de tels graphes est riche, voir par exemple le graphe illustré dans la figure I.2 ou les exemples de graphes non-compacts de la section I.3.4.

Si $\mathcal{G}$ est un graphe métrique avec un nombre fini d'arêtes et de sommets, son *cœur compact*[84] $\mathcal{K}$ est défini comme le sous-ensemble de $\mathcal{G}$ formé de toutes les arêtes bornées de $\mathcal{G}$ (voir par exemple [21, 297]).

Considérons le problème suivant.

$$\begin{cases} -u'' + \lambda u = \kappa(x)|u|^{p-2}u & \text{sur chaque arête } e \text{ du graphe } \mathcal{G}, \\ u \text{ est continue} & \text{en chaque sommet } v \text{ de } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{en chaque sommet } v \text{ de } \mathcal{G}, \end{cases} \tag{$\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$}$$

où κ est la fonction caractéristique du cœur compact $\mathcal{K}$ de $\mathcal{G}$. Autrement dit, $\kappa(x)$ vaut 1 si x appartient à $\mathcal{K}$, 0 sinon.

La *localisation de la non-linéarité* est due à la présence de κ. Ainsi, sur les demi-droites du graphe, l'équation devient simplement

$$-u'' + \lambda u = 0 \tag{I.21}$$

et est linéaire. On observe que lorsque $\lambda \leq 0$, aucune des solutions non-nulles de l'EDO (I.21) ne converge vers 0 en l'infini. Donc, si une solution $u \in L^2(\mathcal{G})$ du problème ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) existe, elle est nécessairement identiquement nulle sur les demi-droites, c'est-à-dire que son support est inclus dans le cœur compact.[85] Lorsque $\lambda > 0$, seuls les multiples de $\mathrm{e}^{-\sqrt{\lambda}x}$ satisfont l'équation (I.21) ainsi que la condition en l'infini.

Pour ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$), bien qu'on puisse définir une fonctionnelle d'action et la variété de Nehari associée, la littérature traite principalement des solutions normalisées. En procédant comme pour les solutions normalisées « classiques », on montre que les solutions du problème de norme L^2 fixée sont des points critiques de l'énergie $E_{\mathrm{loc}} : H^1(\mathcal{G}) \to \mathbb{R}$ définie par

$$E_{\mathrm{loc}}(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{K}} |u|^p \,\mathrm{d}x,$$

sous la contrainte de masse

$$\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x = \mu,$$

pour un certain $\mu > 0$ (voir par exemple [297, Proposition 2.3]). Comme dans la section I.5, la valeur de la masse μ est prescrite alors que λ, qui apparaît comme multiplicateur de Lagrange, est laissé libre.

[84] Cette définition fait sens et peut être pertinente pour des graphes ayant un nombre infini d'arêtes. Néanmoins, dans le cas général, cet ensemble n'est pas nécessairement compact et on l'appellera $\mathcal{B}$ pour rappeler qu'il s'agit de l'ensemble des arêtes *bornées* du graphe (voir en particulier les théorèmes 2.27 et 2.29 dans le chapitre 2).

[85] Si le principe du maximum implique qu'il n'existe pas de solutions positives de ce type, il peut exister des solutions nodales avec $\lambda \leq 0$, voir [298, Theorem 4.2 et Remark 4.6] et la discussion au niveau de la figure 4.1 dans le chapitre 4.

L'article [166] (voir aussi les discussions dans [255, page 18] et [322, Section 1, Introduction]) détaille l'intérêt du point de vue physique d'étudier des modèles où les équations différentielles s'écrivent comme $-u'' + \lambda u = c_e|u|^2u$, où la constante c_e peut dépendre des arêtes et peut être nulle sur certaines d'entre elles (voir [166, Equation (2)]). On pourra par exemple penser à un réseau formé de fibres optiques différentes, ne menant pas toutes aux mêmes effets non-linéaires, certaines pouvant même ne pas en posséder du tout. La richesse du modèle réside dans l'interaction entre la non-linéarité et la diffusion (« scattering »).

Finalement, mentionnons l'article d'exposition [80] dédié à l'étude du problème ($\text{NLS}_\mathcal{G}^{\text{loc}}$) ainsi qu'à des généralisations de ce problème pour l'équation de Dirac.

I.8.2 Travaux dans les régimes L^2-sous critique $(2 < p < 6)$ et L^2-critique $(p = 6)$

Les questions d'existence et de non-existence d'energy ground states pour ($\text{NLS}_\mathcal{G}^{\text{loc}}$) ont été traitées dans [322]. De manière remarquable, l'exposant $p = 4$ y joue un rôle « d'exposant critique » (voir [322, Theorem 3.3 et Theorem 3.4]), celui-ci n'ayant aucun équivalent pour les problèmes standards où la non-linéarité n'est pas localisée.

Dans [297], un résultat de multiplicité arbitraire de solutions normalisées pour ($\text{NLS}_\mathcal{G}^{\text{loc}}$) est obtenu par une méthode basée sur la parité de la fonctionnelle E_{loc} (i.e. sur l'égalité $E_{\text{loc}}(u) = E_{\text{loc}}(-u)$). Il met en œuvre des arguments de théorie du genre (voir par exemple [284, Chapters 7 and 8]) et avait déjà été utilisé dans [124] pour obtenir un résultat de multiplicité sur les graphes compacts.

L'article [298] précise le rôle joué par l'exposant $p = 4$ et par les propriétés du graphe (aussi bien métriques que topologiques) dans les résultats d'existence et de non-existence d'energy ground states pour ($\text{NLS}_\mathcal{G}^{\text{loc}}$).

En particulier, les travaux de Serra et Tentarelli impliquent que (voir [298, Theorem 1.1]) :

- si $p \in (2, 4)$, alors pour tout $\mu > 0$, le problème ($\text{NLS}_\mathcal{G}^{\text{loc}}$) admet un energy ground state de masse μ ;
- si $p \in (2, 6)$, alors pour tout μ suffisamment grand, le problème ($\text{NLS}_\mathcal{G}^{\text{loc}}$) admet de nombreuses solutions de masse μ ;
- si $p \in [4, 6)$, alors pour tout μ suffisamment grand, le problème ($\text{NLS}_\mathcal{G}^{\text{loc}}$) admet un energy ground state de masse μ ;
- si $p \in [4, 6)$, alors pour tout μ suffisamment petit, le problème ($\text{NLS}_\mathcal{G}^{\text{loc}}$) n'admet pas d'energy ground state de masse μ.

Finalement, signalons que les articles [132, 133] traitent du cas critique $(p = 6)$.

Il apparaît que le problème avec non-linéarité localisée est une sorte d'hybride entre le cas compact et le cas non-compact. En particulier, grâce à la localisation de la non-linéarité, E_{loc} a des propriétés de compacité que la fonctionnelle

$$E(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \,\mathrm{d}x,$$

ne possède pas. Ainsi, dans [297, Proposition 4.4], les auteurs prouvent que E_{loc} vérifie la condition de Palais-Smale pour les niveaux d'énergie négatifs.

I.8.3 Travaux dans le régime L^2-supercritique $(6 < p)$

Lorsque $p > 6$, la fonctionnelle E_{loc} est non-bornée inférieurement sur la contrainte de masse, mais possède une *géométrie de col* sur celle-ci. Cela permet de chercher une solution comme *point de selle* de la fonctionnelle sous la contrainte.

Dans [82], les auteurs adaptent leur méthode développée dans [102] au cas d'un graphe non-compact avec non-linéarité localisée. Ainsi, ils obtiennent l'existence d'une solution positive normalisée pour toute masse. Pour ce faire, ils utilisent un résultat d'existence de suites de Palais-Smale bornées avec information sur l'indice de Morse approché (voir la section I.5.7) ainsi qu'une *analyse de blow-up* dédiée à l'étude du comportement des solutions lorsque λ converge vers l'infini.

Dans le chapitre 4 nous montrons que, pour toute masse $\mu > 0$, $(\text{NLS}_{\mathcal{G}}^{\text{loc}})$ possède *une infinité* de solutions normalisées (qui ne sont pas forcément positives). Pour cela, nous utilisons [81, Theorem 1.12], un résultat d'existence de suites de Palais-Smale bornées avec information sur l'indice de Morse approché.

I.9 Solutions nulles sur des arêtes

Comme observé sur l'exemple du graphe têtard dans la section I.3.4, il n'y a en général pas de *principe de continuation unique*[86,87] sur les graphes métriques. Ce phénomène se manifeste aussi pour les problèmes spectraux, où l'on peut rencontrer des *fonctions propres nulles sur les arêtes*. Nous en reparlerons dans le chapitre 5.

Dans le chapitre 2, nous montrons que les nodal ground states peuvent être nuls sur des arêtes des graphes et que leurs ensembles nodaux peuvent être très riches.

[86]Rappelons qu'un principe de continuation unique pour une équation aux dérivées partielles affirme que, si une solution de l'équation s'annule identiquement sur un sous-domaine alors elle est identiquement nulle sur le domaine tout entier. Ce principe s'applique à l'équation elliptique (NLS), nous renvoyons à [286, Theorem XIII.63] pour un énoncé précis.

[87]Nous remercions Matthias Täufer pour son séminaire sur les principes de continuation uniques quantitatifs donné à l'UPHF et l'envoi de références pour les problèmes de contrôle sur les graphes.

La présence possible de solutions à support compact de ($\text{NLS}_\mathcal{G}$) (qui peuvent exister même lorsque $\lambda \leq 0$) sur des graphes non-compacts, nous posera quelques difficultés techniques dans le chapitre 4.

Finalement, mentionnons qu'on dispose d'un *principe du maximum* sur les graphes (voir l'annexe D). Dès lors, même si les problèmes sur graphes peuvent réserver des surprises en ce qui concerne les solutions nodales, on pourra utiliser le fait qu'une solution positive non-nulle de ($\text{NLS}_\mathcal{G}$) est strictement positive.

I.10 Solutions d'équations versus minimiseurs de problèmes sous contraintes

I.10.1 Différents types de solutions pour (NLS)

Dans les sections précédentes, nous avons rencontré la plupart des notions ci-dessous, relatives aux solutions de (NLS) (par exemple sur un domaine Ω avec la condition au bord de Dirichlet). À présent, résumons-les et intéressons-nous aux relations entre elles.

Étant donné une fonction $u \in H_0^1(\Omega) \setminus \{0\}$, on dit que u est :

1. *un action ground state* si u appartient à la variété de Nehari $\mathcal{N}_\lambda$ et minimise la fonctionnelle d'action J_λ parmi les fonctions de $\mathcal{N}_\lambda$;
2. *une solution d'action minimale* si u est une solution de (NLS) qui minimise la fonctionnelle d'action J_λ parmi les solutions non-nulles de (NLS) ;
3. *un nodal ground state* si u appartient à l'ensemble de Nehari nodal $\mathcal{N}_\lambda^{nod}$ et minimise la fonctionnelle d'action J_λ parmi les fonctions de $\mathcal{N}_\lambda^{nod}$;
4. *une solution nodale d'action minimale* si u est une solution nodale de (NLS) qui minimise la fonctionnelle d'action J_λ parmi les solutions nodales de (NLS) ;
5. *un energy ground state de masse (prescrite)* $\mu \geq 0$ si u a une masse μ et minimise la fonctionnelle d'énergie E parmi les fonctions de masse μ ;
6. *une solution d'énergie minimale de masse* $\mu \geq 0$ si u est une solution de masse μ de (NLS) pour un certain $\lambda \in \mathbb{R}$ et qui minimise la fonctionnelle d'énergie E parmi les solutions de (NLS) de masse μ (où $\lambda \in \mathbb{R}$ n'est pas fixé et peut varier entre deux solutions de même masse) ;
7. *une solution nodale d'énergie minimale de masse* $\mu \geq 0$ si u est une solution nodale de masse μ de (NLS) pour un certain $\lambda \in \mathbb{R}$ et qui minimise la fonctionnelle d'énergie E parmi les solutions nodales de (NLS) de masse μ (où $\lambda \in \mathbb{R}$ n'est pas fixé).

Les problèmes de minimisation permettent de trouver des solutions de (NLS). Ainsi lorsqu'ils existent (par exemple lorsque Ω est borné), les action ground states, les nodal ground states et les energy ground states de masse μ sont respectivement des solutions d'action minimale, des solutions nodales d'action minimale et des solutions d'énergie minimale de masse μ.

Plusieurs questions se posent.

(Q1) Que se passe-t-il lorsqu'on travaille sur des domaines non-compacts, auquel cas les problèmes de minimisation n'ont pas forcément de solutions ?

(Q2) Quel est le lien entre les solutions d'action minimale et les solutions d'énergie minimale ?

(Q3) Comment prouver l'existence de solutions normalisées dans le régime masse-supercritique, auquel cas les energy ground states n'existent pas ?

(Q4) Comment trouver des solutions nodales normalisées ?

Toutes ces questions seront abordées dans les sections suivantes.

I.10.2 Action ground states vs solutions d'action minimale

Dans cette section, nous allons étudier la question (Q1) en s'intéressant à la relation entre les action ground states et les solutions d'action minimale. Pour ce faire, utilisons un graphe métrique $\mathcal{G}$ non-compact comme domaine[88].

On définit le niveau d'action

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u).$$

On définit également l'ensemble

$$\mathcal{S}_\lambda(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \setminus \{0\} \;\middle|\; u \text{ est une solution de } (\mathrm{NLS}_{\mathcal{G}}) \right\}$$

et le niveau d'action

$$\sigma_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u).$$

Remarquons qu'un action ground state est une fonction $u \in \mathcal{N}_\lambda(\mathcal{G})$ telle que $J_\lambda(u) = \mathcal{J}_{\mathcal{G}}(\lambda)$ et qu'une solution d'action minimale est une fonction $u \in \mathcal{S}_\lambda(\mathcal{G})$ telle que $J_\lambda(u) = \sigma_{\mathcal{G}}(\lambda)$.

Les raisons amenant à considérer les notions d'action ground state et de solution d'action minimale sont différentes.

[88] La raison pour laquelle il est avantageux de travailler sur des graphes et non sur des ouverts de $\mathbb{R}^N$ sera éclaircie par la suite.

Dans le premier cas, on s'intéresse au niveau $\mathcal{J}_{\mathcal{G}}(\lambda)$ et on essaie de prouver que l'infimum est atteint en montrant qu'il existe une fonction d'action minimale *parmi toutes les fonctions de* $\mathcal{N}_\lambda(\mathcal{G})$.

Dans le second cas, on se demande si *parmi les solutions de* ($\mathrm{NLS}_{\mathcal{G}}$), il y en a une d'action minimale. Remarquons qu'il peut y avoir de nombreuses fonctions dont l'action est inférieure à $\sigma_{\mathcal{G}}(\lambda)$ dans $\mathcal{N}_\lambda(\mathcal{G})$, aucune n'étant une solution du problème ($\mathrm{NLS}_{\mathcal{G}}$) si $\mathcal{J}_{\mathcal{G}}(\lambda)$ n'est pas atteint. Même si $\mathcal{S}_\lambda(\mathcal{G})$ est un ensemble bien plus petit que $\mathcal{N}_\lambda(\mathcal{G})$, il n'existe peut-être pas de fonction dans $\mathcal{S}_\lambda(\mathcal{G})$ qui atteigne $\sigma_{\mathcal{G}}(\lambda)$.

Quatre situations peuvent se présenter.

A1) $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\mathcal{G}}(\lambda)$ et ces deux niveaux sont atteints ;

A2) $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\mathcal{G}}(\lambda)$ et ils ne sont pas atteints ;

B1) $\mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_{\mathcal{G}}(\lambda)$, $\sigma_{\mathcal{G}}(\lambda)$ est atteint mais pas $\mathcal{J}_{\mathcal{G}}(\lambda)$;

B2) $\mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_{\mathcal{G}}(\lambda)$ et aucun des deux n'est atteint.

On peut se demander si ces quatre situations sont effectivement possibles. Afin de le prouver il faut, pour chaque cas parmi A1–B2, produire un exemple de graphe $\mathcal{G}$ ayant le comportement prescrit.

Remarquons que dans les cas A2 et B2, le niveau $\sigma_\lambda(\mathcal{G})$ n'est pas atteint, ce qui impose au problème d'admettre une infinité de solutions de niveaux différents.

Dans le chapitre 1, nous construisons des graphes réalisant les quatre cas parmi A1–B2 pour ($\mathrm{NLS}_{\mathcal{G}}$), ce qui montre une richesse dans les comportements possibles.

Travailler sur des graphes s'avère particulièrement commode pour étudier les relations entre les notions d'action ground state et de solution d'action minimale. En effet, il est possible de construire des solutions localisées sur des arêtes (voir la section I.7) et de contrôler leurs niveaux d'action en utilisant des arguments de réarrangement (voir l'annexe B).

Remarque. D'une certaine façon, nous faisons de la théorie de la preuve. En effet, puisqu'il existe des graphes qui réalisent A1–B2, cela montre qu'il n'est pas possible d'exclure ces phénomènes à un niveau[89] « abstrait ».

On s'attend d'ailleurs à ce qu'il existe des domaines de $\mathbb{R}^N$ réalisant A1–B2 mais, à ce jour, ceci reste un problème ouvert.

[89]Un peu comme l'existence de modèles de géométrie hyperbolique montre qu'il n'est pas possible de déduire l'axiome des parallèles des quatre autres axiomes d'Euclide, voir le livre [324] (en particulier le chapitre 7) et le site `https://mathshistory.st-andrews.ac.uk/HistTopics/Non-Euclidean_geometry` grâce auquel nous avons découvert la référence précédente.

I.10.3 Action ground states vs energy ground states

Dans [129], Dovetta, Serra et Tilli étudient les liens entre les action ground states et les energy ground states. Même si ces deux notions de « ground states » sont classiques pour trouver des solutions de (NLS), la relation entre elles n'avait jamais été étudiée sous cet angle avant la parution de leur article.

Les résultats de [129] sont plus généraux mais nous allons ici les énoncer en considérant l'équation (NLS) posée sur un domaine borné $\Omega \subseteq \mathbb{R}^N$ avec la condition de Dirichlet.

Soit $p \in (2, 2 + \frac{4}{N})$. Pour tout $\lambda \in \mathbb{R}$, on considère la variété de Nehari

$$\mathcal{N}_\lambda := \left\{ u \in H_0^1(\Omega) \;\middle|\; \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \right\}$$

et pour tout $\mu \geq 0$, on considère l'ensemble

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \;\middle|\; \|u\|_{L^2(\Omega)}^2 = 2\mu \right\}$$

associé à la contrainte de masse[90] μ.

Les action ground states sont les fonctions $u \in \mathcal{N}_\lambda$ telles que $J_\lambda(u) = \mathcal{J}(\lambda)$, où

$$\mathcal{J}(\lambda) := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v).$$

Similairement, les energy ground states de masse μ sont les fonctions $u \in \mathcal{M}_\mu$ telles que $E(u) = \mathcal{E}(\mu)$, où

$$\mathcal{E}(\mu) := \inf_{v \in \mathcal{M}_\mu} E(v).$$

Dans [129, Theorem 1.2], les auteurs montrent que $-\mathcal{E}(\mu)$ est la transformée de Legendre-Fenchel de $\mathcal{J}(\lambda)$, c'est-à-dire que pour tout $\mu \geq 0$, on a

$$\mathcal{E}(\mu) = \inf_{\lambda \in \mathbb{R}} (\mathcal{J}(\lambda) - \lambda\mu). \tag{I.22}$$

Un autre résultat frappant de l'article affirme que, si $u \in \mathcal{M}_\mu$ est un energy ground state de masse μ et de multiplicateur de Lagrange λ, alors u est aussi un action ground state dans $\mathcal{N}_\lambda$ (voir [129, Theorem 1.3] pour un énoncé plus complet).

L'article [129] n'a pas pour but d'obtenir de nouveaux résultats d'existence de solutions. D'ailleurs, les auteurs y supposent *a priori* que les action ground states et les energy ground states existent (voir [129, Assumption A]).

Nous allons montrer que *ce n'est pas nécessaire* : pour obtenir des solutions normalisées, il suffit de supposer que les action ground states existent et d'utiliser le « pont » (I.22) qui relie les niveaux d'action et d'énergie. Cette démarche, qui sera menée à bien dans le chapitre 3, est précisée dans la section suivante.

[90] Signalons que μ vérifie ici l'égalité $\|u\|_{L^2(\Omega)}^2 = 2\mu$ et non l'égalité $\|u\|_{L^2(\Omega)}^2 = \mu$ comme c'est le cas dans la section I.5. L'utilisation de différentes conventions de normalisation a pour but d'éviter l'apparition de constantes dans les résultats de dualité.

I.10.4 La méthode du chapitre 3

La philosophie de la méthode est la suivante.

Trouver des solutions (nodales) normalisées de (NLS)
en étudiant les masses des (nodal) action ground states.

Sur un domaine $\Omega \subseteq \mathbb{R}^N$ borné, les action ground states existent pour tout $\lambda > -\gamma_1(\Omega)$ et pour tout $p \in (2, 2^*)$ où $\gamma_1(\Omega)$ est la première valeur propre du laplacien sur Ω avec la condition de Dirichlet (voir par exemple [35, Section 2.3.2]).

Similairement, les nodal ground states existent pour tout $\lambda > -\gamma_2(\Omega)$ et pour tout $p \in (2, 2^*)$, $\gamma_2(\Omega)$ étant la seconde valeur propre du laplacien sur $H_0^1(\Omega)$. Dans ce cas, les résultats d'existence sont plus délicats que ceux des action ground states. Lorsque $\lambda > -\gamma_1(\Omega)$, l'existence des nodal ground states peut se prouver par la méthode directe du calcul des variations (voir par exemple [93] ou [318, Theorem 18]). Le cas $\lambda \in (-\gamma_2(\Omega), -\gamma_1(\Omega)]$ a, quant à lui, été traité par T. Bartsch et T. Weth dans [53] en supposant que le bord de Ω est $\mathcal{C}^\infty$. Nous verrons dans le chapitre 3 que nous pourrons nous passer des hypothèses de régularité sur Ω.

Considérons les *niveaux d'action*[91]

$$\mathcal{J}(\lambda) := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v) \quad \text{et} \quad \mathcal{J}^{nod}(\lambda) := \inf_{v \in \mathcal{N}_\lambda^{nod}} J_\lambda(v).$$

On montre alors (sous certaines hypothèses) que, $\mu_* \in (0, +\infty)$ étant donné, la fonction réelle $f_{\mu_*} : \mathbb{R} \to \mathbb{R}$ définie par

$$f_{\mu_*}(\lambda) := \mathcal{J}(\lambda) - \mu_* \lambda$$

admet un minimum (local). Il s'avère que l'action ground state correspondant est une solution normalisée de masse μ_*.

De plus, dans le régime masse sous-critique, ou en supposant que Ω est étoilé, on peut montrer que la solution obtenue est d'énergie minimale.

De manière analogue, on montre que la fonction $f_{\mu_*}^{nod} : \mathbb{R} \to \mathbb{R}$ définie par

$$f_{\mu_*}^{nod}(\lambda) := \mathcal{J}^{nod}(\lambda) - \mu_* \lambda$$

admet un minimum, qui correspond à une solution nodale normalisée de masse μ_*.

La grande différence entre cette méthode et les résultats présentés dans I.10.3 est d'utiliser le lien entre l'énergie et l'action comme *point de départ* et non comme conséquence. En un certain sens, on obtient les solutions normalisées *par dualité*[92].

[91] On montre que $\mathcal{J}(\lambda) = 0$ lorsque $\lambda \le -\gamma_1(\Omega)$ et que $\mathcal{J}^{nod}(\lambda) = 0$ lorsque $\lambda \le -\gamma_2(\Omega)$ (voir la proposition 3.11). Ainsi, $\mathcal{J}$ et $\mathcal{J}^{nod}$ sont des fonctions continues définies sur $\mathbb{R}$.

[92] Cela rappelle le lien entre les approches lagrangiennes et hamiltoniennes en mécanique classique, voir par exemple [33, Sections 14 et 15] ou [306, Section 3.2].

L'approche que nous venons de présenter nous permet de donner des éléments de réponse aux questions (Q2), (Q3) et (Q4) de la section I.10.1, à savoir :

- éclaircir le lien entre les solutions d'action minimale et les solutions normalisées d'énergie minimale ;
- trouver des solutions normalisées dans le régime masse-supercritique, y compris si le domaine Ω n'est pas étoilé. Dans le cas étoilé, on montre de plus que les solutions normalisées obtenues sont d'énergie minimale ;
- trouver des solutions nodales normalisées ayant deux zones nodales[93].

À notre connaissance, il n'existe aucun problème de minimisation correspondant aux solutions nodales normalisées d'énergie minimale[94]. La méthode du chapitre 3 est ainsi la première méthode variationnelle qui permet de les trouver.

Dans cette thèse, nous appliquons cette démarche à l'étude de l'équation (NLS) avec la condition de Dirichlet sur les domaines bornés de $\mathbb{R}^N$.

I.10.5 Suites minimisantes, suites de Palais-Smale ou suites de solutions ?

Dans nos preuves, nous utiliserons différents types de suites afin d'obtenir des solutions en passant à la limite faible. Ainsi, nous considérerons :

- des suites minimisantes, par exemple dans la preuve du résultat d'existence de solutions concentrées (théorème 1.12) ;
- des suites de solutions sur des domaines tronqués, par exemple dans la preuve du théorème « abstrait » du chapitre 2 (théorème 2.3) ;
- des suites de Palais-Smale après utilisation du « monotonicity trick », voir la section 4.5 ;
- des suites de solutions pour des problèmes perturbés lorsque nous ferons tendre le paramètre « ρ » du monotonicity trick vers 1, voir la section 4.6.

En pratique, le type de suite à considérer dépend grandement des situations, les différentes notions ayant leurs avantages et leurs inconvénients.

À titre d'exemple, si $(u_n)_n \subseteq \mathcal{N}_\lambda$ est une suite minimisante pour $\inf_{v\in\mathcal{N}_\lambda} J_\lambda(v)$, il en est de même de la suite des valeurs absolues $(|u_n|)_n \subseteq \mathcal{N}_\lambda$ (car $|u_n|$ appartient à $\mathcal{N}_\lambda$ et on a $J_\lambda(|u_n|) = J_\lambda(u_n)$). Autrement dit, on peut supposer sans perte de généralité qu'on travaille avec une suite de fonctions positives.

Il en va autrement pour une suite de solutions : si u est une solution nodale, alors on a toujours l'égalité $J_\lambda(|u|) = J_\lambda(u)$, mais $|u|$ n'est en général *pas* une solution du problème.

[93]Une zone nodale étant une composante connexe de l'ensemble $\{x \in \Omega \mid u(x) \neq 0\}$.
[94]Autrement dit, il n'y a pas de notion de « *nodal energy ground state* ».

I.11 Régime faiblement superlinéaire ($p \approx 2$)

I.11.1 Présentation

Lorsque $p > 2$ est proche de 2, l'équation différentielle dans ($NLS_{\mathcal{G}}$) devient « presque linéaire ». Par conséquent, on s'attend à pouvoir relier ses solutions aux fonctions propres du problème spectral ($Spec_{\mathcal{G}}$) :

$$\begin{cases} -u'' + \lambda u = \gamma u & \text{sur chaque arête } e \text{ du graphe } \mathcal{G}, \\ u \text{ est continue} & \text{en chaque sommet } v \text{ de } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{en chaque sommet } v \text{ de } \mathcal{G}. \end{cases}$$

L'étude de ce régime « *faiblement superlinéaire* » sera réalisée dans le chapitre 5.

Une des motivations principales pour étudier le régime faiblement superlinéaire est l'obtention de résultats d'unicité pour certaines classes de solutions (telles que les solutions positives, les action ground states, les nodal ground states). Afin de mettre en perspective les résultats que nous allons obtenir sur les graphes, il nous semble utile de présenter les travaux consacrés à l'étude de l'*unicité des solutions* de (NLS).

I.11.2 Unicité des solutions de (NLS) sur $\mathbb{R}^N$ et les boules

Présentons les travaux existants sur $\mathbb{R}^N$ et les boules $B(0, R)$ avec la condition au bord de Dirichlet.

Solutions positives sur les domaines à symétrie radiale

La stratégie typique pour prouver l'unicité de la solution positive de (NLS) sur une boule ou sur $\mathbb{R}^N$ consiste en deux étapes.

1. Montrer que la solution est *à symétrie radiale*[95], autrement dit qu'il existe une fonction réelle U telle que, pour tout x, $u(x) = U(|x|)$. De nos jours, cette étape est plutôt bien comprise grâce à l'argument dit du *moving plane*, voir l'article fondateur [161] de B. Gidas, W.M. Ni et L. Nirenberg.
2. Montrer que l'*équation différentielle*

$$-\partial_{rr}U - \tfrac{N-1}{r}\partial_r U + \lambda U = |U|^{p-2}U \qquad (EDO_U)$$

obtenue en écrivant (NLS) en coordonnées polaires[96] possède une *unique* solution positive (vérifiant les conditions aux limites appropriées, par exemple $U'(0) = 0$ et $U(R) = 0$ si le domaine est la boule $B(0, R)$).

[95]Dans le cas de $\mathbb{R}^N$, on montre que la solution est radiale *à translation près*. La solution positive ne sera unique qu'à translation près également.

[96]C'est-à-dire en posant $r := |x|$.

En résumé, voici la démarche à effectuer.

Solution positive de (NLS) → *Moving plane* → Solution *radiale* positive de (NLS) → *Analyse EDO* → *Unique* solution positive de (EDO_U)

FIGURE I.33 : Démarche pour prouver l'unicité de la solution positive de (NLS) sur un domaine à symétrie radiale.

Concernant l'analyse de (EDO_U), mentionnons :

- le travail pionner de C.V. Coffman [106] ;
- le célèbre article de M.K. Kwong [213] prouvant l'unicité de la solution positive en toutes dimensions et pour toutes les valeurs de l'exposant p ;
- l'article [236] de K. McLeod qui généralise et simplifie la preuve de Kwong ;
- les références [319, Appendix B] et [157, Section 5] où l'on pourra apprendre les techniques de preuves utilisées dans les travaux susmentionnés ;
- l'article [105] de C.B. Clemons et de C.K.R.T. Jones pour une démarche plus géométrique ainsi que les travaux plus récents de B.L. Maultsby [232].

Signalons que le fait de savoir que les solutions sont radiales est un point de départ essentiel de toutes ces méthodes.

Solutions nodales radiales

En ce qui concerne les solutions nodales, l'argument du moving plane ne s'applique plus et certaines solutions peuvent être non-radiales[97].

Néanmoins, concentrons-nous sur les solutions radiales dont la classification est donnée par la conjecture suivante (voir [177, Sections 19.3 et 19.6]).

Conjecture. *Pour tout naturel k, il existe une* ***unique*** *solution U de* (EDO_U) *telle que $U(0) > 0$ et telle que U a exactement k racines et converge vers* 0 *lorsque $r \to +\infty$.*

Concernant l'existence de ces solutions, il est connu que pour tout naturel k, il existe *au moins une* solution de (EDO_U) ayant k racines et convergeant vers 0 lorsque $r \to +\infty$ (voir [237] pour une preuve utilisant les équations différentielles et [54] pour une preuve variationnelle).

Concernant leur unicité, la conjecture est ouverte pour la plupart des valeurs de p et de N (y compris lorsque $k = 1$). Une *preuve assistée par ordinateur*[98] a été effectuée par A. Cohen, Z. Li et W. Schlag [107] lorsque $p = 4$, $N = 3$ et $k \leq 20$. Celle-ci nécessite d'effectuer des calculs de solutions d'équations différentielles en utilisant l'*arithmétique d'intervalles*, pour p, N et k fixés.

[97] Il existe même des solutions de (NLS) n'ayant *aucune symétrie*, voir par exemple [30] pour une telle construction sur $\mathbb{R}^2$.

[98] Nous reparlerons des preuves assistées par ordinateur dans la section I.11.5.

Ainsi, les résultats d'unicité pour des problèmes non-linéaires sont généralement assez délicats à obtenir. De plus, contrairement aux résultats d'existence, ceux d'unicité n'emploient typiquement pas de méthodes variationnelles.[99]

I.11.3 Unicité et symétries dans le régime $p \approx 2$

Un des objectifs du chapitre 5 est le suivant.

Étudier l'unicité et les symétries des solutions de $(\mathrm{NLS}_{\mathcal{G}})$ *lorsque* $p \approx 2$.

Nous mènerons à bien cette démarche sur des graphes *compacts*.

À cette fin, nous emploierons une méthode de *réduction de Lyapunov-Schmidt* adaptée au régime asymptotique $p \to 2$.

Nous obtiendrons notamment *l'unicité de la solution positive de* $(\mathrm{NLS}_{\mathcal{G}})$ *lorsque* p *est suffisamment proche de* 2 *(théorème 5.9)*. Signalons que ce résultat n'est pas spécifique aux graphes. Ainsi, il est démontré sur des domaines quelconques[100] $\Omega \subset \mathbb{R}^N$ bornés et à bord $\mathcal{C}^\infty$ (voir [117, Lemma 1]).

À propos des solutions nodales, nous étudierons le comportement asymptotique des nodal ground states lorsque $p \to 2$. Une étude analogue sur les domaines bornés de $\mathbb{R}^N$ a été réalisée par D. Bonheure, V. Bouchez, C. Grumiau et J. Van Schaftingen dans [74]. Elle s'applique également sur les graphes. Elle montre que les fonctions propres qui correspondent à des points limites de suites de nodal ground states lorsque $p \to 2$ appartiennent au second espace propre E_2 et minimisent une *fonctionnelle réduite* sur une *variété de Nehari réduite*. Comme pour l'unicité de la solution positive lorsque $p \approx 2$, nous n'observerons pas de différence entre les domaines et les graphes au niveau de la théorie « abstraite ».

Un phénomène propre aux graphes va cependant se manifester au cours de notre étude : la présence possible de solutions s'annulant identiquement sur des arêtes, présentées dans la section I.9. Ceci va nous poser des problèmes « techniques » (à cause d'un manque de régularité) et nous ne pourrons pas étudier ces solutions grâce à la méthode de Lyapunov-Schmidt.

Nous sommes donc amenés à nous pencher davantage sur ce phénomène.

Ainsi, pour les *graphes compacts en étoile*, nous présenterons des conditions sur les longueurs des arêtes pour lesquelles les nodal ground states s'annulent identiquement sur des arêtes.

[99] Il est néanmoins naturel de tenter d'obtenir des résultats d'unicité en conservant le point de vue variationnel. Un travail s'inscrivant dans cette philosophie est [75].

[100] Notons que l'unicité des solutions positives n'est pas forcément vérifiée *pour tout* $p \in (2, 2^*)$ sur un domaine Ω quelconque, voir par exemple [118].

I.11.4 Le graphe tétraèdre, un exemple riche

La dernière section du chapitre 5 est dédiée à l'étude du « graphe tétraèdre ».

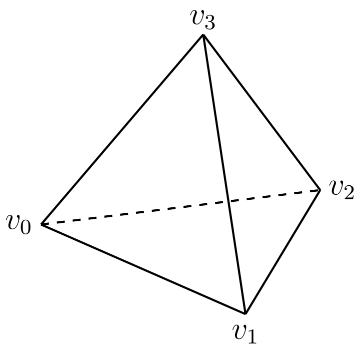


FIGURE I.34 : Le graphe tétraèdre, un graphe formé de quatre sommets et de six arêtes de même longueur.

Cet exemple est intéressant à plusieurs égards. Ainsi, bien qu'il ne possède que quatre sommets, son *groupe de symétries* est particulièrement riche. Ceci fournira des espaces propres de dimension plus grande que 1 ainsi que la présence de points critiques associés aux symétries (à travers le *principe de criticité symétrique*[101]).

De plus, nous verrons que, bien qu'il existe des fonctions propres s'annulant identiquement sur certaines arêtes dans l'espace propre E_2, les nodal ground states ne s'annulent identiquement sur aucune arête lorsque p est suffisamment proche de 2. Afin de le prouver, nous devrons étudier le problème de minimisation de la fonctionnelle réduite sur la variété de Nehari réduite (voir la section 5.2.5 pour des définitions). Ce problème s'avère assez difficile à traiter[102] « à la main ». Dès lors, nous avons eu recours à une *preuve assistée par ordinateur*.

I.11.5 Preuves assistées par ordinateur

Erreurs numériques

Les calculs numériques faisant intervenir des nombres à virgule flottante[103] mènent à des erreurs (« d'arrondis ») dues au fait que les machines ne peuvent manipuler qu'un nombre fini de chiffres.

Ainsi, dans le langage Python3, le résultat retourné par `math.sin(math.pi)` est `1.2246467991473532e-16` et non `0`. Ces erreurs peuvent engendrer de sérieux problèmes, parfois spectaculaires (voir [248, Section 1.3]).

[101] « *Principle of symmetric criticality* », voir l'article [261] de R.S. Palais.

[102] C'est aussi le cas pour les problèmes équivalents sur des domaines de $\mathbb{R}^N$, voir [294] pour l'étude de la fonctionnelle réduite associée à la limite $p \to 2$ sur le carré $(0,1)^2$.

[103] L'implémentation la plus couramment utilisée afin de représenter informatiquement des grandeurs « continues », voir [248, Section 2.1] pour des définitions précises.

Dès lors, une question se pose naturellement : *comment obtenir des preuves rigoureuses qui se basent sur des calculs numériques réalisés par ordinateur ?*

Arithmétique d'intervalles[104]

L'idée de base de l'arithmétique d'intervalles est très simple : *remplacer les nombres par des intervalles de sorte que le résultat d'une opération appartienne à l'intervalle retourné.*

Illustrons ceci en revenant au calcul de $\sin(\pi)$.
Si on utilise la librairie[105] « `mpmath` » en Python3, le résultat de `iv.pi` est

`mpi('3.1415926535897931','3.1415926535897936')`

ce qui signifie que

$$3.1415926535897931 \leq \pi \leq 3.1415926535897936.$$

De plus, le résultat de `iv.sin(iv.pi)` est

`mpi('-3.2162452993532732e-16','1.2246467991473532e-16')`

ce qui signifie que

$$-3.2162452993532732 \cdot 10^{-16} \leq \sin(\pi) \leq 1.2246467991473532 \cdot 10^{-16}.$$

De tels encadrements rigoureux sur les grandeurs peuvent être obtenus par des calculs impliquant uniquement « un nombre fini de chiffres », donc implémentables sur ordinateur. Néanmoins, l'arithmétique d'intervalles a ses limites.

- Elle peut permettre de prouver que des valeurs sont non-nulles, mais elle ne peut pas prouver que des valeurs sont nulles. Par exemple, l'évaluation de `iv.sin(1.)` renvoie `mpi('0.8414709848078965','0.84147098480789662')`, ce qui implique que $\sin(1) \neq 0$. Par contre, les calculs donnent $|\sin(\pi)| \leq 10^{-15}$, mais on ne peut pas garantir que $\sin(\pi) = 0$.
- Si un intervalle retourné est « trop grand », il est valide mais sans intérêt. Ainsi, `iv.sin(x)` pourrait retourner `[-1, 1]` peu importe la valeur de `x`, mais cet encadrement est inutile. En pratique, on cherchera à obtenir des intervalles « suffisamment petits ».

Parmi les applications de l'arithmétique d'intervalles en analyse, citons l'étude de l'état fondamental d'un système quantique comprenant un potentiel de Thomas-Fermi par C.L. Fefferman et L.A. Seco (voir [148, 149]) ou celle de l'attracteur étrange de Lorenz par W. Tucker [107]. On peut également se référer au livre [249] consacré aux preuves assistées par ordinateur en analyse.

[104]Un grand merci au Pr. Christophe Troestler pour son cours d'arithmétique d'intervalles donné à l'UPHF et pour son application à l'étude du graphe tétraèdre.

[105]Voir en particulier le module `iv`, dédié à l'arithmétique d'intervalles, dans `https://www.mpmath.org/doc/1.0.0/contexts.html`.

I.12 Résultats principaux des chapitres

Le but de cette section est double. D'une part, nous allons souligner l'*intérêt d'étudier des problèmes sur les graphes métriques*, que nous pourrions résumer de la manière suivante.

> *Les graphes métriques permettent d'étudier des problèmes en dimension un dans une classe de domaines bien plus riche que celle des intervalles réels.*

D'autre part, nous mettrons en évidence les résultats principaux des chapitres, du point de vue de l'auteur.

Le chapitre 1 correspond à un article qui a été publié dans la revue « Calculus of Variations and Partial Differential Equations » écrit conjointement avec Colette De Coster, Simone Dovetta et Enrico Serra (voir [120]).

Nous y prouvons un résultat d'*existence de solutions concentrées de* ($\mathrm{NLS}_{\mathcal{G}}$) (voir les théorèmes 1.12 et 1.13). Grâce à celui-ci, nous sommes à même de répondre aux questions posées dans la section I.10.2.

Ainsi, nous comparons les deux niveaux d'action $\mathcal{J}_{\mathcal{G}}(\lambda)$ et $\sigma_{\mathcal{G}}(\lambda)$ définis par

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) \quad \text{et} \quad \sigma_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u)$$

où $\mathcal{S}_\lambda(\mathcal{G})$ est l'ensemble des solutions non-nulles de ($\mathrm{NLS}_{\mathcal{G}}$) et $\mathcal{N}_\lambda(\mathcal{G})$ est la variété de Nehari. On dit qu'un de ces niveaux est *atteint* si l'infimum correspondant est un minimum. Si $\mathcal{J}_{\mathcal{G}}(\lambda)$ est atteint, il existe des action ground states. Si $\sigma_{\mathcal{G}}(\lambda)$ est atteint, il existe des solutions d'action minimale.

Le résultat suivant montre que de nombreuses relations peuvent exister entre les deux niveaux (voir le théorème 1.3).

> **Théorème.** *Étant donné deux réels $p > 2$ et $\lambda > 0$, il existe quatre graphes métriques $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ et $\mathcal{G}_4$ (dépendant de p et de λ) tels que :*
>
> *A1) $\mathcal{J}_{\mathcal{G}_1}(\lambda) = \sigma_{\mathcal{G}_1}(\lambda)$ et ces deux niveaux sont atteints ;*
>
> *A2) $\mathcal{J}_{\mathcal{G}_2}(\lambda) = \sigma_{\mathcal{G}_2}(\lambda)$ et ils ne sont pas atteints ;*
>
> *B1) $\mathcal{J}_{\mathcal{G}_3}(\lambda) < \sigma_{\mathcal{G}_3}(\lambda)$, $\sigma_{\mathcal{G}_3}(\lambda)$ est atteint mais pas $\mathcal{J}_{\mathcal{G}_3}(\lambda)$;*
>
> *B2) $\mathcal{J}_{\mathcal{G}_4}(\lambda) < \sigma_{\mathcal{G}_4}(\lambda)$ et aucun de ces deux niveaux n'est atteint.*

La *version précisée de l'inégalité de Polyá-Szegő*[106]*, propre à la dimension un,* permet de prouver le résultat d'existence des solutions localisées et de contrôler les niveaux d'action des différentes solutions.

[106]Voir le théorème B.23.

Le chapitre 2 correspond à une prépublication en commun avec Colette De Coster, Simone Dovetta, Enrico Serra et Christophe Troestler (voir [121]). Nous y :

- prouvons *des résultats « abstraits » d'existence d'action ground states et de nodal ground states*, en fonction d'un « niveau à l'infini » (théorème 2.3) ;
- introduisons des *conditions topologiques* généralisant « l'hypothèse (H) » et nous montrons que celles-ci impliquent la *non-existence des action ground states et des nodal ground states* (théorème 2.6) ;
- caractérisons les cas où les ground states et les nodal ground states existent sur les *graphes périodiques* et les *arbres infinis* (théorèmes 2.7 et 2.8) ;
- montrons une grande richesse dans les *zones nodales* des nodal ground states sur les graphes (théorème 2.9).

Précisons le dernier point grâce à l'énoncé suivant.

Théorème. *Pour tous les entiers $k \geq 0$, $m \geq 2$ et $n \geq 0$, il existe un graphe métrique $\mathcal{G}$ et un nodal ground state u sur $\mathcal{G}$ tel que l'ensemble nodal $u^{-1}(\{0\})$ est l'union de k points isolés, m demi-droites et n arêtes bornées.*

Lors de notre étude, les graphes métriques nous ont permis d'*expérimenter avec différentes sources de non-compacité* (graphes avec un nombre fini d'arêtes et de demi-droites, graphes périodiques, arbres infinis, etc.). Comme dans le chapitre 1, l'*inégalité de Polyá-Szegő précisée* joue un grand rôle, notamment dans l'utilisation des hypothèses topologiques sur le domaine. L'existence de nodal ground states pour ($\text{NLS}_{\mathcal{G}}$) qui *s'annulent identiquement sur des arêtes* est un phénomène propre au cadre des graphes.

Les résultats du chapitre 3 proviennent, eux aussi, d'une collaboration avec Colette De Coster, Simone Dovetta et Enrico Serra.

Nous développons une *nouvelle méthode* d'étude des solutions normalisées de (NLS) sur des domaines bornés de $\mathbb{R}^N$, avec la condition au bord de Dirichlet. Cette technique permet de montrer l'existence de *solutions nodales normalisées* et de traiter le *régime L^2-supercritique*. Voici le fruit de nos recherches (voir les théorèmes 3.1 et 3.4, où l'on trouvera des énoncés plus précis).

Théorème. *Si Ω est un domaine borné de $\mathbb{R}^N$ et si $p \in (2, 2^*)$, alors l'équation* (NLS) *possède une solution positive de masse μ et une solution avec deux zones nodales de masse μ pour toute valeur suffisamment petite de la masse μ.*

Si de plus Ω est à bord $\mathcal{C}^\infty$ et étoilé, alors si la masse $\mu > 0$ est suffisamment petite, il existe des solutions (nodales) normalisées d'énergie minimale parmi les solutions du problème.

Notre méthode est « abstraite » et ne nécessite pas de travailler seulement en dimension un. Nous l'avons donc présentée dans le cadre des ouverts bornés de $\mathbb{R}^N$.

Le chapitre 4 correspond à une prépublication commune avec Pablo Carrillo, Louis Jeanjean et Christophe Troestler (voir [91]), dédiée à la preuve du résultat suivant.

> **Théorème.** *Soit $\mathcal{G}$ un graphe métrique avec un nombre fini d'arêtes dont au moins une arête bornée et au moins une demi-droite. Étant donné deux réels $p > 6$ et $\mu > 0$, le problème $(\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}})$ possède une suite de solutions normalisées de masse μ dont les niveaux d'énergie convergent vers $+\infty$.*

Dans la preuve de ce théorème (théorème 4.2), nous utilisons lors de certains passages des méthodes basées sur la théorie des *équations différentielles ordinaires*. De plus, la *localisation de la non-linéarité*, essentielle dans les résultats, se formule de façon particulièrement commode sur les graphes ayant un nombre fini d'arêtes, où l'on pourra distinguer le cœur compact et les demi-droites.

Quant au chapitre 5, il a été développé en collaboration avec mes directeurs de thèse Colette De Coster et Christophe Troestler et ne correspond pas encore à une prépublication. Il contient davantage de détails dans les preuves que les chapitres précédents.

Nous y étudions le comportement de $(\mathrm{NLS}_{\mathcal{G}})$ sur des graphes compacts dans le régime $p \approx 2$ et étudions les branches de solutions émanant des valeurs propres. Pour cela, nous utilisons une *réduction de Lyapunov-Schmidt*. Nous verrons que la dimension un sera utilisée de façon essentielle afin de *prouver la régularité*[107] *de certaines applications entre espaces fonctionnels*.

Une conséquence générale de la théorie est le théorème suivant (théorème 5.9).

> **Théorème.** *Si $\mathcal{G}$ est un graphe compact et si le réel $p > 2$ est suffisamment proche de 2, alors le problème $(\mathrm{NLS}_{\mathcal{G}})$ possède une unique solution positive.*

Nous étudions aussi le comportement des *nodal ground states* lorsque $p \approx 2$.

Les *fonctions propres s'annulant identiquement sur des arêtes* ont un statut particulier lorsqu'on étudie le régime $p \approx 2$. Cela nous a amené à étudier l'existence de *solutions de $(\mathrm{NLS}_{\mathcal{G}})$ s'annulant identiquement sur des arêtes* sur des graphes compacts en étoile, complétant ainsi des résultats du chapitre 2.

Finalement, nous appliquons la méthode basée sur la réduction de Lyapunov-Schmidt à un *exemple riche en symétries : le graphe tétraèdre*. Grâce à une *preuve assistée par ordinateur*, nous obtenons l'unicité, aux symétries près, des nodal ground states pour ce graphe dans le régime $p \approx 2$.

[107] Au niveau technique, nous utiliserons le fait que les racines des fonctions propres qui ne s'annulent identiquement sur aucune arête sont nécessairement de multiplicité un. Cet argument est basé sur le *théorème d'existence et d'unicité pour le problème de Cauchy associé à une EDO*.

I.13 Présentation des annexes

Six annexes succèdent à l'introduction et aux chapitres.

L'annexe A présente la *notion de graphe métrique* et détaille les structures d'*espaces métriques* et d'*espaces mesurés* dont les graphes sont munis. Nous y présentons également la notion de *dérivée faible*, l'*espace de Sobolev* $H^1(\mathcal{G})$ et quelques *résultats de plongement*. Finalement, nous y prouvons une *formule de la co-aire*, utilisée dans l'annexe B. Son contenu est extrait de documents rédigés avec l'aide de Colette De Coster.

L'annexe B est dédiée à l'étude du procédé de *réarrangement décroissant*, en particulier à une preuve « auto-contenue » de l'*inégalité de Polyá-Szegő précisée* grâce au *nombre de préimages* des fonctions (théorème B.23). Nous y référerons de façon systématique lors de l'usage de ces arguments dans le document. Si ces résultats ne sont pas nouveaux, l'approche suivie y est originale. Les preuves ont été détaillées avec l'aide de Colette De Coster lors de notre étude des équations posées sur les graphes.

L'annexe C centralise diverses informations à propos de l'*équation différentielle* $-u'' + \lambda u = |u|^{p-2}u$. Son contenu sera régulièrement utilisé dans le manuscrit et a été influencé par des discussions avec mes directeurs de thèse.

L'annexe D présente un résultat de *principe du maximum* pour les graphes métriques, issu de la collaboration avec Pablo Carillo, Colette De Coster, Louis Jeanjean et Christophe Troestler.

L'annexe E présente deux énoncés de *théorèmes des fonctions implicites* (l'un au niveau « topologique », l'autre au niveau « différentiable »), sous des hypothèses assez faibles de régularité. Il se base sur des notes écrites par Christophe Troestler et retravaillées en collaboration avec Colette De Coster.

L'annexe F commente quelques résultats concernant l'*équation d'évolution* de Schrödinger non-linéaire sur $\mathbb{R}^N$. Cela nous permet de poursuivre des discussions entamées dans cette introduction. Son contenu est en grande partie basé sur mon mémoire de M2, dirigé par Christophe Troestler et ayant bénéficié de nombreux échanges avec Colette De Coster lors de sa rédaction.

Dernières remarques

- Nous avons voulu rendre les chapitres aussi « auto-contenus » que possible. Ainsi, nous n'hésiterons pas à y réintroduire quelques concepts déjà rencontrés dans les chapitres précédents.

- Les chapitres 1, 2 et 4 contiennent des hypothèses différentes sur les graphes étudiés. Nous avons décidé d'appeler $\mathbf{G_1}$, $\mathbf{G_2}$ et $\mathbf{G_4}$ les classes de graphes considérées, en référence à la numérotation des chapitres, de façon à supprimer toute ambiguïté potentielle.

II. Introduction (in English)

Before tackling the precise questions studied in this thesis and the results we obtained, it is appropriate to present some notions and historical elements.

This introduction tries to be, when possible, more accessible than the following chapters. Pictures, examples and intuitions will be used, and we will in general find neither precise statements nor proofs.

The table hereunder indicates which themes from the following sections are present in the different chapters.

Subjects (sections) \ Chapters	1	2	3	4	5
Nonlinear Schrödinger equation (II.1)	✓	✓	✓	✓	✓
Metric graphs (II.2 and II.3)	✓	✓		✓	✓
(Nodal) action ground states (II.4)	✓	✓	✓		✓
Nodal solutions (II.4)		✓	✓	(✓)	✓
Normalized solutions (II.5)			✓	✓	
Noncompact domains (II.6)	✓	✓		✓	
Role of the domain geometry and topology (II.6)	✓	✓	✓		(✓)
Concentrated solutions (II.7)	✓				
Problems with localized nonlinearity (II.8)				✓	
Solutions vanishing on edges (II.9)		✓		(✓)	✓
Solutions versus minimizers (II.10)	✓		✓		
$p \to 2$ regime (II.11)					✓
Uniqueness, symmetries and computer-assisted proofs (II.11)					✓

II.1 The nonlinear Schrödinger equation

The aim of this thesis is to study nonlinear elliptic partial differential equations, essentially the nonlinear Schrödinger equation[1]

$$-\Delta u + \lambda u = |u|^{p-2}u, \qquad \text{(NLS)}$$

where $\Delta := \sum_{1\le i\le N} \partial_{ii}$ denotes the laplacian operator, $p > 2$ and λ are real parameters, $u : \Omega \to \mathbb{R}$ and $\Omega \subseteq \mathbb{R}^N$ is an open domain, bounded or not.

[1]In the sequel, the equation (NLS) will systematically be referred to as the "nonlinear Schrödinger equation". In the literature, this expression also refers to the evolution equation $\boldsymbol{i}\partial_t\Psi(t,x) + \Delta\Psi(t,x) + |\Psi(t,x)|^{p-2}\Psi(t,x) = 0$. The *stationary wave* ansatz $\Psi(t,x) = e^{\boldsymbol{i}\lambda t}u(x)$ relates the evolution equation and the elliptic equation (see Appendix F).

It then remains to specify boundary conditions, like the Dirichlet one (imposing that u vanishes on the boundary of Ω) or the Neumann one (imposing that the normal derivative of u vanishes on the boundary of Ω).

If the domain Ω is unbounded, for instance if $\Omega = \mathbb{R}^N$, then the conditions at infinity also play a role. In this case, we will be interested in the solutions which are "sufficiently small" at infinity, such as the square-integrable ones.

An important property of equation (NLS) is to admit a *variational formulation*. Indeed, if we define the *action functional*[2]

$$J_\lambda(u) := \frac{1}{2}\int_\Omega \|\nabla u\|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x,$$

then critical points of J_λ on the Sobolev space $H^1(\Omega)$ correspond to solutions of the equation satisfying the Neumann condition, and critical points of J_λ on $H^1_0(\Omega)$ correspond to solutions of the equation satisfying the Dirichlet condition. The variational viewpoint will play a big role in most chapters.

The literature on the subject is vast and it would be vain to try to present it exhaustively. Nevertheless, let us mention the fundamental articles [65, 66, 213, 307, 332] which study the properties of equation (NLS) on $\mathbb{R}^N$.

The reader willing to be introduced to the subject can, for instance, consult the overview articles [52, 214] and the book [35].

The treatises [284, 310, 335] are precious references about the variational point of view.

Let us also cite the reference works [98, 152, 317] which study the nonlinear Schrödinger equation in a broad sense, including the theory of associated nonlinear evolution problems.

Let us summarize the object of our works (except those of chapter 3) in one sentence.

This thesis is devoted to the study of equivalents of equation (NLS) *on metric graphs.*

As for Chapter 3, it is dedicated to the study of (L^2-normalized, a notion we will specify in section II.5) solutions of the equation (NLS) on bounded domains Ω.

Before going any further, let us specify what metric graphs are.

[2] In the text, we will call *functional* a function defined on a function space.

II.2 Why use metric graphs as domains?

II.2.1 What are metric graphs?

A *metric graph* is a *one-dimensional* domain made of *vertices* (or *nodes*) and *edges* joining the vertices between them or joining a vertex and infinity.

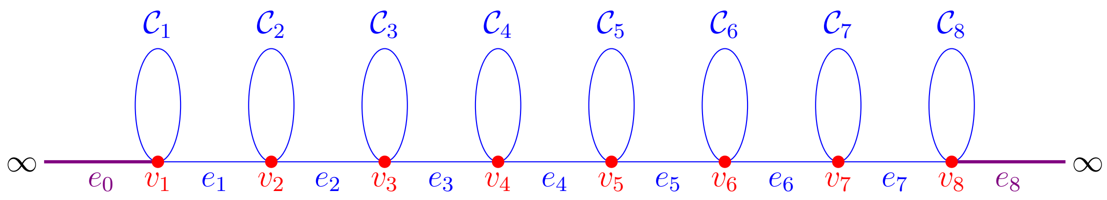


Figure II.1: A first example of metric graph with 8 vertices $v_1, \ldots, v_8$; 7 edges $e_1, \ldots, e_7$ with finite length joining distinct vertices; 8 loops (edges) $\mathcal{C}_1, \ldots, \mathcal{C}_8$ joining a vertex to itself; two half-lines e_0 and e_8.

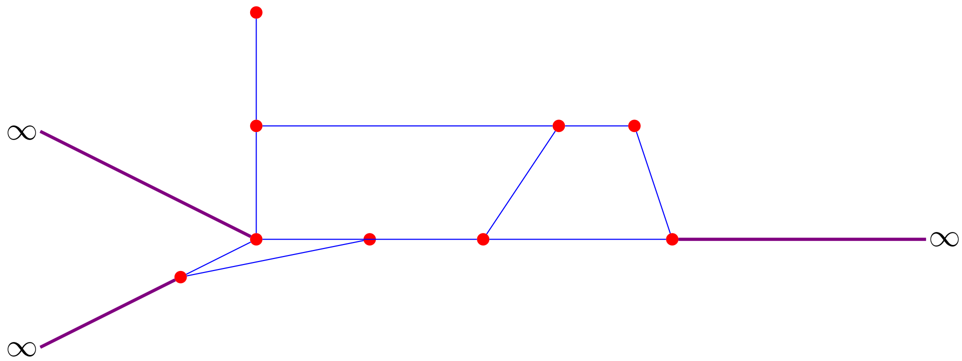
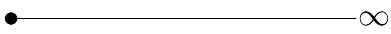


Figure II.2: Second example of metric graph

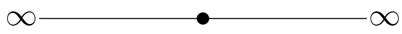
A formal definition of the notion of metric graph is available in Appendix A. Let us already remark that:

- the graphs under study are *metric*: the lengths of edges are important and will sometimes play a role in the results;
- the edges of graphs going to infinity are half-lines and have an *infinite length*;
- in this thesis, we will say that a graph is *compact* if it is made of a finite number of edges of finite lengths (indeed, in this case, the graph is compact as a metric space, see Proposition A.1).

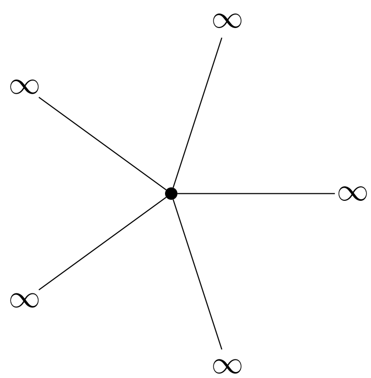
Using only some half-lines, we can form the family of *star graphs*, examples of noncompact graphs having a single vertex.



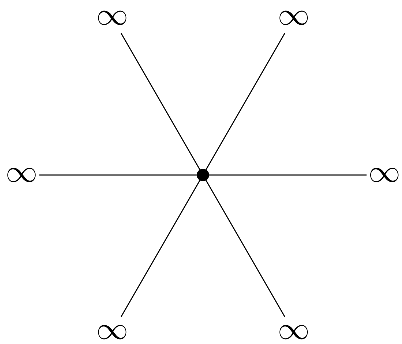
The half-line



The real line



The 5-star graph



The 6-star graph

Figure II.3: Constructions based on half-lines

The class of metric graphs is very rich. One can for instance consider *periodic graphs* (see Figures II.4 and II.5).

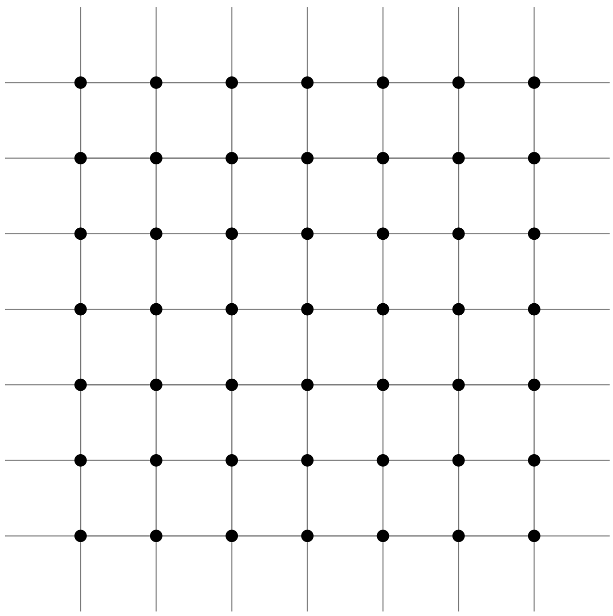


Figure II.4: The infinite grid in the plane

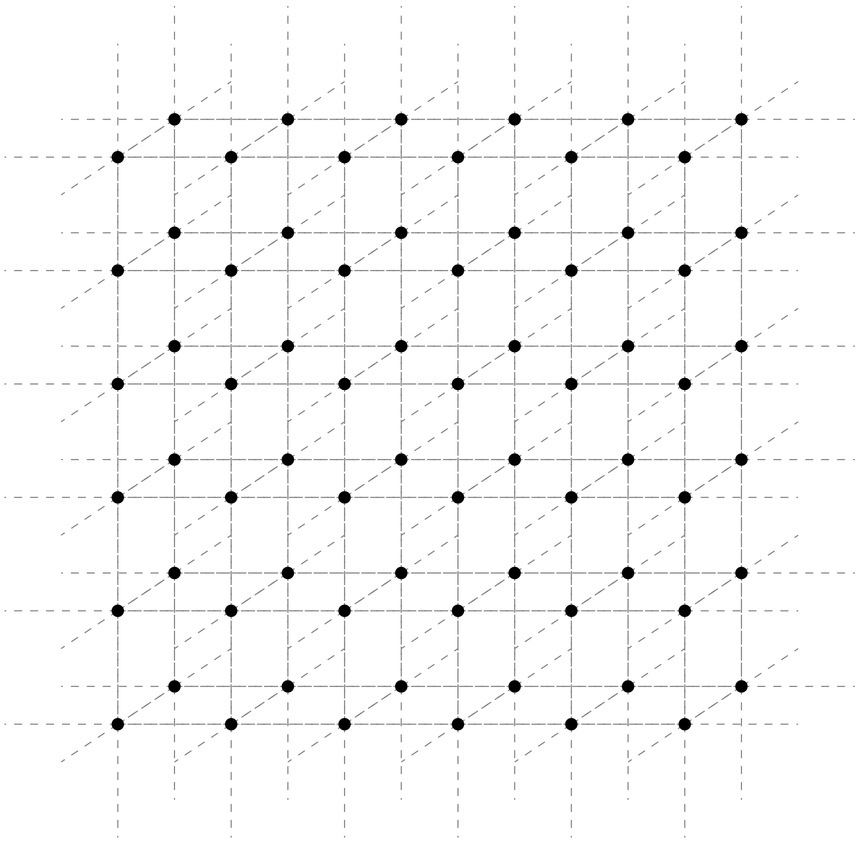


Figure II.5: The infinite grid in space

We will also consider *infinite trees* (see Figure II.6).

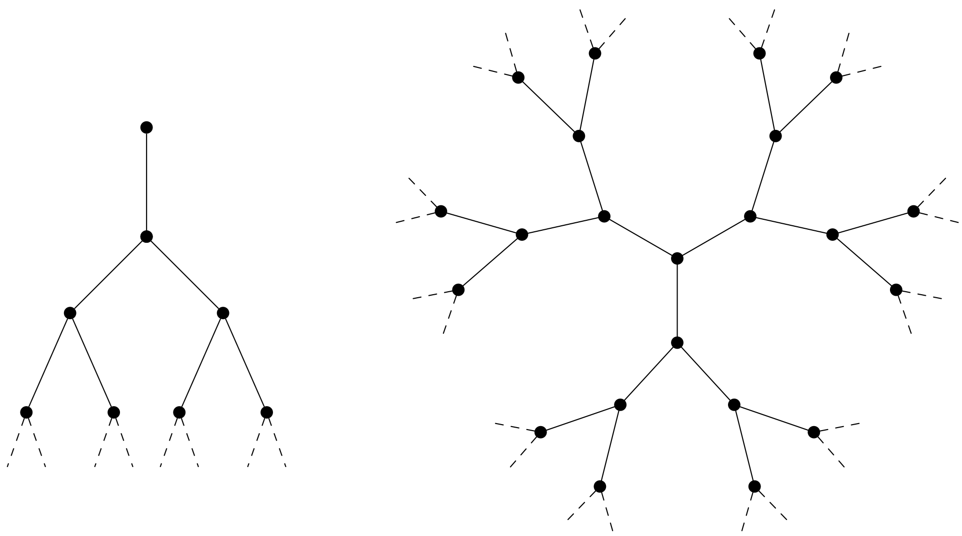


Figure II.6: Infinite trees

In this thesis, metric graphs will play the role of *domains* on which the nonlinear problems will be studied. Our main objects of study will thus be *functions* defined on metric graphs.

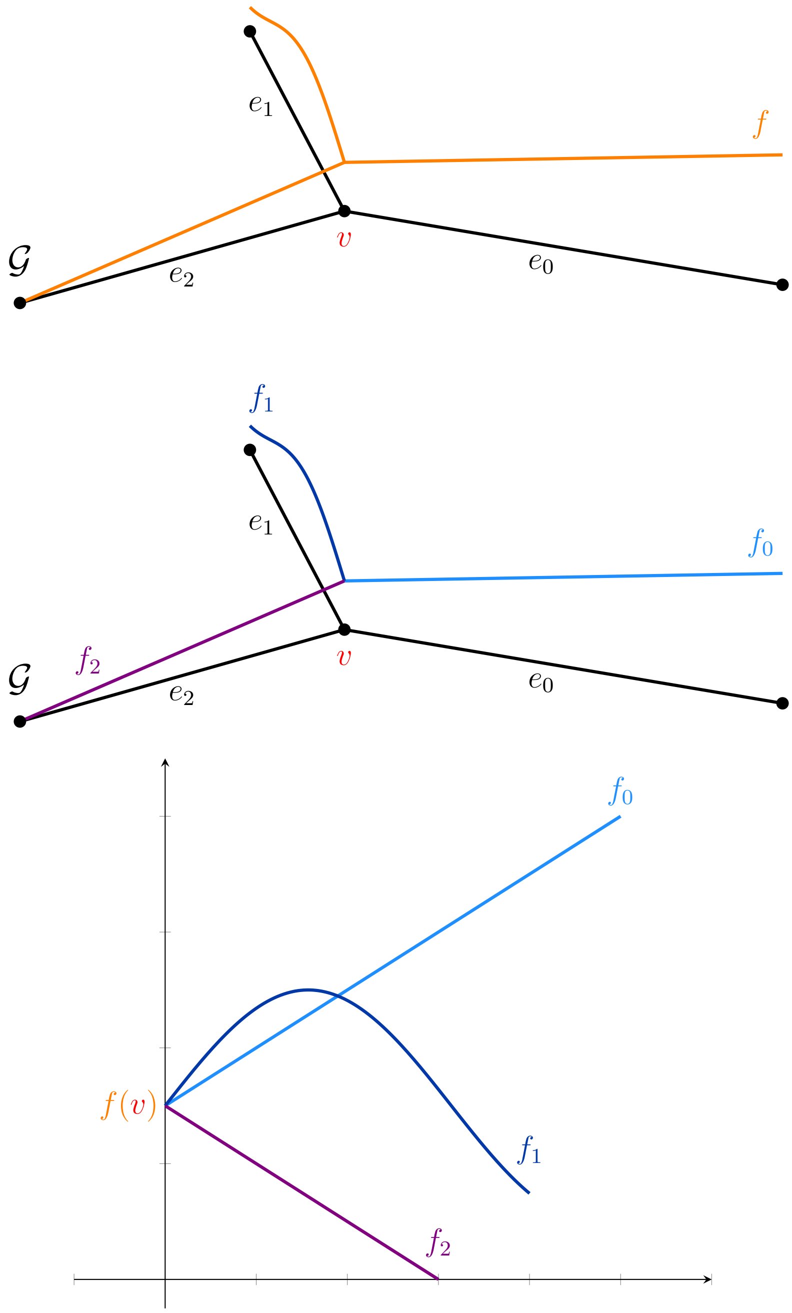


Figure II.7: A metric graph $\mathcal{G}$ with three edges e_0 (of length 5), e_1 (of length 4), e_2 (of length 3) and a function $f : \mathcal{G} \to \mathbb{R}$ with the three associated real functions f_0, f_1, f_2. *Let us remark that* $f_0(0) = f_1(0) = f_2(0) = f(v)$.

The classical operations of analysis can be defined naturally on graphs working edge by edge. For instance, in Figure II.7, we have

$$\int_{\mathcal{G}} f \, \mathrm{d}x := \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x.$$

Having a theory of integration at our disposal allows us to define Lebesgue spaces in the usual way, that we will note $L^p(\mathcal{G})$. We can also endow $\mathcal{G}$ with a metric space structure by saying that the distance between two points of the graph is given by the smallest length of a continuous path joining those two points. Thus, metric graphs form a class of *metric measured spaces* (see sections A.2 and A.3 for precise definitions of the structures defined on graphs).

Now, let us see some "pragmatic" reasons to study problems on graphs.

II.2.2 Dimension reduction

Metric graphs are relevant to the study of problems for which *only one spatial direction is important.*

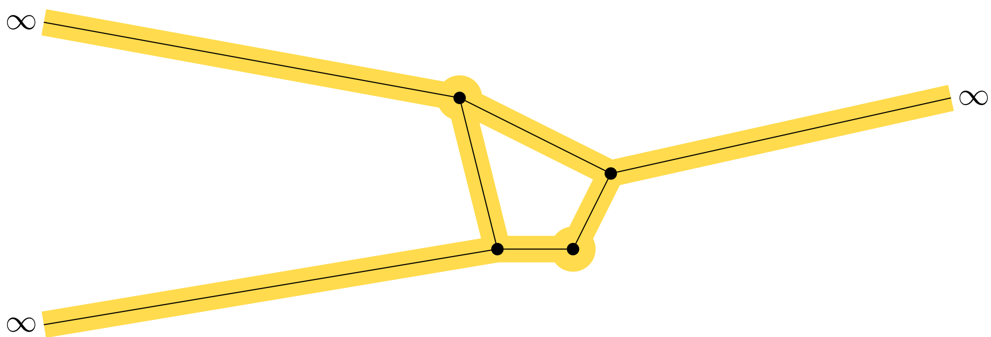


Figure II.8: A "fat graph" and the underlying metric graph

It is the case in the example depicted by Figure II.8 where a domain in $\mathbb{R}^2$ (a "fat graph") is shown. The situation is analogous for cables in $\mathbb{R}^3$ (optical fibers, electric cables, etc.). Common sense tells us that, when the thickness of the domain is small and that transverse directions to the cable do not play any important role, the problem "becomes one-dimensional".

In this way, in physics courses which study electric circuits, one may introduce more or less heuristically *reduced models* of circuits by starting from fundamental principles of electromagnetism. In such models, the precise geometric form of the cables in $\mathbb{R}^3$ does not play any role (see for instance [151, Volume II, Chapter 22]).

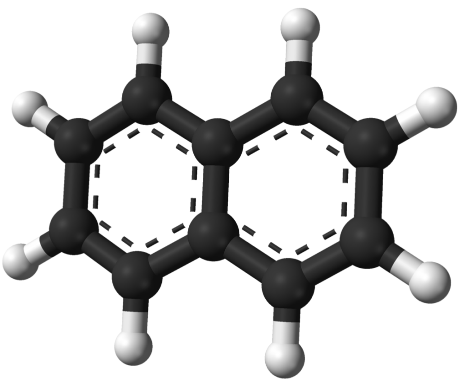
On this topic, let us recall *Kirchhoff's junction rule*, stipulating that the sum of electric intensities at each node of the circuit vanishes, using a good sign convention (see e.g. [64, Chapter 7]).

II.2.3 Genesis of quantum graphs

A dimension reduction such as the one presented in the examples of Section II.2.2 is also relevant to quantum models. Let us present a few historical cases illustrating our point. They are based[3] on the article [113, Section 8], in which the reader will find more details on the early days of the quantum graph theory.

As early as 1930, E. Hückel [180] has shown that the quantum description of hydrocarbons could be reduced to the study of a model set on the graph associated to the structure of the molecule.

In the 1950s, K. Ruedenberg and C.W. Scherr [290] used the same method to study the dynamics of valence electrons in naphthalene $C_{10}H_8$ (see Figure II.9). In the same spirit, we can also cite works of C.A. Coulson [112] and later works of Ruedenberg (see e.g. [289]).



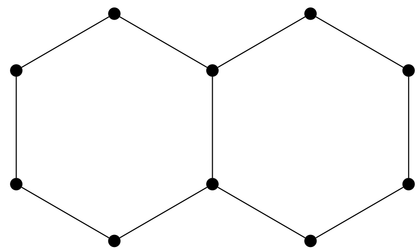


Figure II.9: Representation[4] of a molecule of naphthalene $C_{10}H_8$ and the associated metric graph.

The aforementioned quantum chemistry works show that the eigenvalues of hamiltonian operators[5] associated with a molecule are related to the *spectrum of the metric graph corresponding to the molecule.*

Let us specify what this notion means.

[3]Many thanks to Prof. Delio Mugnolo for presenting the history of the theory during his course given during the *Nonlinear Quantum Graphs* summer school in Valenciennes in June 2024 (see `https://nqg.sciencesconf.org/`), in particular for highlighting the reference [113].

[4]Image from `https://commons.wikimedia.org/wiki/File:Naphthalene-from-xtal-3D-balls.png`, public domain.

[5]Which correspond to *energy levels*, according to the quantum theory.

II.2.4 The spectral problem and Kirchhoff's condition

If $\mathcal{G}$ is a metric graph, the spectral problem on $\mathcal{G}$ reduces to finding couples (u, γ) for which the *eigenfunction* $u : \mathcal{G} \to \mathbb{R}$ and the *eigenvalue* $\gamma \in \mathbb{R}$ lead to a solution of the differential system

$$\begin{cases} -u'' = \gamma u & \text{on every edge } e \text{ of the graph } \mathcal{G}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}. \end{cases} \tag{Spec$_\mathcal{G}$}$$

The notation $e \succ v$ means that the sum ranges over all edges of vertex v and $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ is the outgoing derivative of u at v. If we parametrize e by a coordinate $x_e \in [0, |e|]$ (where $|e|$ denotes the length of e), we thus have

$$\frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = \begin{cases} u'(v) & \text{if } v \text{ corresponds to } x_e = 0, \\ -u'(v) & \text{if } v \text{ corresponds to } x_e = |e|. \end{cases}$$

By analogy with Kirchhoff's junction rule, the condition

$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0$$

is known as *Kirchhoff's condition.*

The observation hereunder will follow us all along our study of problems set on graphs.

A differential problem set on a graph consists in: 1) *a differential equation edge by edge;* 2) *compatibility conditions at the vertices.*

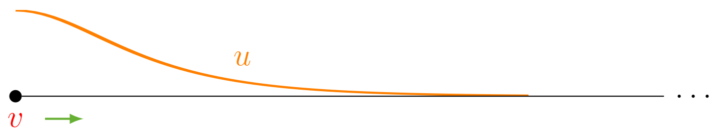
In the case of (Spec$_\mathcal{G}$), the continuity condition and Kirchhoff's condition are the conditions at the vertices.

In Section II.3, we will encounter the same conditions during the study of the nonlinear Schrödinger equation on graphs.

Now, let us see what Kirchhoff's condition means by illustrating it through a few simple examples. In the sequel, we will call *degree of* v the number of edges adjacent to a given vertex v.

Case of a node of degree one

Let us consider a node v of degree one and a real-valued function u. Locally, the situation is the following:



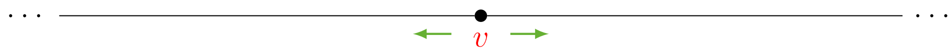
The condition becomes

$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(v+t)-u(v)}{t} = 0.$$

Put another way, Kirchhoff's condition requires that the derivative of u vanishes at the vertex v: we recover the usual *Neumann condition*.

Case of a node of degree two

This time, the situation is (locally) illustrated as follows:

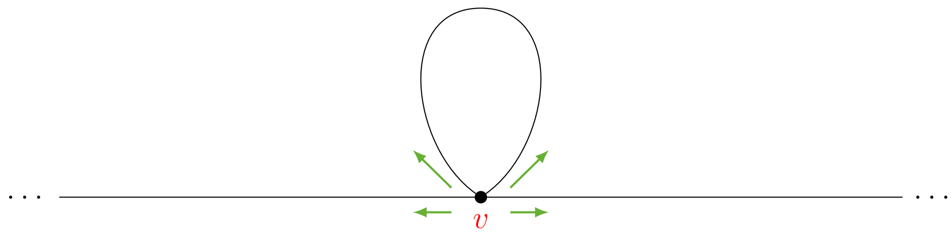


Let us recall that we consider functions which are *continuous at the nodes*. Thus, we may consider the function u defined on the graph as a function defined on a real interval containing v in its interior.

In this case, Kirchhoff's condition imposes that the left and right derivatives of u at v are equal, which means that u (still considered as a function defined on a real interval) is differentiable at v. This differentiability property is satisfied by solutions of second order differential equations 2, which explains why we will generally identify two graphs which only differ by the presence of nodes having degree two.

Illustration of the general case

Let us consider a situation where the graph possesses a node v of degree four:



In this case, four derivatives are evaluated at the vertex v and Kirchhoff's condition claims that the sum of those four derivatives vanishes.

II.2.5 Quantum graphs[6]

Two founding articles about analysis problems on metric graphs are those of B.S. Pavlov–M.D. Faddeev [266] and S. Nicaise [254][7]. In those works, the authors show that the laplacian operator on graphs is self-adjoint in L^2 when coupled with continuity and Kirchhoff's conditions at vertices. The reader wanting to learn more about the study of differential operators on graphs can consult the book [68] by G. Berkolaiko and P. Kuchment. One can, in particular, find therein the definition of a *quantum graph* ([68, Definition 1.4.1]), notion adapted to the study of the physicochemical problems mentioned before:

> A **quantum graph** is a metric graph equipped with a differential operator $\mathcal{H}$ (Hamiltonian), accompanied by "appropriate" vertex conditions. That is, a quantum graph Γ **is a triple** **{metric graph Γ, Hamiltonian $\mathcal{H}$, vertex conditions}**.

In particular, if we couple the second derivative operator with the continuity and Kirchhoff's conditions on a graph, the spectral problem of the corresponding quantum graph takes the form ($\text{Spec}_{\mathcal{G}}$).

Aside from [68], several treatises aiming to study differential problems on metric graphs exist, see e.g. [246] of D. Mugnolo, [108] of Y. Colin de Verdière or [210] of P. Kurasov.

Among questions related to the domain of quantum graphs, let us cite the study of the spectrum of quantum graphs (Weyl formulas, Cheeger and Faber-Krahn inequalities, etc.) in [24, 158, 254], isospectral problems ("*Can one hear the shape of a network?*") in [59], the obtention of spectral estimates using "surgery" techniques in [67], "quantum chaos" in [165], etc. A more detailed overview of the subject can be found in the lecture notes [202].

Remark. We will only consider one-dimensional models, where the quantities of interest are defined on the *edges* of the graph, which are identified to intervals. Some "zero-dimensional" models, where the quantities of interest are defined on the *vertices*, also exist, see e.g. [246, Section 2.1.4] for a definition of the *discrete Laplacian.* If all edges of a graph have the same length, there exists a profound link between the one-dimensional model and the discrete model, as shown by J. von Below [57] and S. Nicaise [253] in the mid-1980s (see also [264] and [68, Section 3.6]). We will use those considerations during our study of the "tetrahedron graph" in Chapter 5.

---

[6]Once again, we thank Prof. Mugnolo, thanks to whom we gained a better understanding of the *ad hoc* literature.

[7]Let us also mention the note [229] of G. Lumer in the "Comptes Rendus de l'Académie des Sciences de Paris". Therein, the author considers graphs (that he calls "topological networks") as examples of *ramified spaces*, spaces obtained by gluing more elementary structures together, namely real intervals in the case of graphs. We refer the interested reader to [26, 60, 251, 252] and to the references therein for more information.

II.3 The nonlinear Schrödinger equation on metric graphs

The procedure of dimensional reduction can be applied as well to situations described by the *nonlinear* Schrödinger equation (NLS).

Given a metric graph $\mathcal{G}$ and two real numbers $p > 2$ and λ, we couple the nonlinear Schrödinger equation

$$-u'' + \lambda u = |u|^{p-2}u$$

on each edge with the continuity and Kirchhoff's conditions.

The problem is then given by

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on every edge } e \text{ of the graph } \mathcal{G}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}. \end{cases} \tag{NLS$_\mathcal{G}$}$$

The problem (NLS$_\mathcal{G}$) *(and its variants) is the main topic under study in this thesis.* It possesses applications in physics that we will describe in section II.3.2.

First, let us consider some examples.

II.3.1 Examples: the (half-)line, star graphs

In this section, let us study the nonlinear Schrödinger equation on simple graphs: the real line, the half-line and star graphs. Those examples will serve as important reference points in the sequel. *Hereunder, we take* $\lambda > 0$.

The affirmations stated in this section are proved in [10, Section 2], see also Section 1.4.3. The arguments are elementary and are based on the fact that nonzero solutions converging to 0 of the differential equation

$$-u'' + \lambda u = |u|^{p-2}u \tag{II.1}$$

are (up to sign) translated portions of the *soliton*[8] ϕ_λ (see Proposition C.2 and Figure II.10) which explicit expression is

$$\phi_\lambda(x) = \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}}x\right)^{\frac{-2}{p-2}}.$$

[8] The term *soliton* comes from the term "solitary wave" and is used in the study of several dispersive equations (see [320, Section 2]). Here, we will only use it to refer to ϕ_λ.

The real line: $\mathcal{G} = \mathbb{R}$

The set of nonzero solutions of $(\mathrm{NLS}_{\mathcal{G}})$ on the real line is $\left\{\pm\phi_\lambda(x+a) \mid a \in \mathbb{R}\right\}$, where ϕ_λ is the soliton. In this case, the set of solutions of the problem is *invariant by*[9] the map $u \mapsto -u$ and *invariant by spatial translations*.

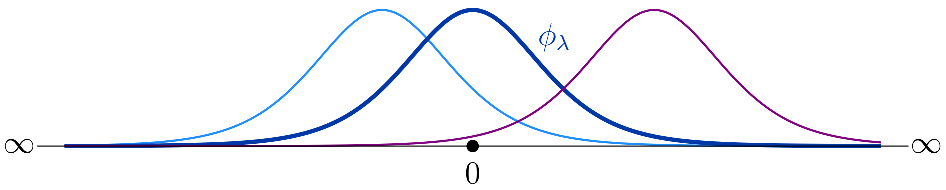


Figure II.10: Three solutions of $(\mathrm{NLS}_{\mathcal{G}})$ on the real line

The half-line: $\mathcal{G} = \mathbb{R}^+ := [0, +\infty)$

On $\mathbb{R}^+$, there are only two opposed solutions, given by half-solitons. In particular, there is no continuous family of solutions.

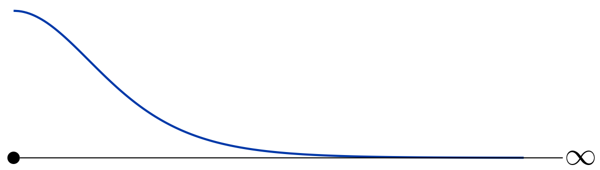


Figure II.11: The positive solution of $(\mathrm{NLS}_{\mathcal{G}})$ on the half-line

Star graphs with an odd number of half-lines

One can show that, on a star graph with an odd number of half-lines, $(\mathrm{NLS}_{\mathcal{G}})$ only has two nonzero solutions. Those solutions are opposed, one of them is positive and is obtained by pasting together half-solitons in their maximum point, as Figure II.12 shows.

Let us remark that Kirchhoff's condition is satisfied for those solutions because all derivatives at the node vanish.

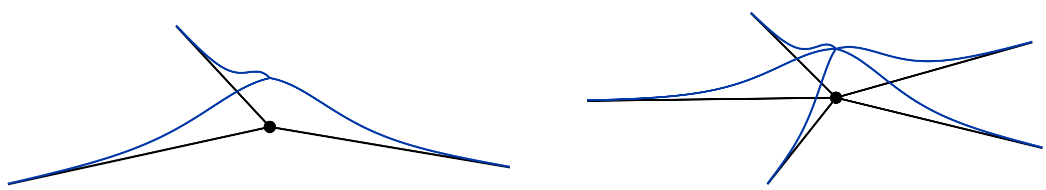


Figure II.12: Positive solutions on the 3-star and the 5-star graphs

[9] Which means that if u is a solution, so is $-u$.

Star graphs with an even number of half-lines

When the number of half-lines is even, it is possible to group them two by two and to put an entire soliton on them, as Figure II.13 shows. Thus, we observe again the presence of continuous families of solutions.

Kirchhoff's condition is satisfied for those solutions since the derivatives at the node cancel each other two by two.

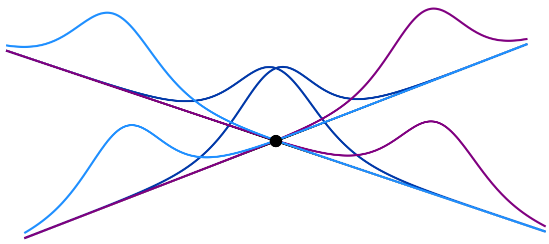


Figure II.13: A continuous family of solutions on the 4-star graph

II.3.2 From Bose-Einstein condensates to atomtronics[10]

Two physical situations, a priori quite different, are described by the nonlinear Schrödinger equation on graphs: the transmission of signals in optical fibers and the study of Bose-Einstein condensates. Let us cite Y.S. Kivshar and G.P. Agrawal in [207, Preface, page xv]:

> In particular, the remarkable similarities between the matter-wave solitons and optical solitons emphasize the intimate connection between classical nonlinear optics and coherent atom optics and may lead to many discoveries in other, seemingly different fields.

One can show that the propagation of signals in some optical fibers leads to consider a focusing nonlinear Schrödinger equation (see [207, Section 1.2.2]). The *cubic* term therein comes from the *Kerr effect*, referring to a nonlinear change in the refraction index of an optical material. The focusing nonlinearity can compensate the dispersive effects. This leads to various notions of *solitons* (spatial, temporal, bright, dark[11], etc.).

[10]For an introduction to physical aspects of the nonlinear Schrödinger equation on metric graphs, one may watch the course of Prof. Riccardo Adami given for the "Expository Quantum Lecture Series 8" at the Institute for Mathematical Research (INSPEM) from the University Putra Malaysia (UPM), see `https://einspem.upm.edu.my/equals8/`. The first video of this series of lectures is available at the address `https://www.youtube.com/watch?v=4ZsIV7i0wgI`.

[11]Those "dark" solitons converge to *nonzero* constants at infinity and thus are not square integrable. We refer to [73, 325] for more information and thank André de Laire for presenting this subject from the mathematical point of view during a seminar at Valenciennes and for highlighting the two aforementioned references.

To know more about the role played by the nonlinear Schrödinger equation in optics, let us cite the books of D.E. Pelinovsky [268], G. Fibich [152] and C. Sulem–P.L. Sulem [317], at the interface between mathematics and their applications. Let us also cite the papers [166, 291] which explain how the nonlinear Schrödinger equation on graphs can model *networks* of fibers.

Now, let us consider Bose-Einstein condensates.

When several identical bosons[12] are cooled down at a temperature very close to absolute zero, they occupy a unique lowest energy quantum state.

In 1925, A. Einstein published [138] in which he predicted this phenomenon (now known under the name of *Bose-Einstein* condensation), relying on works of S.N. Bose [83].[13]

A *macroscopic* quantum phenomenon?

Bose-Einstein condensation is a *macroscopic* phenomenon which can even arise in the presence of many particles, more than 1000 during the first experimental realizations described below.

This seems to contradict the *decoherence* principle, claiming that a "sufficiently large" system loses its quantum properties. This is illustrated by the famous example of *Schrödinger's cat*. Like the authors of [171, Section 12.5][14], we suggest reading [296, Chapter 1, "Introducing Decoherence"] to know more about those aspects.

Regarding Bose-Einstein condensates, let us highlight that this phenomenon arises in a *very low energy* regime, which explains at least partially why we can expect to observe quantum effects even for a relatively large number of particles (see also [296, Section 6.4.1]).

Experimental aspects

Despite the predictions in the 1920s, it is only during the later 1990s that the teams of C. Wieman, E. Cornell and W. Ketterle succeeded in producing Bose-Einstein condensates experimentally. Those three researchers will obtain the Nobel prize in physics 2001 for those works[15].

[12]A *boson* is an integer spin particle. Concerning Bose-Einstein condensation, we will mostly be interested in composite bosons, such as atoms. Particles which are not bosons have an half-integer spin and are *fermions*. These follow *Pauli's exclusion principle*, which claims that two identical fermions cannot occupy the same quantum state. Thus, there does not exist any equivalent of Bose-Einstein condensation for fermions. To know more about bosons, fermions and the exclusion principle, one can for instance consult [171, Section 5.1.1].

[13]Let us signal that a pedagogical video presentation can be consulted at the address `https://toutestquantique.fr/en/bose-einstein-condensate/` (produced by the research group "La Physique Autrement", `https://vulgarisation.fr/`).

[14]Where a discussion about Schrödinger's cat can be found.

[15]See the press release [282] and the presentation [281] from which most of the information presented in the section about experimental aspects comes.

From an experimental standpoint, one of the major challenges is to cool down matter to extremely low temperatures. To this end, Wieman, Cornell and Ketterle notably used advances made in the *laser cooling of atoms*.

In this way, Wiemann and Cornell realized condensation in a gas consisting of rubidium atoms cooled down to approximately 20 nanokelvins[16]. Soon after, Ketterle produced a condensate in a gas of rubidium atoms.

To learn more about the experimental aspects of the realization of condensates, one can refer to the overview article [137] and to the "Nobel Conferences" [110, 203, 333].

Bose-Einstein condensates, N-body quantum systems[17]

Let us denote by Ω the region of $\mathbb{R}^3$ in which the Bose-Einstein condensate is confined and by N the number of bosons. To each boson corresponds a position $x_1, \ldots, x_N \in \Omega$.

The quantum Hamiltonian operator associated with the system writes as

$$H_N = -\Delta + \sum_{1 \leq j \leq N} W(x_j) + \sum_{1 \leq i < j \leq N} V_N(x_i - x_j),$$

where $W : \Omega \to \mathbb{R}$ and $V_N : \mathbb{R}^3 \to \mathbb{R}$. The Hamiltonian represents the energy of the boson system.

The *ground state* of the quantum system is the eigenfunction $\psi_N(x_1, \ldots, x_N)$ of H_N associated with the lowest energy level E_N. In other words, we have

$$H_N \psi_N = E_N \psi_N, \tag{II.2}$$

where $E_N \in \mathbb{R}$ is the first eigenvalue of H_N.

If the system is in the ground state, it means[18] that for every measurable set $S \subseteq \Omega^N$, if we perform a measure of positions, then the probability that the N bosons are such that the vector of positions $(x_1, \ldots, x_N)$ belongs to S is equal to

$$\int_S |\psi_N(x_1, \ldots, x_N)|^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

When the number N of particles present in the system becomes large, Bose-Einstein condensation implies that ψ_N gets close to a *factorized* state:

$$\psi_N(x_1, \ldots, x_N) \approx \varphi(x_1) \cdots \varphi(x_N). \tag{II.3}$$

In a certain way, it means that the system behaves like a unique quantum particle occupying state φ.

[16] By way of comparison, the temperature in outer space is approximately 2.7 kelvins (under the effect of the cosmic microwave background, see [123, Page 3, Figure 1.2], [156]).

[17] This section and the following are strongly inspired by [20, Sections 1.1 and 1.2], that we recommend to the reader interested in aspects related to quantum physics.

[18] See [171, Section 1.2] for more explanations of the statistical interpretation (or "Born's rule") of *states of a quantum system* (also called *wave functions*).

Emergence of a *nonlinear* quantum model?

It turns out (see [20, Section 1.1]) that the common quantum state φ appearing in (II.3) belongs to $H^1(\Omega)$ and is such that the product $N\varphi$ minimizes the *Gross-Pitaevskii* functional

$$E_{GP}(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x + 8\pi\alpha \int_\Omega |u(x)|^4 \,\mathrm{d}x \tag{II.4}$$

under the constraint

$$\int_\Omega |u(x)|^2 \,\mathrm{d}x = N. \tag{II.5}$$

The constant α appearing in (II.4) depends on the intensity of the interaction between particles[19]. We will only be interested in the case[20] $\alpha < 0$, said *focusing*.

One can show that if $u \in H^1(\mathcal{G};\mathbb{R})$ is a critical point of E_{GP} under the constraint $\|u\|^2_{L^2(\Omega)} = N$, then u is a solution of the nonlinear Schrödinger equation[21]

$$-\Delta u + \lambda u = -32\pi\alpha|u|^2 u \tag{II.6}$$

for a certain $\lambda \in \mathbb{R}$. Thus, we aim to determine *normalized solutions* (for the L^2 norm) of nonlinear Schrödinger equations. We will get back to this notion in section II.5 of this introduction.

The presence of the term of degree four in the functional (II.4) makes the Gross-Pitaevskii model *nonlinear*, as can be seen in the equation (II.6). This is quite surprising. Indeed, quantum mechanics[22] is formulated in terms of *linear* equations (see [171, Sections 2.1 and 3.1]). As such, we described in the preceding section the quantum system by a wave function ψ_N, *eigenfunction* of the *linear operator* H_N (see Equation (II.2)).

Nevertheless, we observe a nonlinearity in the expression (II.3), where ψ_N is not linear in φ. The apparition of the functional (II.4) and the cubic nonlinearity in equation (II.6) results from several approximations performed on the N-body quantum system, notably a passage to the *mean field limit* (in the asymptotic regime $N \to \infty$). We will not present its technical details here and refer to [114, 279, 293] for references from the point of view of physics and to[23] [2, 3, 15, 141, 220, 221, 222, 223, 270, 295], among others, for references concerning convergence proofs (see also [20, Section 1.2]).

[19] Only pairwise interactions between particles are taken into account in the analysis.

[20] One may expect the parameter α to be positive in the experiences, which corresponds to a repulsive interaction. This is for instance the case for the rubidium used by Wieman and Cornell. Nevertheless, the value of the constant α can be modulated experimentally thanks to an external magnetic field (phenomenon of "Feshbach resonance"). Thus, it is possible to perform experiments corresponding to the $\alpha < 0$ regime (see e.g. [182]).

[21] At least as far as minima of E_{GP} are concerned, one may replace u with $|u|$ without changing the value of the functional and thus assume that u is real-valued.

[22] At least in the sense of the Schrödinger equation. We will not mention here more modern quantum theories, for instance those of relativistic quantum mechanics.

[23] Many thanks to Prof. Adami for having provided the reference [3].

Ramified structures and atomtronics

In the above, we have described Bose-Einstein condensation in an arbitrary domain Ω. It is relevant to consider the case of a "quasi one-dimensional" domain Ω, made of "quantum wires" (in the spirit of [208]).

Those can be realized experimentally, notably thanks to *Josephson junctions* (see[24] for instance [94, 204, 228, 328]). This leads us to *atomtronics*, a recent research domain aiming to study circuits guiding the propagation of ultracold atoms (see [29]). The general idea is to produce components such as those found in electronic circuits by using (cold) *matter circuits* and, if possible, to realize "purely quantum" effects in those circuits.

Thus, the perspective of atomtronics leads to the study of equation ($\mathrm{NLS}_{\mathcal{G}}$) on metric graphs. Notably, it will be important to understand the role played by the topological and metrical properties of the graphs, which has drawn the attention of several researchers in recent years. For more information about the role of ($\mathrm{NLS}_{\mathcal{G}}$) in physics, one may consult the overview articles [20, 255].

II.3.3 Genesis of ($\mathrm{NLS}_{\mathcal{G}}$)

Let us first mention the pioneering works by F. Ali Mehmeti (for instance his book [25]) and by J. von Below (see e.g.[25] [58]), in which those authors study *semilinear* evolution equations on graphs.

Regarding ($\mathrm{NLS}_{\mathcal{G}}$), even though it is a quite recent subject, the literature is vast. One may for instance consult with interest the overview articles [20, 199, 255].

As seen in the preceding section, the case $p = 4$ (for which the nonlinearity in ($\mathrm{NLS}_{\mathcal{G}}$) is *cubic*) is particularly important from the physics point of view. The first works about ($\mathrm{NLS}_{\mathcal{G}}$) have thus naturally studied this case[26].

One of the first articles which considered the nonlinear Schrödinger (evolution) equation on metric graphs is [56], where there are more generally studied "quantum fields" on star graphs. The modeling of "quantum wires" is a reason to consider the question (see the discussion about atomtronics).

Later, the three works [92], [305] and [7] were published. They study the Schrödinger cubic (evolution) equation on star graphs[27], notably the dynamics of solitons.

[24] We discovered the references [204, 328] thanks to the introduction of [19], which also contains elements of discussion about the physics of condensates.

[25] Many thanks to Prof. Serge Nicaise for having provided this reference.

[26] Let us signal that the case $p = 4$ is convenient in dimension 1 thanks to the *integrability* properties of the evolution equation (see section F.8).

[27] And a few other graphs, see [305, Section IV].

The previous articles highlight a richness in the phenomena as well as the *importance of transmission conditions at the node.* Let us remark that in [7], a "δ-type condition" (or simply δ-condition) is used at the vertex. This condition is more general than Kirchhoff's condition and is written as

$$\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = \alpha u(v), \tag{II.7}$$

where α is a parameter[28]. In the language of partial differential equations, this is essentially a *Robin condition* at the vertex.

Remark. The presence of a term $\alpha \neq 0$ in (II.7) corresponds to a term of *pointwise interaction* (or *pointwise defect*) at the vertex, in other words[29] to a Dirac δ. This is the reason why the "δ-condition " is named as such.

Let us signal that the δ-condition generalizes to other graphs than the star graphs, in which case the coefficient α may vary from one vertex to another (see e.g. [199, Equation (1.5)] and [255, Section 1 (a)]).

II.3.4 Study of $(\mathrm{NLS}_{\mathcal{G}})$ on specific graphs

In this section, we are interested in the study of$(\mathrm{NLS}_{\mathcal{G}})$ on specific graphs.

The results presented hereunder are based, for the most part, on the theory of ordinary differential equations (ODEs).

The intervals, the half-line and the circle

If we consider graphs made of a single edge, we recover the bounded intervals of $\mathbb{R}$, the half-line or even the circle. Thus, the boundary problems coupling the ODE $-u'' + \lambda u = |u|^{p-2}u$ with Neumann boundary conditions[30] or periodic conditions are (simple) instances of $(\mathrm{NLS}_{\mathcal{G}})$.

Compact graphs[31]

Let us cite [292], one of the first works that studies the elliptic problem $(\mathrm{NLS}_{\mathcal{G}})$ (and not the evolution equation). In this article, the graphs under study are "compact stars", made of a central node linked to several nodes of degree one, at which the Dirichlet condition is imposed and not Kirchhoff's one[32].

Let us now move to the study of examples of *noncompact* graphs. We have already encountered some of them in section II.3.1, in which we determined the solutions of $(\mathrm{NLS}_{\mathcal{G}})$ on star graphs.

[28] Let us note that we recover Kirchhoff's condition if we take $\alpha = 0$.

[29] The α term then corresponds to the intensity of this interaction. Let us signal that pointwise defects may be modeled also on the real line. This is the case in [179], an article which served as a source of inspiration for the authors of [7].

[30] Or even the Dirichlet condition, as we will do in Chapter 2.

[31] Let us recall that those are the graphs made of a finite number of edges of finite length.

[32] In other words, the solution vanishes in those nodes and not its derivative.

The “tadpole graph” and flower graphs

The[33] “*tadpole graph*” possesses a single vertex to which are linked a loop of length ℓ and a half-line. It is depicted in Figure II.14.

Figure II.14: The tadpole

The tadpole is the simplest example of noncompact graph having a loop.

Nevertheless, it can offer a few surprises. For instance, let us consider[34] a solution $v_0 : \mathbb{R} \to \mathbb{R}$ of period ℓ to the ODE

$$-v_0'' = |v_0|^{p-2} v_0$$

such that $v_0(0) = 0$. Let us define a function u on the tadpole graph, equal to v_0 on the loop of length ℓ (whose vertex is identified with 0) and vanishing on the half-line. Then, by construction, u is a solution to equation ($\mathrm{NLS}_{\mathcal{G}}$) whose support in $\mathcal{G}$ is compact.

In a spectacular way, this shatters all hope of *uniqueness continuation principle* for solutions of ($\mathrm{NLS}_{\mathcal{G}}$). We will come back to this issue in section II.9.

Up to considering sufficiently oscillating solutions, we can perform the same construction for every $\lambda \in \mathbb{R}$ (even when $\lambda < 0$). We deduce that, unlike the real line for which the ODE $-u'' + \lambda u = |u|^{p-2}u$ does not have any nontrivial solution with $u(x) \xrightarrow[x\to\pm\infty]{} 0$ when $\lambda \leq 0$ (see Proposition C.2), problem ($\mathrm{NLS}_{\mathcal{G}}$) admits solutions for every $\lambda \in \mathbb{R}$. Nevertheless, let us signal that those solutions necessarily vanish on the half-line when $\lambda \leq 0$.

The tadpole graph was used in order to show that solutions of ($\mathrm{NLS}_{\mathcal{G}}$) can vanish on some edges. Detailed studies of problem ($\mathrm{NLS}_{\mathcal{G}}$) on this graph exist. For instance, C. Cacciapuoti, D. Finco and D. Noja classify all solutions when $p = 4$ in [90]. Moreover, D. Noja, D.E. Pelinovsky and G. Shaikhova determine branches of solutions and study their stability in [256]. As for the case $p = 6$, it was studied by Noja and Pelinovsky in [257].

[33] We will use the singular to refer to the tadpole because, from the topological point of view, this is the only graph made of a single vertex to which are linked a loop and a half-line. Nevertheless, let us recall that our graphs are metric and that there exist infinitely many “metric tadpoles” depending on the length of the loop. In practice, no confusion should arise.

[34] It is standard to show that such solutions exist, see Lemma 4.21.

More generally, let us consider "*flower graphs*" having $N \geq 2$ petals[35], made of a single vertex from which emanate a half-line as well as N loops having same length, as can be seen in Figure II.15.

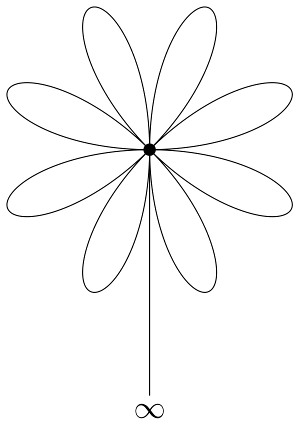


Figure II.15: A flower with eight petals

In [198], A. Kairzhan, R. Marangell, D.E. Pelinovsky and K. Xiao analyze *bifurcation phenomena* on those graphs (when $p = 4$), notably thanks to arguments from the ODE theory (analysis of the *period function*). We can also refer to [199, Section 6.3] for a presentation of their results.

The double-bridge

The "*double-bridge*" is depicted in Figure II.16.

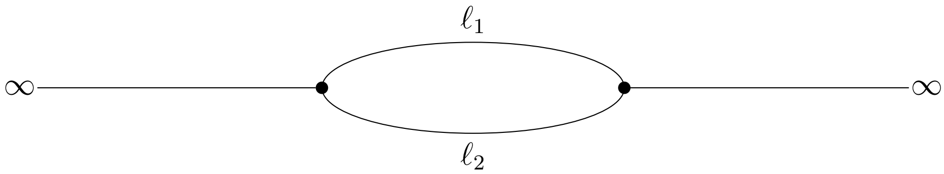


Figure II.16: The double-bridge

In [258], D. Noja, S. Rolando and S. Secchi study solutions of $(\text{NLS}_\mathcal{G})$ on the double-bridge for a cubic nonlinearity. In this case, the solutions of differential equations on the edges are expressed through *Jacobi elliptic functions*[36], which is used in an important way in the reasoning.

[35]In the case $N = 1$, we recover the tadpole graph.

[36]The course of Prof. Diego Noja during the *Nonlinear Quantum Graphs* summer school at Valenciennes taught us a lot about the works about $(\text{NLS}_\mathcal{G})$ in the case $p = 4$. In particular, it is on this occasion that we discovered the Jacobi elliptic functions. Several references presented in this introduction come from this course. To know more about the use of Jacobi elliptic functions, we refer to [209] and to [199, Section 5].

The study happens to be very rich. As it turns out, the description of the set of solutions depends in an important way on the ratio $r := \frac{\ell_1}{\ell_2}$ between the lengths of the two bounded edges. In particular, whether r is rational or not is important.

Even if it is not obvious "visually", there is a big difference between the study of $(\mathrm{NLS}_{\mathcal{G}})$ on the real line and on the double-bridge!

The $\mathcal{T}$-graph

The $\mathcal{T}$-graph is depicted in Figure II.17.

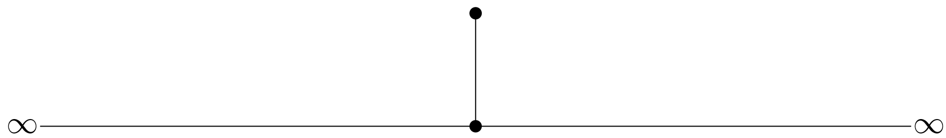


Figure II.17: The $\mathcal{T}$-graph

In [22], the authors completely classify solutions of $(\mathrm{NLS}_{\mathcal{G}})$ on the $\mathcal{T}$-graph and give information about their stability and their variational characterization. We will come back to this discussion in section II.6.4, where the $\mathcal{T}$-graph will be an emblematic example of a noncompact graph admitting *ground states*.

II.3.5 A variational formulation of $(\mathrm{NLS}_{\mathcal{G}})$

In section II.3.1, we showed that we can determine *all* solutions to the nonlinear Schrödinger equation on star graphs. To do so, one should combine the analysis of the ODE on the edges with a "gluing" argument at the vertex.

We just saw that such an analysis was sometimes possible on specific graphs[37]. Nevertheless, it becomes quickly very complex as soon as the number of vertices increases or that bounded edges are present.

In order to study problem $(\mathrm{NLS}_{\mathcal{G}})$ in a more general way, we should use a better suited formulation which is less sensitive to the specifics of the graph or to the value of the exponent.

Thus, we will adopt a *variational approach*, such as the one already encountered in section II.3.2 dedicated to Bose-Einstein condensates where we saw that is was important to minimize the Gross-Pitaveskii functional on the L^2-norm constraint.

Now, let us introduce a Hilbert space and a functional used in the variational approach.

[37]Up to assuming that $p = 4$, see section II.3.4.

The $H^1(\mathcal{G})$ space and the action functional J_λ

On the Sobolev space[38]

$$H^1(\mathcal{G}) := \Big\{u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G})\Big\},$$

we define the *action functional* $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$ by

$$J_\lambda(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \,\mathrm{d}x.$$

Let us remark that the directional derivatives of J_λ are given by

$$J'_\lambda(u)[v] = \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x + \lambda\int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x - \int_{\mathcal{G}} |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x$$

for every couple of functions u and v in $H^1(\mathcal{G})$.

Now, let us show that the critical points of J_λ on $H^1(\mathcal{G})$ are solutions[39] to ($\mathrm{NLS}_{\mathcal{G}}$). The reasoning is classical and analogous to the one relating a partial differential equation and its weak formulation. Nevertheless, we will present the details in order to highlight the natural appearance of Kirchhoff's condition.

The differential equation edge by edge

If φ is a $\mathcal{C}^\infty$ function whose compact support is included inside an edge e joining vertices a and b (see Figure II.18), we have[40]

$$\begin{aligned} 0 &= J'_\lambda(u)[\varphi] \\ &= \int_e u'(x)\varphi'(x)\,\mathrm{d}x + \lambda\int_e u(x)\varphi(x)\,\mathrm{d}x - \int_e |u(x)|^{p-2}u(x)\varphi(x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}u}{\mathrm{d}x_e}(b)\underbrace{\varphi(b)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_e}(a)\underbrace{\varphi(a)}_{=0} \\ &\quad + \int_e \Big(-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x)\Big)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

Therefore, we have

$$\int_e \Big(-u''(x) + \lambda u(x) - |u(x)|^{p-2}u(x)\Big)\varphi(x)\,\mathrm{d}x = 0$$

for every test function φ of class $\mathcal{C}^\infty$ with compact support in e. We deduce that the equation $-u'' + \lambda u = |u|^{p-2}u$ is satisfied inside edge e.

[38] See section A.4 for more details about the structure of the space $H^1(\mathcal{G})$. Since we are working in dimension one, *all the functions of* $H^1(\mathcal{G})$ *are continuous*, as proved in Proposition A.7.

[39] The converse of this claim is true and is proved similarly.

[40] Here, we assume that u is smooth enough in order to justify the integrations by parts. More rigorously, one has to understand the equation as being satisfied in the weak sense and prove an elliptic regularity result. The details are classical and we omit them.

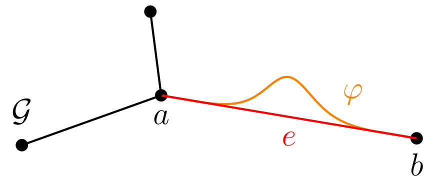


Figure II.18: Test function whose compact support is included inside an edge.

Kirchhoff's condition

Let a be a vertex of degree D of $\mathcal{G}$ and let $b_1, \ldots, b_D$ be the vertices adjacent to a.

We define a function φ, affine on the edges of $\mathcal{G}$ and such that $\varphi(a) = 1$ and $\varphi(v) = 0$ for every vertex $v \neq a$ (see Figure II.19). Denoting e_i the edge joining a and b_i, we obtain

$$\begin{aligned} 0 &= J'_\lambda(u)[\varphi] \\ &= \sum_{1\leq i\leq D} \left(\int_{e_i} u'\varphi' \,\mathrm{d}x + \lambda \int_{e_i} u\varphi \,\mathrm{d}x - \int_{e_i} |u|^{p-2}u\varphi \,\mathrm{d}x \right) \\ &= \sum_{1\leq i\leq D} \left(\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(b_i) \underbrace{\varphi(b_i)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a) \underbrace{\varphi(a)}_{=1} \right) \\ &\quad + \sum_{1\leq i\leq D} \int_{e_i} \underbrace{\left(-u'' + \lambda u - |u|^{p-2}u \right)}_{=0} \varphi(x) \,\mathrm{d}x \end{aligned}$$

so that $\sum_{1\leq i\leq D} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(a_i) = 0$, which is Kirchhoff's condition.

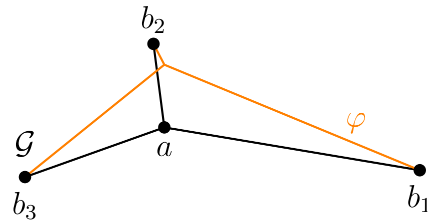


Figure II.19: Affine function used to recover Kirchhoff's condition.

And now?

We just saw that solutions of $(\mathrm{NLS}_\mathcal{G})$ correspond to critical points of the *action functional* $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$, defined by

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},$$

where

$$H^1(\mathcal{G}) = \left\{ u : \mathcal{G} \to \mathbb{R} \;\middle|\; u \text{ is continuous and } u, u' \in L^2(\mathcal{G}) \right\}.$$

Let us note that the action functional J_λ is not bounded from below on $H^1(\mathcal{G})$. Indeed, if $u \neq 0$, then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t\to\infty]{} -\infty.$$

A possible strategy to find critical points of J_λ consists in studying *constrained minimization problems* in order to obtain solutions of ($\mathrm{NLS}_\mathcal{G}$) which are minima of J_λ under the constraint. We will present a way to carry out this process in the following section II.4.

II.3.6 A few words about numerical simulations

When one wants to study in detail a specific example of graph, the computations can turn out to be very complex. Therefore, numerical simulations are a precious tool.

On this subject, let us cite the Python library "GraFiDi[41]" developed by C. Besse, R. Duboscq and S. Le Coz (see [71, 72]) as well as the MATLAB package "QGLAB[42] " developed by R. Goodman[43], G. Conte and J. Marzuola (see [169]).

We will also use numerical tools in Chapter 5 in which we will develop a *computer-assisted proof* (see section II.11.5).

II.4 (Nodal) action ground states

II.4.1 Nehari manifold and action ground states

Definitions and first properties

Let us consider the *Nehari manifold* associated to ($\mathrm{NLS}_\mathcal{G}$) and defined by

$$\begin{aligned}\mathcal{N}_\lambda(\mathcal{G}) &:= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \;\middle|\; J'_\lambda(u)u = 0\right\}\\ &= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \;\middle|\; \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})}\right\}.\end{aligned}$$

The Nehari manifold[44] contains all nonzero critical points of J_λ and one can show that a critical point of J_λ constrained to $\mathcal{N}_\lambda(\mathcal{G})$ is a solution to ($\mathrm{NLS}_\mathcal{G}$) (see e.g. [35, Remark 2.3.13]).

[41] Available at the address `https://plmlab.math.cnrs.fr/cbesse/grafidi`.

[42] Available at `https://github.com/manroygood/Quantum-Graphs/tree/master`.

[43] Whom we thank again for his course given during the *Nonlinear Quantum Graphs* summer school. The reader can consult the support used during the lectures at the address `https://roygoodman.net/course/nqg_valenciennes/`.

[44] One can show that the Nehari manifold is a $\mathcal{C}^2$ Hilbert manifold modeled on $H^1(\mathcal{G})$. In general, we will not need to use such considerations. We refer to [28, Chapter 6] to know more about the point of view of constraints as manifolds.

According to the definition of $\mathcal{N}_\lambda(\mathcal{G})$, we remark that

$$u \in \mathcal{N}_\lambda(\mathcal{G}) \quad \Longrightarrow \quad J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^p_{L^p(\mathcal{G})}.$$

In particular, J_λ is positive on $\mathcal{N}_\lambda$, thus bounded from below. We define

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u).$$

An *action ground state* for (NLS$_{\mathcal{G}}$) is a function $u \in \mathcal{N}_\lambda(\mathcal{G})$ such that

$$J_\lambda(u) = \mathcal{J}_{\mathcal{G}}(\lambda).$$

If an action ground state exists, one can show that it is a constant sign solution of (NLS$_{\mathcal{G}}$), see e.g. [35, Remark 2.3.13] or [318, Corollary 10, (c)] for a proof. Let us remark that if $\mathcal{G}$ is not compact, the existence of action ground states is not guaranteed in general.

Even though we just introduced the Nehari manifold and the notion of action ground state on graphs, one can follow the same approach in many situations: differential equations, elliptic equations on open sets of $\mathbb{R}^N$, systems of elliptic equations, etc. The founding article of Z. Nehari [250] studies an ODE problem. The reader willing to be introduced to the subject can in particular consult [35, Section 2.3] or the overview article [318].

A few works studying action ground states on graphs

Up to now, the action ground states for (NLS$_{\mathcal{G}}$) have not been thoroughly studied. Indeed, most works are concerned with normalized solutions (see section II.5).

Nevertheless, let us mention [10, 11, 126, 211, 218, 263, 302], which consider the action functional and the Nehari manifold on graphs.

In [10, 11], considering action ground states (on star graphs, with the δ-condition) allows us to use the method to study orbital stability of solutions developed by M.I. Weinstein and M. Grillakis–J. Shatah–W. Strauss (see [172, 173, 330, 331] and section F.7).

The article [218] shows that some arguments developed for normalized solutions on graphs (see section II.5) adapt to the case of the Nehari manifold.

The phenomenon of *concentration* of action ground states is studied in the articles [211, 302]. We will return to this in section II.7.

The article [263] studies (NLS$_{\mathcal{G}}$) on periodic graphs using techniques based on action ground states. We will consider similar problems in section 2.5.1.

Finally, in [126], S. Dovetta studies the asymptotic behavior of action ground states on an entire grid (see Figure II.4 of page 78) whose lengths converge to 0. The author proves convergence results between the action ground state on grids (which are metric graphs) and the soliton of the partial differential equation (NLS). It is thus a manifestation of the "dimensional reduction" phenomenon by which we introduced the model ($\text{NLS}_\mathcal{G}$) in section II.3.

Chapters 1 and 2 of this thesis study the notion of action ground state on graphs in detail (both on compact and on noncompact ones), by providing especially existence and non-existence theorems. In Chapter 3, we will study the L^2 masses of action ground states on bounded domains of $\mathbb{R}^N$ in order to obtained normalized solutions.

II.4.2 Nodal Nehari set and nodal ground states

The action ground states that were presented in the previous section are *constant sign* solutions of ($\text{NLS}_\mathcal{G}$). Now, let us consider sign-changing solutions, also called *nodal solutions*.

Given a real-valued function u, we define

$$u^+ := \max(u, 0) \quad \text{and} \quad u^- := \min(u, 0).$$

A *nodal solution* is, by definition, a solution u of ($\text{NLS}_\mathcal{G}$) such that $u^+ \neq 0$ and $u^- \neq 0$. All nodal solutions of ($\text{NLS}_\mathcal{G}$) belong to *the nodal Nehari set*[45]

$$\mathcal{N}_\lambda^{nod}(\mathcal{G}) := \Big\{u \in H^1(\mathcal{G}) \;\Big|\; u^\pm \in \mathcal{N}_\lambda(\mathcal{G})\Big\} = \Big\{u \in H^1(\mathcal{G}) \;\Big|\; u^\pm \neq 0,\ J_\lambda'(u)u^\pm = 0\Big\}.$$

A function $u \in \mathcal{N}_\lambda^{nod}(\mathcal{G})$ is a *nodal ground state* of ($\text{NLS}_\mathcal{G}$) if

$$J_\lambda(u) = \inf_{v \in \mathcal{N}_\lambda^{nod}(\mathcal{G})} J_\lambda(v).$$

When they exist, nodal ground states are sign-changing solutions to ($\text{NLS}_\mathcal{G}$) (see e.g. [318, Proof of Theorem 18]). More precisely, they are *minimal action nodal solutions* of the problem.

The founding article studying the minimization method on the nodal Nehari set is the one of A. Castro, J. Cossio et J.M. Neuberger [93]. For a more detailed overview, we refer to [53, 318] and to chapters 2 and 3.

To the best of our knowledge, nodal ground states had never been studied on graphs before the works presented in this thesis.

[45] In contrast to the Nehari manifold, the nodal Nehari set $\mathcal{N}_\lambda^{nod}$ is in general not endowed with a manifold structure. See [53, Introduction and Lemma 3.2] for a discussion and the presentation of a strategy to address this, using the H^2 norm and not the H^1 norm.

Nodal ground states will play a big role in chapter 2 (where we will study their existence on noncompact graphs) as well as in chapter 3 (where we will study their L^2 masses on open sets of $\mathbb{R}^N$ in order to obtain normalized nodal solutions). They will also be considered in chapter 5 in which we will determine their qualitative properties in the case of compact graphs.

II.5 Normalized solutions[46]

II.5.1 What are normalized solutions?

A *normalized* solution to ($\mathrm{NLS}_\mathcal{G}$) is a solution whose L^2-norm is prescribed but where the value of the parameter λ is not. This notion is important when studying certain physical models (see section II.3.2). For instance, when modeling a Bose-Einstein condensate, we have seen in section II.3.2 that the L^2-norm corresponds to the number of particles in the described system (see constraint (II.5)).

Seeking normalized solutions is also useful when studying evolution equations (see Appendix F and in particular section F.7).

For the variational viewpoint, normalized solutions correspond to critical points of the *energy functional*[47]

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})},$$

on the *mass constraint* $\|u\|^2_{L^2} = \mu$, μ being the mass[48]. In this case, the parameter λ shows up as a Lagrange multiplier associated to the constraint.

On a metric graph $\mathcal{G}$, given $\mu > 0$, one can show that the infimum

$$\inf_{\substack{u\in H^1(\mathcal{G})\\ \|u\|^2_{L^2}=\mu}} E(u)$$

is finite when $2 < p < 6$ and is equal to $-\infty$ when $p > 6$ (see for example [20, Section 2.1]).[49]. The *mass-critical exponent* of the problem appears here: it is equal to $p = 6$.

[46]Many thanks to Prof. Louis Jeanjean for the discussion about the history of the study of normalized solutions.

[47]Already encountered in the case $p = 4$ of section II.3.2, where it was called the *Gross-Pitaevskii functional* E_{GP} and defined by (II.4). Let us recall that the parameter α in (II.4) is negative.

[48]We will sometimes use other normalization conventions "up to a multiplicative constant", for instance in section II.10.3 as well as in chapter 3.

[49]When $p = 6$, the functional E is bounded from below on the mass constraint if and only if μ is small enough.

More generally, if Ω is a bounded open set in $\mathbb{R}^N$, the Gagliardo–Nirenberg inequality (see e.g. [332])

$$\|u\|_{L^p}^p \leq K_p \|u\|_{L^2}^{p-N\left(\frac{p}{2}-1\right)} \|\nabla u\|_{L^2}^{N\left(\frac{p}{2}-1\right)}, \qquad \forall u \in H^1(\mathbb{R}^N),$$

which holds for all[50] $p \in (2, 2^*)$, implies that the energy functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x$$

is bounded from below on the mass constraint when $p < 2 + \frac{4}{N}$. This is not the case when $p > 2 + \frac{4}{N}$ (see [183]). Thus, the value $2 + \frac{4}{N}$ is the *mass-critical exponent* in dimension N. Let us note that we find again $p = 6$ in dimension $N = 1$.

The mass-critical exponent also plays an important role in the study of the nonlinear Schrödinger evolution equation (see sections F.3 to F.5).

When the energy is bounded from below on the constraint, we call *energy ground state* a minimum of the energy under the mass constraint. We then show that this is necessarily a constant sign solution to (NLS). In all cases, the notion of *minimal energy solution*, namely a solution minimizing the energy functional among the set of solutions of (NLS) having mass μ (for a certain λ that can differ between solutions), is available. We will return to these notions in section II.10.

II.5.2 Genesis of normalized solutions[51]

Bifurcation from the essential spectrum

In [316], C.A. Stuart studies bifurcation phenomena. Let us illustrate this concept on an example, by considering the differential equation

$$-u''(x) - \frac{|u(x)|^{p-2}u(x)}{x} = \lambda u(x) \tag{II.8}$$

where $x \in (0, +\infty)$, $p \in (2, +\infty)$ and $\lambda \in \mathbb{R}$. We define the Hilbert space

$$\tilde{H} := H^2(0, +\infty) \cap H_0^1(0, +\infty),$$

endowed with the H^2 norm, and the set

$$E := \left\{(u, \lambda) \in \tilde{H} \times \mathbb{R} \;\middle|\; u \text{ is a solution of (II.8)}, u \neq 0\right\}.$$

Following Stuart, we say that $\lambda \in \mathbb{R}$ is a *bifurcation point* if $(0, \lambda)$ belongs to the adherence of E in $\tilde{H} \times \mathbb{R}$.

[50] Let us recall that 2^* is the critical Sobolev exponent, equal to $+\infty$ when the dimension N is equal to 1 or 2, and equal to $2N/(N-2)$ when $N \geq 3$.

[51] Many thanks to Prof Charles A. Stuart for sharing his work on the subject and for clarifying its chronology. Reading [185, Section 4, Solutions de normes prescrites] was also very useful in writing this section.

When [316] was published in the "Comptes Rendus de l'Académie des Sciences de Paris" in 1977, such bifurcation phenomena were studied in the literature, but only starting from eigenvalues (see e.g. [285] and references in [316] for more information).

Those do not always exist, notably on noncompact domains. For instance, on $[0, +\infty)$, the operator $v \mapsto -v''$ with the Dirichlet condition at the origin has $[0, +\infty)$ as its spectrum but has no eigenvalues.

Thus, the originality in Stuart's approach consists in studying the phenomenon of *bifurcation from the essential spectrum*. The first results obtained in this way are presented in the paper [104] by R. Chiappinelli and C.A. Stuart.

Now, let us see what the link is between the phenomenon of bifurcation from the essential spectrum and the search for normalized solutions.

From bifurcation problems to normalized solutions

The works [311, 312, 313, 314, 315] by Stuart highlight a relation between the bifurcation phenomenon and normalized solutions: *if a problem admits energy ground states, whose energy is negative, for every L^2-mass small enough, then* 0 *is a bifurcation point.* The reader can refer to the aforementioned references (see notably [313, Theorem 2.1] and [312, Theorem 3.3]).

In those papers, techniques to prove existence of energy ground states are developed.

P.L. Lions will later obtain in [224, 226, 227] his results on concentration-compactness, particularly well-suited to study the problem of energy minimization under the mass constraint.

II.5.3 Normalized solutions on star graphs in the mass-subcritical regime ($2 < p < 6$)

After their first works [7, 10], R. Adami, C. Cacciapuoti, D. Finco and D. Noja continued their study of solutions on star graphs.

The variational characterization of the stationary solution (see Figure II.12 at page 87) on the 3-star graph with Kirchhoff's condition and a cubic nonlinearity is clarified in [8]. It appears that the solution is a *saddle point* of the energy on the mass constraint, and not a minimum.

Let us mention [6], devoted to the existence of normalized solutions and the study of their orbital stability on star graphs. This work improves the results of [10] and leads Adami, Cacciapuoti, Finco and Noja to adapt the concentration-compactness methods of Lions [224, 226, 227] to the case of star graphs. Let us mention that the stability of those solutions was subsequently described precisely by A. Kairzhan in [197], relying on *Sturm theory* techniques.

To conclude, let us highlight that in [9], the authors prove that stationary solutions on star graphs, *in presence of an attractive δ-interaction at the vertex*, are always *local*[52] minima of the energy on the mass constraint. This remarkable result implies orbital stability of those solutions.

The aforementioned works highlight a perhaps unexpected mathematical depth. And this, *only on star graphs*!

Questions naturally arise. *What happens on graphs whose structure is more complex than star graphs? Is there a richness of phenomena in this case, even in the absence of the δ-interaction at the vertices?*

We have already partially answered those questions during our studies of $(\mathrm{NLS}_{\mathcal{G}})$ on specific graphs in section II.3.4. We will return to them in section II.6 devoted to *noncompact* domains.

There also exist pertinent questions on compact graphs, as we will see in the following section.

II.5.4 Normalized solutions on compact graphs

If $\mathcal{G}$ is a compact graph of total length $|\mathcal{G}|$ and if μ is a positive real number, then the constant function $u : \mathcal{G} \to \mathbb{R}$ equal to $c := \sqrt{\frac{\mu}{\mathcal{G}}}$ at every point of the graph is such that $\|u\|_{L^2}^2 = \mu$ and is a solution of $(\mathrm{NLS}_{\mathcal{G}})$ if we take $\lambda = c^{p-2}$. Up to sign, it is the only constant solution of the problem.

Two questions arise. *Are there other solutions than the constant one? Is the constant solution an energy ground state?*

The first question is answered positively in all cases. Indeed, there always exist infinitely many solutions when $2 < p < 6$, as shown in [124] (where the author also considers the critical case $p = 6$).

The answer to the second question generally depends on the parameters. An analysis of variational properties of the constant solution and its orbital stability is performed in [89] for the cases $p < 6$ and $p = 6$. A more precise study of this second question was performed by J.L. Marzuola and D.E. Pelinovsky in [231] on the "*dumbbell*" graph shown below in Figure II.20.

Figure II.20: The dumbbell

[52]Let us recall that there are no energy ground states on star graphs with $N \geq 3$ half-lines, which means that the energy has no global minima under the mass constraint.

These two authors consider *bifurcation phenomena* for solutions on the dumbbell thanks to methods closer to the works mentioned in section II.3.4. This allows us to determine in which case the constant solution of ($\mathrm{NLS}_{\mathcal{G}}$) is a ground state. This approach on the dumbbell was pursued by R. Goodman in [168]. The reader may refer to [199, Section 6.3] for a presentation of the result obtained in the aforementioned works.

Remark. It is not surprising that a dumbbell shape is considered in order to study the symmetry breaking of ground states. Indeed, "fattened" versions (similarly to Figure II.8 on page 81) of the dumbbell graph are classical examples of open domains in $\mathbb{R}^N$ which illustrate this phenomenon for superlinear elliptic partial differential equations, see e.g. [118] and [175, Page 18, Figure 3].

Remark. On compact graphs, the constant solution is present in all three regimes $p < 6$, $p = 6$ and $p > 6$. If we develop a result showing existence of solutions, we should make sure to find non-constant solutions, under the risk of obtaining a useless result.

To conclude, let us signal that compact graphs provide a convenient setting in which one can study *qualitative properties* or *uniqueness* of certain types of solutions to ($\mathrm{NLS}_{\mathcal{G}}$). One can for instance see the role played by Dirichlet vertices, or the phenomenon of solutions vanishing identically on edges (see section II.9). Those questions will be at the heart of Chapter 5.

II.5.5 A few important concepts

In the sequel of section II.5, we will consider the mass-supercritical regime, in which the energy is not bounded from below on the mass constraint anymore.[53] We will present the works on $\mathbb{R}^N$ and several "abstract" methods. We will then come back to the case of bounded domains and metric graphs, which will be studied in chapters 3 and 4.

Before going any further, we have to introduce several classical concepts in the theory of semilinear elliptic partial differential equations. We have followed [136, Chapter 1] and [115] to present them.

Stability, Morse indices, non-degeneracy

Given a solution[54] $u \in H_0^1(\Omega)$ to (NLS), namely a critical point of J_λ on $H_0^1(\Omega)$, we consider the quadratic form Q_u defined by

$$Q_u(\varphi) := J''_\lambda(u)[\varphi, \varphi] = \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x + \lambda \int_\Omega \varphi^2 \,\mathrm{d}x - (p-1) \int_\Omega |u|^{p-2} \varphi^2 \,\mathrm{d}x.$$

[53]In this case, it is no longer possible to study a constrained minimization problem in order to find normalized solutions.

[54]We chose to present the concepts for the problem set on a domain $\Omega \subseteq \mathbb{R}^N$ with the Dirichlet boundary condition. The definitions can be adapted, *mutatis mutandis*, to the other frameworks: problems set on $\mathbb{R}^N$, on metric graphs, etc.

We then say that the solution u:

- is *stable* if $Q_u(\varphi) \geq 0$ for all[55] $\varphi \in \mathcal{C}_c^1(\Omega)$ ([136, Definition 1.1.2]);
- is *stable outside the compact set* $K \subset \Omega$ if $Q_u(\varphi) \geq 0$ for all $\varphi \in \mathcal{C}_c^1(\Omega \setminus K)$ ([136, Definition 1.5.1]);
- has *Morse index* k if k is the maximal dimension of a vector subspace V of $\mathcal{C}_c^1(\Omega)$ such that $Q_u(\varphi) < 0$ for all $\varphi \in V \setminus \{0\}$ ([136, Definition 1.5.2]);
- is *nondegenerate* if 0 is not an eigenvalue of the linearized operator around u. This operator[56] $L_u : H_0^1(\Omega) \to H^{-1}(\Omega)$ is associated to the quadratic form Q_u in the sense that
$$\Big\langle L_u[\varphi] \mid \psi \Big\rangle := \frac{1}{2} Q_u'(\varphi)[\psi]$$
for every function $\psi \in H_0^1(\Omega)$. More concretely, this means that
$$L_u[\varphi] = -\Delta\varphi + \lambda\varphi - (p-1)|u|^{p-2}\varphi,$$
see e.g. [115, (3.21), page 94].

The notions of stability and of Morse index (as well as the one of *approximate Morse index*) will play a big role in Chapter 4. The one of nondegenerate solution will be very important in Chapter 5.

To know more about the role played by the notions of stability and of Morse index in the study of semilinear elliptic equations, one may for instance consult [41, 115, 136, 147].

Palais-Smale sequences and condition

In infinite dimension, compactness is often lacking. Thus, before obtaining critical points of functionals, we will often need to consider *sequences of approximate critical points.*

Thus, we say[57]that a sequence $(u_n)_n \subseteq H_0^1(\Omega)$ is a *Palais-Smale sequence at level* $c \in \mathbb{R}$ for the functional J_λ if
$$J_\lambda(u_n) \xrightarrow[n\to\infty]{} c \quad \text{and} \quad J_\lambda'(u_n) \xrightarrow[n\to\infty]{H^{-1}(\Omega)} 0.$$

Moreover, we say[58] that J_λ satisfies the *Palais-Smale condition at level* c if all Palais-Smale sequences at level c possess a convergent subsequence.

This notion turned out to be very fruitful in nonlinear analysis. It admits several generalisations. We refer to [234] for more information.

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[55]There is some latitude in the choice of the regularity to impose to functions φ, that we could for instance assume to be $\mathcal{C}_c^\infty$.

[56]As in [85, page 291], we will note $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$. Even if $H_0^1(\Omega)$ is a Hilbert space, we will not identify it to its dual, see [85, Chapter 5, Remark 3].
We will note $\langle T \mid \psi\rangle := T[\psi]$ the *duality bracket* between those spaces (see [85, Notation, page 3]).

[57]Definition taken from [335, Introduction].

[58]Definition taken again from [335, Introduction].

Mountain pass geometry

In section II.4, we have seen that the action functional J_λ is not bounded from below on H^1. We have also presented a way to treat this issue by introducing the Nehari manifold $\mathcal{N}_\lambda$ and by considering the minimization problem of J_λ on $\mathcal{N}_\lambda$.

Let us illustrate here an alternative method. It will show how to exploit the *mountain pass geometry* of the functional. A similar method will play an important role in the following sections in which we will study the mass-supercritical regime.

Let us return for a moment to the geometry of the action functional. First, let us look "direction by direction". If $u \in H^1 \setminus \{0\}$, then there exists a unique $t_u > 0$ such that $t_u u$ belongs to $\mathcal{N}_\lambda$. The number t_u is characterized by

$$J_\lambda(t_u u) = \max_{t>0} J_\lambda(tu).$$

Thus, the situation looks like the one illustrated by Figure II.21. In this picture, the H^1 space corresponds to the horizontal plane and the action levels correspond to the vertical axis.

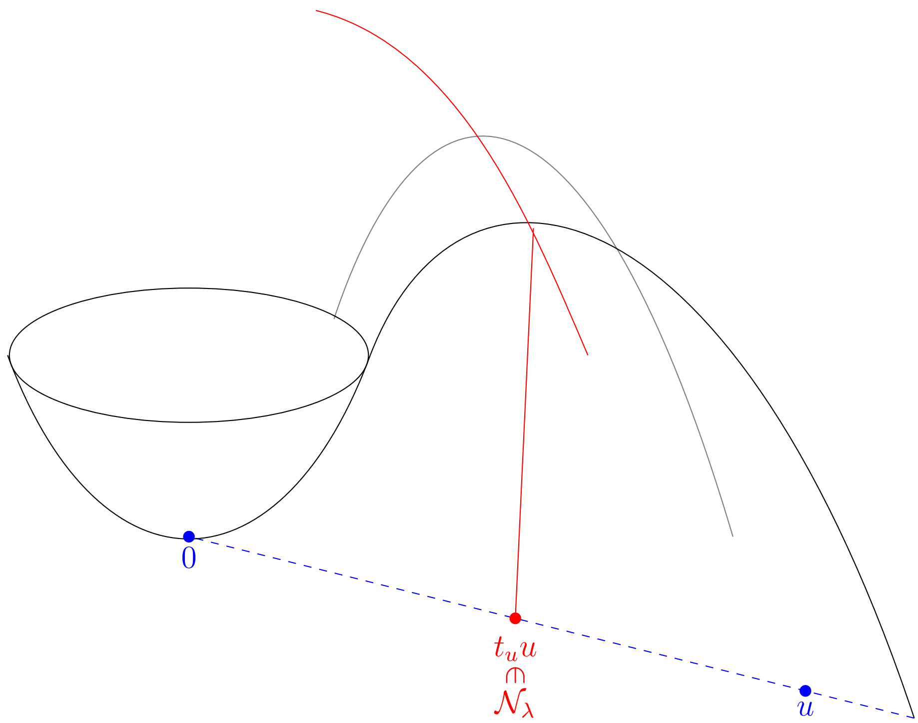

Figure II.21: Geometrical representation of the Nehari manifold[59]

On this figure, we can see that minimizing the functional J_λ on the Nehari manifold $\mathcal{N}_\lambda$ amounts to find the "mountain pass" of the "mountain" $\mathcal{N}_\lambda$.

[59]Many thanks to Prof. Christophe Troestler who supplied this figure.

Intuitively, looking for the mountain pass boils down to understand how to go from the origin to "the other side of the mountain" (that we will characterize as the region where the action is negative) while "increasing as little as possible the action levels".

This leads us to consider the *class of paths*

$$\Gamma := \left\{ \gamma \in \mathcal{C}([0,1], H^1) \;\middle|\; \gamma(0) = 0, J_\lambda(\gamma(1)) < 0 \right\}$$

as well as the *mountain pass action level*

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

The presence of a mountain pass geometry is traduced by the inequality $c > 0$: "to cross the mountain, it is necessary to move to higher action levels".

It implies the existence of a Palais-Smale sequence at level c for J_λ (which corresponds intuitively to the saddle point at the mountain pass level). We refer to [214, Section 3], [35, Chapter 4] and [335, Section 1.3] for more information.

For the action functional J_λ, this approach is equivalent to the minimization on the Nehari manifold (see [214, Proposition 3.12]).

The use of the mountain pass geometry can be adapted to many situations in which one does not necessarily have an equivalent of the Nehari manifold. We will observe this in the subsequent sections.

II.5.6 The L^2-supercritical regime on $\mathbb{R}^N$

Existence of a normalized solution of (NLS)

The first article devoted to normalized solutions of (NLS) in the mass-supercritical regime ($p > 2 + \frac{4}{N}$) is the one of L. Jeanjean [183].

The situation there is very different from the mass-subcritical regime. Indeed, in this case, the energy is no more bounded from below on the L^2-mass constraint. Then, one has to search for the critical point by exploiting the mountain pass geometry that the energy has on the mass constraint.

As we have seen in the previous section, the presence of a mountain pass geometry allows to obtain the existence of a Palais-Smale sequence. Nevertheless, a problem arises[60]: *it is not clear that such a sequence is bounded in* $H^1(\mathbb{R}^N)$. Once that we have shown that the sequence is bounded, we may assume that it converges weakly[61] and try to show that the weak limit is a normalized solution.

[60] This problem does not occur when searching for critical points on the Nehari manifold or energy ground states in the mass-subcritical regime.

[61] By taking a subsequence.

The strategy used to show that the Palais-Smale sequence is bounded is based on the *Pohožaev identity*[62] that solutions of (NLS) on $\mathbb{R}^N$ satisfy:

$$(N-2)\|\nabla u\|_2^2 + \lambda N \|u\|_2^2 = \frac{2N}{p}\|u\|_p^p. \tag{II.9}$$

One can for instance consult [183, Section 2.2], which presents details on how to use the Pohožaev identity to show that Palais-Smale sequences are bounded.

After having solved this technical difficulty, Jeanjean proves that equation (NLS) admits a normalized solution of mass μ on $\mathbb{R}^N$ for every mass $\mu > 0$, in the mass-supercritical regime.

Infinitely many normalized solutions to (NLS)

In [51], T. Bartsch and S. de Valeriola show the existence of *infinitely many radial normalized solutions* in the L^2-supercritical case[63].

To this end, they used the *Mountain pass theorem*, in a rather general version, for variational problems invariant by the map $u \mapsto -u$. Let us note that in [51], the proof that Palais-Smale sequences are bounded relies on the Pohožaev identity in the same way as in [183, Section 2.2].

Extensions: more general autonomous equations, systems, problems with potential

The article [183] treats *autonomous* equations[64] more general than (NLS)[65]. This is possible because the Pohožaev identity is well-suited to those cases on $\mathbb{R}^N$.

One may consult [189, 191, 192] for later works and [27, 187, 188] for extensions involving the Sobolev critical exponent 2^*.

Other approaches than the one of [183] may be used to prove the existence of normalized solutions. Let us cite for instance the article [194], where the authors show the existence and study the L^2-masses of branches of solutions in the space of radial H^1 functions.

One may also study L^2-normalized solutions of *systems* of elliptic equations using variational techniques, see e.g. [42, 43, 44, 48, 49, 50, 170, 195]. All these works use Pohožaev identities in an important way.

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[62] Proved originally by S. Pohožaev in [280]. For a proof in the case of $\mathbb{R}^N$, see for example [335, Theorem B.3].

[63] When $\lambda > 0$ is fixed, the infinite multiplicity of radial solutions is proved in [66].

[64] Those of the form $-\Delta u = f(u)$ and not $-\Delta u = f(x,u)$.

[65] On $\mathbb{R}^N$, if a "power-type" nonlinearity is used as in (NLS), we can explicitly express the L^2-mass of action ground states as a function of λ and obtain the same results showing the existence of normalized solutions. In Appendix C, we will present similar computations for the soliton on the real line, see Proposition C.4. If the nonlinearity is modified or if the domain changes, this argument cannot be applied anymore and Jeanjean's method is necessary.

In [46, 245], the authors study generalizations of (NLS) where a *potential* term $V(x)$ is added, which makes the equation

$$-\Delta u + V(x)u + \lambda u = |u|^{p-2}u$$

non-autonomous. The associated Pohožaev identity is[66]

$$\begin{aligned}(N-2)\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \lambda N \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x \\ = \frac{2N}{p}\int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x + 2\int_{\mathbb{R}^N} (x\cdot\nabla u)V(x)u \, \mathrm{d}x. \qquad \text{(II.10)}\end{aligned}$$

In order to show the existence of normalized solutions, one has to impose conditions on the potential, including enough integrability of $V(x)$ and of $x \cdot V(x)$ in order to treat the last term in (II.10).

The works mentioned up to now use a Pohožaev identity in order to establish the boundedness of Palais-Smale sequences as well as their convergence. This can sometimes be restrictive and imposes, for instance, restrictions on the potential in [46, 245].

In order to pursue the study of normalized solutions in the regime $p > 2 + \frac{4}{N}$, we are led to search for methods which are not based on a Pohožaev identity. In the following section, we will present such methods devoted to the existence of bounded Palais-Smale sequences.

II.5.7 "Monotonicity trick", existence of bounded Palais-Smale sequences and Morse indices

The monotonicity trick[67]

In [308, 309], M. Struwe is also confronted with the problem of existence of bounded Palais-Smale sequences, in problems coming from geometry and from the study of Hamiltonian systems. He develops an argument now known as the "*monotonicity trick* " (see also [310, Chapter II, Section 9]).

In [186], Jeanjean shows that the techniques previously developed by Struwe can be generalized in an "abstract" version. Let us cite his result.

[66] At least formally: see [155, Lemma 1.1] where one can find the expression taken by the Pohožaev identity of a non-autonomous equation on a bounded domain and to [335, Appendix B, Section 3] to move from the case of bounded domains to $\mathbb{R}^N$. We will not try to rigorously justify the identity (II.10) and refer to [46, 245] where one can see how identities such as (II.10) play a role in the proofs.

[67] Reading [185, Section 5, Suites de Palais-Smale bornées] was very useful in writing this section. We recommend it to the (French speaking) reader willing to learn more about the question of boundedness of Palais-Smale sequences.

Theorem ([186, Theorem 1.1])**.** *Let $(X, \|\cdot\|)$ be a Banach space.*

Let us consider an interval $I \subset (0, +\infty)$ and a family $\Phi_\rho \colon X \to \mathbb{R}$ of $\mathcal{C}^1$ functionals of the form

$$\Phi_\rho(u) := A(u) - \rho B(u) \quad \textit{where } \rho \in I.$$

We assume that $B(u) \geq 0$ for all $u \in X$ and that

$$A(u) \xrightarrow[\|u\|\to+\infty]{} +\infty \quad \textit{or} \quad B(u) \xrightarrow[\|u\|\to+\infty]{} +\infty.$$

Given $v_0, v_1 \in X$, we define

$$\Gamma := \Big\{\gamma \in \mathcal{C}([0,1], X) \;\Big|\; \gamma(0) = v_0, \gamma(1) = v_1\Big\}.$$

We assume that, for all $\rho \in I$, the number

$$c_\rho := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\rho(\gamma(t)),$$

satisfies the inequality $c_\rho > \max\{\Phi_\rho(v_0), \Phi_\rho(v_1)\}$.

Then, for almost every $\rho \in I$, there exists a bounded Palais-Smale sequence $(u_n)_{n\geq1} \subseteq X$ at level c_ρ for Φ_ρ. In other words:

(i) $(u_n)_n$ *is bounded;*

(ii) $\Phi_\rho(u_n) \to c_\rho$ *as* $n \to \infty$*;*

(iii) $\Phi'_\rho(u_n) \to 0$ *in the dual of* X *as* $n \to \infty$*.*

The proof of this result uses the "monotonicity trick": since $I \subset (0, +\infty)$ and since $B(u) \geq 0$ for all $u \in X$, the map $I \to \mathbb{R} : \rho \mapsto c_\rho$ is nondecreasing. It is thus differentiable almost everywhere. Jeanjean then shows that, when c'_ρ exists, then the functional Φ_ρ possesses a bounded Palais-Smale sequence at level c_ρ (see [186, Section 2]).

In [193], L. Jeanjean and J.F. Toland generalize this result to situations where the map $\rho \mapsto c_\rho$ is not monotonous.

The existence result of bounded Palais-Smale sequences is particularly useful and has found many applications: the existence of solutions to a "Landesman-Lazer type" problem in [186], extension to the L^2-supercritical regime of the bifurcation results studied by Stuart in[68] [160, 184], etc.

We are faced with several questions.

[68]See also [185, Section 5, Retour sur les problèmes de bifurcation].

Given a bounded Palais-Smale sequence, how to prove that it converges? Can one find bounded Palais-Smale sequences for constrained problems[69]*? Can one show the existence of such sequences at various levels in order to obtain multiplicity results?*

As we will see, possessing a "Morse-index type" information about elements of a Palais-Smale sequence is a precious tool in order to prove the convergence of this sequence.

Information on the (approximate) Morse indices

A Palais-Smale sequence is made of *approximate* critical points which are not associated a priori with a notion of Morse index. Nevertheless, G. Fang and N. Ghoussoub have shown in [145, 146] that it is possible to define a notion of *approximate Morse index* and to show the existence of Palais-Smale sequences with information on the approximate Morse index of elements.

This can be useful to study convergence questions, as observed in [39, 41, 225] by A. Bahri and P.L. Lions. In those articles, the authors highlight the role that information on the Morse indices can play in nonlinear elliptic equations, notably when compactness is lacking.

Let us note that Fang and Ghoussoub's works concern Palais-Smale sequences for *unconstrained* variational problems.

In [81], J. Borthwick, X. Chang, L. Jeanjean and N. Soave have shown that there exist *bounded* Palais-Smale sequences, *with information on the approximate Morse index*, in the case of perturbed functionals *constrained on a sphere* (see [81, Theorem 1.5]) having a mountain pass geometry. Thus, the authors refine the monotonicity trick in order to obtain also information on the approximate Morse indices of the elements of the Palais-Smale sequence, as done by Fang and Ghoussoub.

In [81, Theorem 1.12], the authors also prove a "generalized" theorem allowing one to obtain multiplicity results for even functionals[70] (in the spirit of the method used on $\mathbb{R}^N$ by Bartsch and de Valeriola [51]). This theorem will be crucial in Chapter 4.

We refer to the introduction of [81] and also to [139] to learn more about the existence of Palais-Smale sequences with information on the Morse indices of the elements and how to use such results.

[69] The theorem presented above concerns *free* variational problems, which means that the paths in Γ take their values in X and are not constrained to a submanifold of X.

[70] This is the case of the energy functional.

II.5.8 Normalized solutions on graphs when $6 < p$

As we have seen, there is no equivalent on graphs to the arguments used on $\mathbb{R}^N$, which are based on Pohožaev's identity. The same will generally apply in the following section, in which we will study normalized solutions on bounded domains.

Remark. Nevertheless, let us signal that Pohožaev's identity is convenient to use on *star-shaped* domains (as we will see in Chapter 3). For instance, in dimension $N \geq 3$, it implies that there is no nonzero solution to (NLS) when p is greater than or equal to $2^* := \frac{2N}{N-2}$ on star-shaped domains (see e.g. [310, Chapter III, Section 1]).

Therefore, the results on the existence of bounded Palais-Smale sequence with information on the approximate Morse indices reveal their full importance on graphs and bounded domains. In fact, it is the study of normalized solutions on graphs that led Borthwick, Chang, Jeanjean and Soave to develop [81].

When $p > 6$, those results allow one to obtain a non-constant normalized solution when the mass is sufficiently small on compact graphs, see [102]. For the problem with localized nonlinearity, we obtain a normalized solution for all masses, see [82].

In [82, 102], after having shown that a bounded Palais-Smale sequence exists, one needs to show that the associated sequence of "almost Lagrange multipliers" converges to a positive real number. This requires to exclude[71] the convergence of almost multipliers to infinity as well as their convergence to zero[72].

As announced in the previous section, we will extend the result of [82] by showing that the problem with localized nonlinearity possesses *infinitely many* normalized solutions. To this end, we will also have to develop an argument excluding the case $\lambda = 0$. It is more delicate to treat when solutions may change sign due to the possible presence of solutions vanishing identically on edges of the graph (see section II.9). This argument is the object of section 4.3 (see in particular Proposition 4.23).

Concerning normalized solutions on graphs in the $p > 6$ regime, let us also mention [31]. In this article, A.H. Ardila studies normalized solutions on graphs in the mass-supercritical regime, but where the presence of a potential changes the geometry of the energy functional, which allows the author to find solutions by minimization under the mass constraint.

To the best of our knowledge, to this day there exists only one paper which treats the mass-supercritical case on *noncompact* graphs: the one of S. Dovetta, L. Jeanjean and E. Serra [128]. We will return to it in section II.6.

[71] Thanks to a *blow-up analysis* using the information on the Morse indices.
[72] Which is rather easy when seeking *positive* solutions as in [82, 102].

II.5.9 Normalized solutions on bounded domains in the mass-supercritical regime

In [259], B. Noris, H. Tavares and G. Verzini study the existence as well as the orbital stability of normalized solutions of (NLS) on the unit ball with the Dirichlet condition. The uniqueness of the positive solution of (NLS) (for a given λ) plays a big role in this work, which increases the difficulty of a generalization of the results to other domains, where uniqueness is only rarely known.

The study of normalized solutions on bounded domains is futher developed in[73] the articles [273, 274].

Let us also cite [269], which proves existence results of *concentrated* normalized solutions. We will return to those in section II.7.

Let us already signal that in section II.10, we will present a new method to study normalized solutions on bounded domains, including in the mass-supercritical regime. This is the subject of Chapter 3.

II.6 Role of topological and metrical properties of domains

The two most classical settings in which one may study an elliptic equation such as (NLS) are bounded domains and the whole space $\mathbb{R}^N$.

On a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$, there is some compactness in functional spaces. Namely, the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact according to Rellich-Kondrachov's Theorem (see e.g. [85, Theorem 9.16]).

In contrast, we will consider hereunder the case of unbounded domains and to the study of equation ($\text{NLS}_\mathcal{G}$) on noncompact metric graphs. In those cases, the lack of compactness will be one of the challenges to overcome.

Hereunder, we will also mention other sources of noncompactness than those of the domain. Thus, if $\Omega \subseteq \mathbb{R}^N$ is a domain in dimension $N \geq 3$, the injection of $H^1(\Omega)$ into $L^{2^*}(\Omega)$ is not compact, even if Ω is bounded (see [85, Corollary 9.14 and page 286, Remark 14]).

As we have seen at the beginning of the preceding section, the exponent 2^* does not only play a "technical" role and equation (NLS) does not posses any nonzero solutions when $p \geq 2^*$ on star-shaped domains.

[73] Let us signal that the proof of the existence result presented in [273] does not show how to check that the obtained Palais-Smale sequences are bounded. This was commented by the authors, see [274, top of page 3] where this step is detailed.

Now, let us formulate a few remarks.

- On graphs, we do not encounter compactness issues related to a critical Sobolev exponent because graphs are one-dimensional.
- The critical Sobolev exponent naturally appears in some questions coming from Riemannian geometry, for instance in the *Yamabe problem* [336] where one asks if every manifold can be endowed with a Riemannian metric whose scalar curvature is constant. The overview article [216] contains more information on this problem, its history and its resolution.
- The reader interested in a more exhaustive presentation of various sources of noncompactness that may appear in variational problems may refer to [310, Chapter III].

First of all, let us consider the most natural noncompact domain: the whole Euclidean space.

II.6.1 Entire solutions on $\mathbb{R}^N$

We sometimes say that solutions defined on $\mathbb{R}^N$ are *entire*. The literature is very vast (see e.g. the book [212]).

If we study an *autonomous* equation on $\mathbb{R}^N$, namely one of the form

$$-\Delta u = f(u),$$

then the problem is *invariant under translations*, which allows to recover some compactness as illustrated by the Lemma [219, Lemma 6] and its application[74] in [86].

We will also use the translation invariance during our study of *periodic graphs*, which will be presented in section II.6.5.

The role of noncompact group actions in losses of compactness was substantially clarified by the works of P.L. Lions on the principle of *concentration-compactness* [224, 226, 227]. For an introduction to the subject and to its newer developments, we refer to [217].

For autonomous equations on $\mathbb{R}^N$, one can also look for *radial* solutions, which also brings compactness, see e.g. [65, 299, 307].

The two aforementioned techniques (use of translations or search for radial solutions) do not apply to *non-autonomous* equations of the form $-\Delta u = f(x, u)$.

This is for instance the case when a *potential* is present in the equation, like for equations of the type $-\Delta u + V(x)u + \lambda u = |u|^{p-2}u$.

Although the presence of the potential $V(x)$ makes the equation nonautonomous, it can sometimes bring compactness to it, see e.g. [47].

[74] See also [157, Section 3], in which we found the reference [219].

In this thesis, we will mostly focus on the role of the *domain* (and its boundary conditions) and not on the *equation*.

Now, let us consider the "most simple" cases of noncompact domains which are not $\mathbb{R}^N$: the half-spaces.

II.6.2 The half-line and half-spaces

In dimension one, a *half-space* is simply a half-line, for instance $[0, +\infty)$. On it, we study solutions of the ODE, $-u'' + \lambda u = |u|^{p-2}u$ which converge to 0 at infinity.

If we impose the Neumann boundary condition at $x = 0$ (namely $u'(0) = 0$), we have seen in section II.3.1 that the ODE has two nonzero solutions, which are opposed and given by half-solitons $\pm\phi_\lambda(x)_{|\mathbb{R}^+}$.

If we impose the Dirichlet boundary condition at $x = 0$ (namely $u(0) = 0$), the problem does not admit any nonzero solution (see Proposition C.2).

This shows that the boundary condition may play a crucial role.

As we will see in Chapter 2, the presence or not of Dirichlet vertices will be important in the existence result of (nodal) action ground states on metric graphs.

More generally, we may consider a half-space in dimension N, for instance $\mathcal{H} := [0, +\infty) \times \mathbb{R}^{N-1}$.

If we impose the Dirichlet boundary condition at the boundary of $\mathcal{H}$, then the Theorem [143, Theorem I.1] implies that (NLS) has no solution in $H_0^1(\mathcal{H})$.

The situation is very different if a *non-homogeneous* Dirichlet condition $u = c$ is imposed on the boundary of $\mathcal{H}$. In this case, the number of solutions crucially depends on the value of the constant c, see [150].

II.6.3 Domains with a richer geometry

The geometry of the domain plays a role in compactness questions, as we will see in the following situations.

- The seminal papers by J.M. Coron and A. Bahri [36, 37, 111] show that, on certain domains with a non-trivial topology (those which, intuitively, have "holes", see Figure II.22), the equation (NLS) may admit solutions even when p is equal to the critical Sobolev exponent[75] 2^*. Let us note that in the Sobolev-supercritical case, the presence of a non-trivial topology does not always suffice to obtain the existence of solutions, as proved by D. Passaseo [265].

[75] None of those domains is star-shaped due to the presence of holes. This explains why the obstruction of existence of solutions to (NLS) obtained by Pohožaev does not apply.

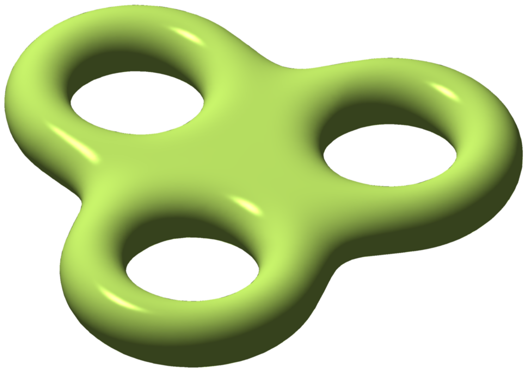


Figure II.22: The "triple torus[76]", a domain with non-trivial topology in $\mathbb{R}^3$

- The work of Bahri and Coron led to several later developments for domains admitting "small holes" (see an illustration in dimension two in Figure II.23). Thus:
  - V. Benci and G. Cerami [61, 63] show links between the domain topology (in particular, notions related to the "number of holes") and the multiplicity of positive solutions of (NLS);
  - O. Rey [287, 288] obtains multiplicity results in function of the number of holes when $p = 2^*$;
  - A. Bahri, Y. Li and O. Rey [38] study concentration points of solutions in a quasi Sobolev-critical regime;
  - M. del Pino and J. Wei [276] show the existence of solutions for certain $p > 2^*$ in such domains;
  - O. Rey et A. Pistoia [278] construct solutions developing several peaks in the regime $p > 2^*$;
  - etc.

Figure II.23: A bounded domain with small holes in $\mathbb{R}^2$

[76]Image from https://upload.wikimedia.org/wikipedia/commons/f/f0/Triple_torus_illustration.png, public domain.

- Several authors have studied solutions of (NLS) on *exterior domains* (the open sets in $\mathbb{R}^N$ whose complement is compact). Let us mention, among others, A. Bahri, V. Benci, G. Cerami, M. Clapp, P.L. Lions, R. Molle, D. Passaseo, etc. Their works show that equations of (NLS)-type:
  - may admit solutions if the "hole" is small enough ([62, Theorem B]) or in the presence of certain potential terms ([40]);
  - admit nodal solutions in the presence of enough symmetries ([100]);
  - admit multiple solutions in function of the number of "holes" ([101]);
  - etc.

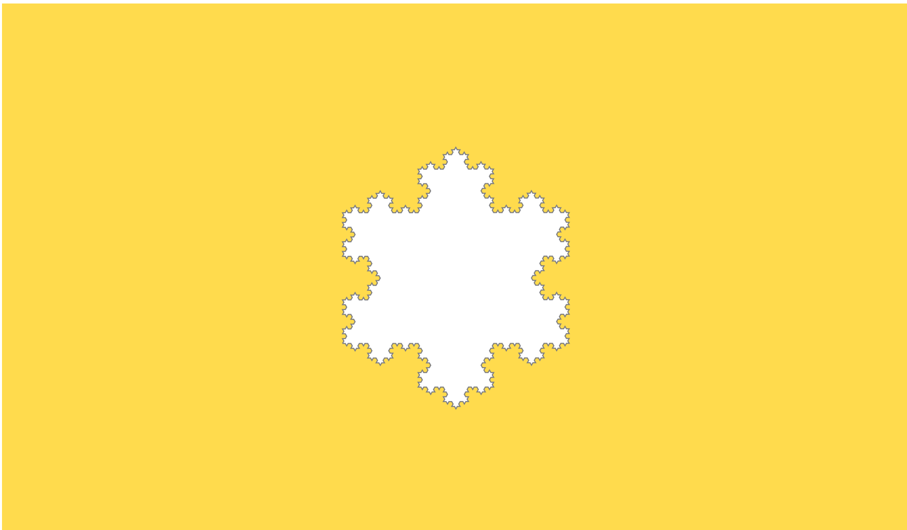


Figure II.24: An exterior domain[77]

- In [275], M.A. del Pino and P.L. Felmer consider domains of $\mathbb{R}^2$ of the form

  $$\Big\{(t,x) \in \mathbb{R}^2 \mid -f(t) < x < f(t)\Big\},$$

  where f is a positive $\mathcal{C}^\infty$ function on $\mathbb{R}$ converging to 0 at infinity.

  Even though such domains are noncompact, the authors prove the existence of positive solutions of (NLS) on these domains as well as on some of their generalizations, including in dimension $N \geq 3$.
- R. Molle has studied in [243, 244] domains $\Omega \subset \mathbb{R}^N$ such that neither Ω nor $\mathbb{R}^N \setminus \Omega$ are bounded.
- M.J. Esteban and P.L. Lions obtain, in [143], non-existence results for nonzero solutions on certain noncompact domains.
- etc.

[77]Picture realized thanks to the TikZ library `decorations.fractals`, using explanations from `https://latex.org/forum/viewtopic.php?t=17902`.

As illustrated by the aforementioned works, certain *geometrical properties* of domains *may, or may not, bring compactness.* We will observe that the same applies to metric graphs, which provide an excellent framework to study in more depth this phenomenon.

II.6.4 (Non-)existence of ground states on graphs with a finite number of edges when $2 < p < 6$

First study on bridge-type graphs

First of all, let us present the article [16] in which R. Adami, E. Serra and P. Tilli prove that there is no energy ground states when $2 < p < 6$ on *bridge-type graphs* such as the one represented in Figure II.25 (another example is the double-bridge represented by Figure II.16 of page 95).

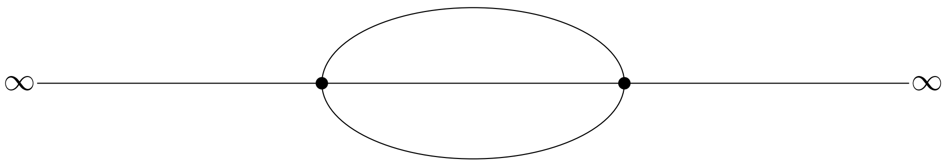


Figure II.25: The triple-bridge

If the number of "bridges" between the two vertices is odd, the non-existence result is based on a technique consisting in "unfolding" functions defined on graphs in order to obtain functions on the real line. Using a comparison of levels between the N-bridge and the $(N-1)$-bridge, the general case follows (see the proof of [16, Theorem 1.2]).

This shows that adding bounded edges to the real line can completely change the situation. Indeed, on $\mathbb{R}$, energy ground states exist for every mass while for bridge-type graphs, they never exist!

The considerations mentioned above are greatly clarified and generalized in the important article [19], "NLS ground states on graphs", of the three authors cited earlier.

In this article, the focus is put on *the graph on which the equation is set* and Adami, Serra and Tilli aim to *develop arguments applying to large classes of graphs.*

Now, let us present several results from the article [19] and let us observe the link between them and the works of this thesis.

Decreasing rearrangement, number of preimages and hypothesis (H)

The real line and the half-line are two "extreme" cases, for which the level of the energy ground state under the mass constraint is maximized or minimized, as claimed by the theorem hereunder (obtained in [19, Theorem 2.2] for graphs with finitely many edges, but true in full generality).

Theorem II.1 ([19, Theorem 2.2]). *Let $p \in (2,6)$ and $\mu > 0$ be two real numbers. If a metric graph $\mathcal{G}$ contains at least one half-line, then*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) \leq \inf_{u \in H^1_\mu(\mathbb{R})} E_{\mathbb{R}}(u) = E_{\mathbb{R}}(\widehat{\phi}_\mu), \tag{II.11}$$

where $\widehat{\phi}_\mu$ is the soliton of mass μ (see Definition C.6). Moreover,[a]*, we have*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) \geq \inf_{u \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(u) = \frac{1}{2} E_{\mathbb{R}}(\widehat{\phi}_{2\mu}). \tag{II.12}$$

[a] Let us note that an energy ground state of mass μ on $[0,+\infty)$ is obtained by "cutting $\widehat{\phi}_{2\mu}$ in half at its maximum point" (see Proposition C.9).

Given the importance, in the two first chapters, of arguments used in the proof of this theorem, we decided to present its main ideas.

Elements of the proof of Theorem II.1. The inequality (II.11) can be easily proved. Indeed, on every graph which has at least one half-line, we can consider "quasi solitons $\widehat{\phi}_\mu$". More precisely, for every $\varepsilon > 0$, there exists $v_\varepsilon \in H^1_\mu(\mathcal{G})$ such that $\|\widehat{\phi}_\mu - v_\varepsilon\|_{H^1} \leq \varepsilon$ and such that v_ε has its compact support included in a half-line of $\mathcal{G}$ (see Figure II.26).

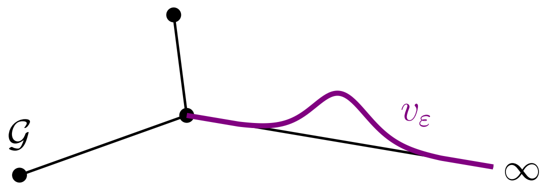


Figure II.26: A truncated soliton on a half-line

The inequality (II.12) is based on an argument of *decreasing rearrangement.* Given $u \in H^1_\mu(\mathcal{G})$, we want to show that

$$E(u) \geq \inf_{v \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(v).$$

Up to replacing u by $|u|$, we may assume that u takes nonnegative values.

Let us consider $u^* : [0, +\infty) \to \mathbb{R}$, the *decreasing rearrangement* of u. In an intuitive way[78], this amounts to "cut the image of u into vertical slices" and to replace them on a half-line, by decreasing order of height. This process is represented in the figure hereunder.

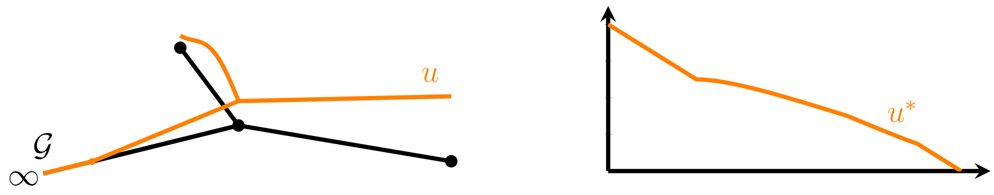


Figure II.27: A function $u : \mathcal{G} \to [0, +\infty)$ and its decreasing rearrangement u^*.

Thus, for every $t > 0$, we have[79]

$$\lambda_{\mathcal{G}}\Big(\{x \in \mathcal{G}, u(x) > t\}\Big) = \lambda_{\mathbb{R}^+}\Big(\{x \in (0, |\mathcal{G}|), u^*(x) > t\}\Big). \tag{II.13}$$

According to (II.13), we have $\|u^*\|_{L^2(0,+\infty)} = \|u\|_{L^2(\mathcal{G})}$ and $\|u^*\|_{L^p(0,+\infty)} = \|u\|_{L^p(\mathcal{G})}$. Moreover, the *Pólya-Szegő inequality* (Lemma B.25) claims that

$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \|u'\|_{L^2(\mathcal{G})}. \tag{II.14}$$

Thus, we obtain

$$\begin{aligned} E(u) &= \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} \\ &\geq \frac{1}{2}\|(u^*)'\|^2_{L^2(0,+\infty)} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} \\ &\geq \inf_{v \in H^1_\mu(0,+\infty)} E_{[0,+\infty)}(v), \end{aligned}$$

which ends the proof of inequality (II.12). □

In dimension one, it turns out that we may (under conditions) refine the Pólya-Szegő inequality (II.14). More precisely, if a function $u : \mathcal{G} \to [0, +\infty)$ is such that

$$\# u^{-1}(\{t\}) \geq N \tag{II.15}$$

for almost every t in its image (where $N \geq 1$ is an integer), then its decreasing rearrangement u^* satisfies

$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \frac{1}{N}\|u'\|_{L^2(\mathcal{G})}. \tag{II.16}$$

In this way, *we improved the Pólya-Szegő inequality by a factor* N *thanks to the hypothesis* (II.15) *on the number of preimages!*

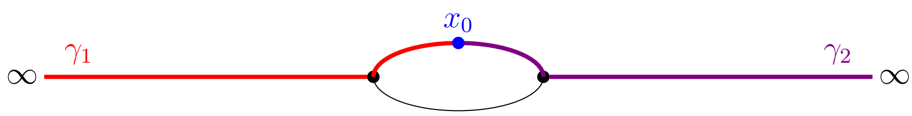
Some graphs are such that *all positive continuous functions satisfy condition* (II.15) *with* $N = 2$. This leads us to the following definition.

[78]Those heuristic explanations are based on [20, Section 5.1]. We refer to Appendix B for rigorous definitions and the proofs of the announced properties.

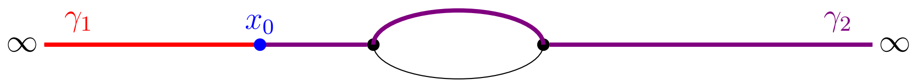
[79]Where $\lambda_{\mathcal{G}}$ and $\lambda_{\mathbb{R}}$ refer respectively to the Lebesgue measures on $\mathcal{G}$ and $\mathbb{R}$, see Section A.3.

Definition. A metric graph $\mathcal{G}$ *satisfies assumption*[80](H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ parametrized by arclength, having disjoint images except for an at most countable number of points and such that $\gamma_1(0) = \gamma_2(0) = x_0$.

Example. Bridge-type graphs satisfy assumption (H). For instance, if x_0 lies on the "top" edge of the graph, we may consider the two curves γ_1 and γ_2 represented in the figure hereunder:



Similarly, if x_0 lies on the "left" half-line, we consider the curves γ_1 and γ_2 as follows:



The graphs which satisfy assumption (H) do not generally hold energy ground states, as claimed by the following statement (see [19, Theorems 2.2, 2.3, 2.5]).

Theorem II.2. *Let $p \in (2, 6)$ and $\mu > 0$ be two real numbers. If a metric graph $\mathcal{G}$ satisfies* (H)*, then*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) = \inf_{u \in H^1_\mu(\mathbb{R})} E_{\mathbb{R}}(u) = E_{\mathbb{R}}(\widehat{\phi}_\mu). \tag{II.17}$$

Moreover, the infimum

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u)$$

is not attained, which means that there are no energy ground states of mass μ, unless $\mathcal{G}$ is isometric to the real line or to a "tower of loops" graph, see Figures II.28, II.29 and II.30.

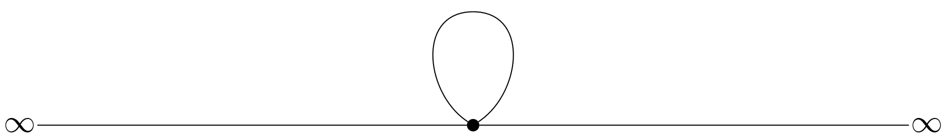


Figure II.28: A tower with one loop

[80] Called (H') in [19]. It turns out that the assumption (H') is equivalent to assumption (H), see [19, Lemma 5.1] (which proves one of the implications of the equivalence).

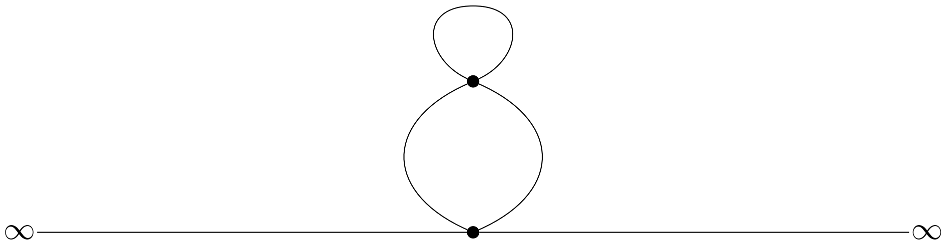


Figure II.29: A tower with two loops

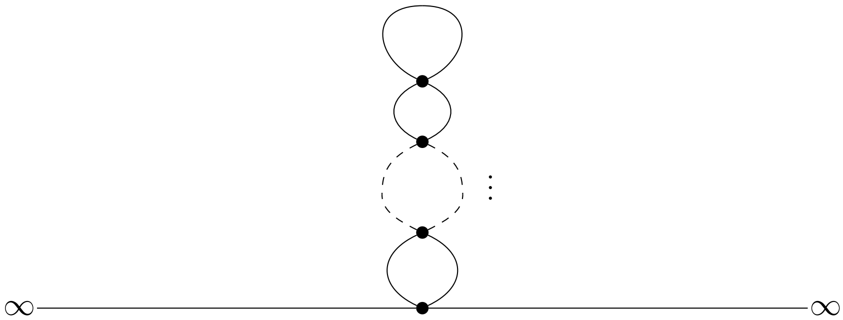


Figure II.30: A tower of loops (general case)

Since bridge-type graphs satisfy assumption (H), the previous theorem allows us to recover the non-existence results mentioned previously.

To conclude the presentation of [19], let us also signal that Adami, Serra and Tilli study the $\mathcal{T}$-graph (see example (a) from Figure II.31 on the following page) and show that it holds energy ground states for all values of the mass (see [19, Theorems 2.6 and 2.7]). This result follows from the inequality

$$\inf_{u \in H^1_\mu(\mathcal{T})} E_{\mathcal{T}}(u) < E_{\mathbb{R}}(\widehat{\phi}_\mu),$$

proved thanks to constructions based on rearrangement techniques and adapted to the structure of the $\mathcal{T}$-graph (see [19, Lemma 6.1]). Let us remark that the $\mathcal{T}$-graph does not satisfy assumption (H).

The study is continued in the paper [21], where one can notably find the next result (see [21, Theorem 3.3]).

Theorem II.3. *Let $\mathcal{G}$ be a metric graph made from a finite number of edges, including at least one half-line. Let $p \in (2,6)$ and $\mu > 0$ be two real numbers. If the inequality*

$$\inf_{u \in H^1_\mu(\mathcal{G})} E_{\mathcal{G}}(u) < E_{\mathbb{R}}(\widehat{\phi}_\mu),$$

is satisfied, then there exists an energy ground state on $\mathcal{G}$.

The authors also prove, using constructions based on techniques such as the decreasing rearrangement process, that some examples of graphs satisfying the assumption of the previous Theorem II.3 (see Figure II.31).

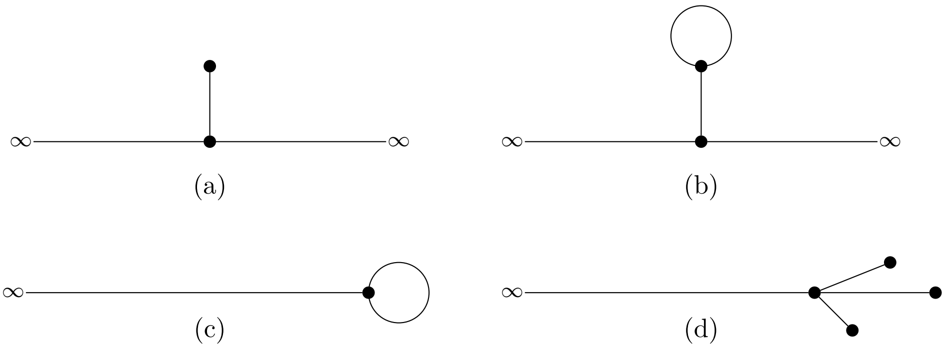


Figure II.31: Examples of graphs admitting energy ground states.
(a) the $\mathcal{T}$-graph; (b) the signpost; (c) the tadpole; (d) the 3-fork

For a presentation of the works of Adami, Serra and Tilli, we recommend the overview articles [1, 20].

The different themes encountered up to now (comparisons with the line and the half-line, rearrangement arguments, existence vs non-existence of ground states, etc.) will also be at the heart of Chapters 1 and 2.

II.6.5 Periodic graphs

Among the simplest examples of periodic graphs are infinite grids such as the one depicted in Figure II.4 of page 78. Such grids (and some of their generalizations) have been studied in [12, 13, 14].

Those works have highlighted the phenomenon of "*dimensional crossover*". It is expressed by the coexistence of one-dimensional phenomena, related to the *local* structure of graphs, and of phenomena typical of higher dimensions, related to the *global* structure of graphs.

For instance, the grid in Figure II.4 is, in a certain sense, 2-dimensional. When studying the existence of energy ground states on it, the critical exponent

$$p_{\text{crit},\, N=1} := 2 + \frac{4}{1} = 6$$

plays a role, as can be expected. Remarkably, there appears a *second* particular exponent, given by

$$p_{\text{crit},\, N=2} := 2 + \frac{4}{2} = 4.$$

For a precise statement about the *dimensional crossover*, we refer for instance to [14, Theorem 1.2]. The persistence of this phenomenon under perturbations of the grid (that is, the erasing of some edges) is studied in [134].

S. Dovetta has made several remarkable contributions to the study of periodic graphs[81]. Thus, in [125, Theorem 1.1], he proves that all $\mathbb{Z}$-periodic graphs[82] admit an energy ground state when $2 < p < 6$. The critical case is more complex and we refer to [125, Theorems 1.2 and 1.3] for more information about it.

Remarque. The invariance of a periodic graph under the action of its group of "translations" plays a big role in its compactness properties, as we will see in section 2.5.1. Therein, we will prove that all periodic graphs possess an action ground state for all $p > 2$ and all $\lambda > 0$, which shows that action ground states are not subject to the phenomenon of dimensional crossover.

II.6.6 Other works

Many other works study problem ($\mathrm{NLS}_{\mathcal{G}}$) and its variants (for a recent overview of the literature, see [199]). They address:

- uniqueness and multiplicity of ground states ([131]);
- infinite trees ([130]);
- problems with several sources of nonlinearity ([4, 77, 78, 271]);
- the mass-critical case $p = 6$ ([18, 272]);
- etc.

II.6.7 Contributions to the subject developed in this thesis

As we have already mentioned, the two first chapters fit within the spirit of the works of Adami, Serra and Tilli. Even though we will study the notions of action ground state and nodal ground state but not the one of energy ground state, many arguments are inspired by the results mentioned before. Thus,

- Theorem 2.3 adapts Theorem II.3 to the study of action ground states (and their nodal equivalents), under an "abstract" form allowing us to study various families of graphs in a unified way. We apply this method to several classes of graphs and obtain in particular the existence of action ground states for the graphs represented in Figure II.31. Regarding *periodic graphs* and *infinite trees*, Theorems 2.7 and 2.8 fully characterize the cases where ground states and nodal ground states exist;
- we introduce *topological conditions* generalizing assumption (H) and we show that they imply *non-existence of action ground states and of nodal ground states* (Theorem 2.6).

[81] For a more detailed presentation of periodic graphs, see the beginning of section 2.5.1 in Chapter 2. We refer to [68, Definition 4.1.1] for a precise definition.

[82] Namely those made of copies of a compact graph "arranged in a line" (we refer to [125] for more details). Therefore, those graphs are both locally and globally uni-dimensional, which explains why we do not observe a dimensional crossover phenomenon.

II.6.8 Normalized solutions on noncompact graphs in the case $p > 6$

To the best of our knowledge, the only[83] work studying normalized solutions of ($\mathrm{NLS}_{\mathcal{G}}$) on noncompact graphs in the mass-supercritical regime is the one of Dovetta, Jeanjean and Serra [128].

Therein, the authors prove the existence of *positive* solutions, having a *positive energy*, for graphs having a finite number of edges and to which are attached at least one edge ending by a degree-one node (as for the $\mathcal{T}$-graph, see Figure II.31 (a)) or a "signpost" (see Figure II.31 (b)), *assuming that the mass is small enough.*

They also prove the existence of solutions of positive energy on periodic graphs, assuming that the mass is large enough.

Those proofs use constructions based both on the mountain pass theorem and the monotonicity trick (see sections II.5.6 and II.5.7), but also arguments specific to noncompact metric graphs.

Thus, for graphs having a finite number of edges, the authors compare the energy levels of functions defined on the graph with the energy levels of the soliton and the half-soliton with the same mass (see [128, Section 3]), extending in a certain way the approach used in the case $2 < p < 6$. Nevertheless, the case $p > 6$ is much more delicate.

II.7 Concentrated solutions

II.7.1 Concentration in the regime $\lambda \to +\infty$

The unique nonzero H^1 solution to the equation (NLS) on the real line, up to a sign and up to translations, is the *soliton*

$$\phi_{\lambda,p}(x) := \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-2}{p-2}}, \tag{II.18}$$

as proved in Appendix C.

The expression above implies that:

- the L^∞ norm of the soliton converges to infinity as $\lambda \to +\infty$;
- for every real $x \neq 0$, we have $\phi_{\lambda,p}(x) \xrightarrow[\lambda \to +\infty]{} 0$.

In Figure II.32 hereunder, we observe that the solitons $\phi_{\lambda,3}$ "*concentrate*" as λ increases.

[83] Let us also signal the paper [31] (see the bottom of the page 114), but whose concerns are different due to the presence of a potential term.

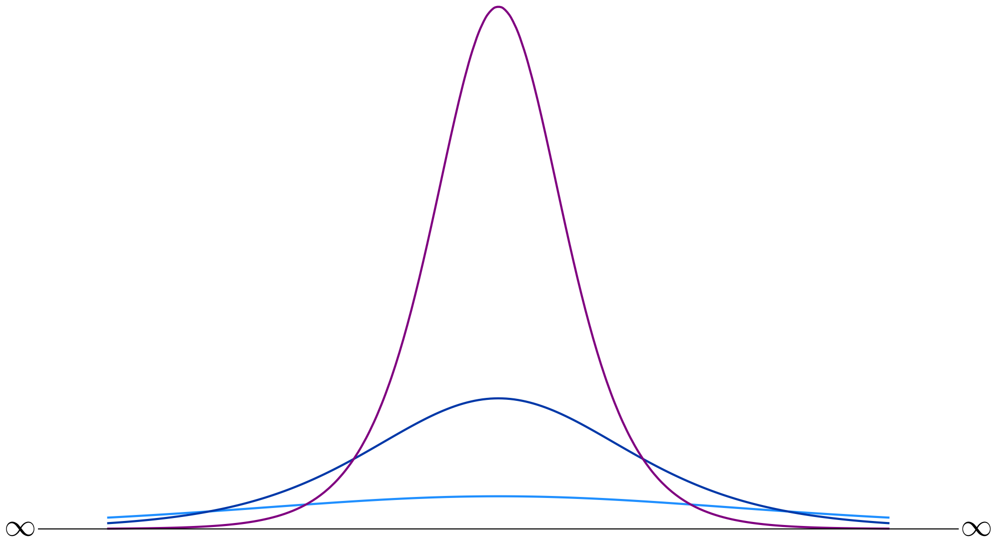


Figure II.32: A portion of solitons $\phi_{1/4,3}$, $\phi_{1,3}$ and $\phi_{4,3}$

More generally, some superlinear elliptic equations such as (NLS) may admit *concentrated* solutions in the neighbourhood of a point of the domain in certain asymptotic regimes.

This is a very active research theme. We refer the reader to [277] (see also [321, Section 2.4]) for a broader overview.

Remark. Many existence results concerning concentrated solutions use the *Lyapunov-Schmidt reduction method*. We will not present it here and refer to the exposition article [277].

Nevertheless, let us signal that we will use this method in order to study (NLS) in the regime $p \approx 2$. We will return to it in section II.11.

Now, let us present results about localized solutions on edges of a graph. They generalize the phenomena observed on the real line.

II.7.2 Solutions localized on edges of a graph

A computation based on the expression (II.18) shows that the L^2 mass of the soliton $\phi_{\lambda,p}$ is given by

$$\|\phi_{\lambda,p}\|_{L^2}^2 = \lambda^{\frac{6-p}{4(p-2)}} \|\phi_{1,p}\|_{L^2}^2.$$

Thus, *in the mass-subcritical regime* $2 < p < 6$, considering a soliton associated with a large value of λ is equivalent to consider a soliton whose mass is large. Moreover, a "sufficiently long" bounded edge may host functions which are "very close" to a given soliton.

This leads to three different regimes in which one may find localized solutions:

($\lambda \to +\infty$) given $p \in (2, +\infty)$, searching for solutions (for a fixed λ) to ($\mathrm{NLS}_\mathcal{G}$) and making λ tend to $+\infty$;

($\mu \to +\infty$) given $p \in (2, 6)$, searching for normalized solutions of mass μ (where, this time, λ is not fixed) to ($\mathrm{NLS}_\mathcal{G}$) and letting μ tend to $+\infty$;

($\ell \to +\infty$) given $p \in (2, +\infty)$, searching for solutions (for a fixed λ) to ($\mathrm{NLS}_\mathcal{G}$) which are "localized on an edge" of length ℓ and making ℓ tend to $+\infty$.

In the literature, there also exist three rather different approaches to prove the existence of solutions localized on an edge of a graph:

(Var) a variational method, consisting in searching solutions as local minima of the energy functional under the mass constraint or local minima of the action functional on the Nehari manifold;

(L-S) the Lyapunov-Schmidt method, which allows us to prove the existence of concentrated solutions for partial differential equations and which may be transposed to graphs, taking into account their local structure;

(ODE) a method based on ODE techniques, in particular an analysis of the period function (in the case $p = 4$).

Let us summarize the articles which study concentrated solutions on graphs in the following table (see also [199, Sections 5 and 6]).

Method / Regime	(Var)	(L-S)	(ODE)
($\lambda \to +\infty$)	[211, 302]	[103, 127]	
($\mu \to +\infty$)	[17]		[70, 200]
($\ell \to +\infty$)	Chapter 1		

Remark. In [211, 302], K. Kurata and M. Shibata consider the $\varepsilon \to 0^+$ limit for the problem

$$-\varepsilon^2 u'' + u = |u|^{p-2} u.$$

Nevertheless, the change of variables $v := \varepsilon^{\frac{-2}{p-2}} u$ transforms this equation into

$$-v'' + \varepsilon^{-2} v = |v|^{p-2} v,$$

for which the $\varepsilon \to 0^+$ limit is of the "$\lambda \to +\infty$" type.

In the articles [103, 127], it is manifest that the local structure of graphs plays a role. Thus, the *parities of the degrees of nodes* play a role in the results of [103]. This makes sense: if we "zoom" around a degree-D node, we observe a D-star graph. The description of solutions on such a star depends on the parity of D, as we have seen in section II.3.1.

To conclude our discussion about localized solutions on an edge of a graph, let us present the result proved in Chapter 1.

Given a bounded edge e of a graph $\mathcal{G}$, we define the set

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}.$$

For every real number $\lambda > 0$, we consider the level

$$\mathcal{J}_{\mathcal{G},e}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e} J_\lambda(u), \tag{II.19}$$

corresponding to a doubly-constrained minimization problem.

The result obtained is the following (in a slightly simplified form, see Theorems 1.12 and 1.13 for more precise statements).

Theorem II.4. *Let $p > 2$ and $\lambda > 2$ be two real numbers. If $\mathcal{G}$ is a metric graph with a finite number of edges satisfying assumption* (H). *If e is a long enough bounded edge of $\mathcal{G}$ (where the length threshold depends on λ and p), then the minimization problem* (II.19) *admits a minimum $u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$. Moreover, u is a constant sign solution to* ($\text{NLS}_\mathcal{G}$) *and we have*

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G} \setminus e)}. \tag{II.20}$$

Remark. The inequality (II.20) traduces, in this case, the fact that the solution u is "localized" on edge e: the absolue value of u atteins its maximum in e, and in e only.

Finally, let us remark that several constructions of localized solutions (whether on graphs or on higher dimensional domains) apply even when the domain is not compact. It is not surprising that those results "support the noncompactness of the domain relatively well" because their constructions are *local* by nature.

Those considerations will help greatly in Chapter 1. Indeed, constructions of concentrated solutions will allow us to prove existence of sequences of solutions on noncompact domains.

II.8 Problems with localized nonlinearity

II.8.1 Definitions and motivations

Let us consider metric graphs having finitely many edges and vertices, including at least one bounded edge and at least one half-line. The class of such graphs is rich, see e.g. the graph depicted in Figure II.2 or the examples of noncompact graphs of section II.3.4.

If $\mathcal{G}$ is a metric graph with a finite number of edges and vertices, its *compact core*[84] $\mathcal{K}$ is defined as the subset of $\mathcal{G}$ made of all bounded edges of $\mathcal{G}$ (see e.g. [21, 297]).

Let us consider the following problem.

$$\begin{cases} -u'' + \lambda u = \kappa(x)|u|^{p-2}u & \text{on every edge } e \text{ of the graph } \mathcal{G}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}, \end{cases} \tag{$\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$}$$

where κ is the characteristic function of the compact core $\mathcal{K}$ of $\mathcal{G}$. In other words, $\kappa(x)$ is equal to 1 if x belongs to $\mathcal{K}$, and to 0 otherwise.

The *localization of the nonlinearity* is due to the presence of κ. Thus, on half-lines of the graph, the equation simply becomes

$$-u'' + \lambda u = 0 \tag{II.21}$$

and is linear. We observe that when $\lambda \leq 0$, none of the nonzero solutions to the ODE (II.21) converges to 0 at infinity. Thus, if a solution $u \in L^2(\mathcal{G})$ of problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) exists, it is necessarily identically zero on half-lines, which means that its support is included in the compact core.[85] When $\lambda > 0$, only the multiples of $\mathrm{e}^{-\sqrt{\lambda}x}$ satisfy the equation (II.21) as well as the condition at infinity.

Regarding ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$), even though one may define an action functional and the associated Nehari manifold, the literature focuses mainly on normalized solutions. Proceeding as for the "classical" normalized solutions, we show that solutions to the problem with a fixed L^2 norm are critical points of the energy $E_{\mathrm{loc}} : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E_{\mathrm{loc}}(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{K}} |u|^p \,\mathrm{d}x,$$

under the mass constraint

$$\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x = \mu,$$

for a certain $\mu > 0$ (see e.g. [297, Proposition 2.3]). As in section II.5, the value of the mass μ is prescribed whereas λ, which appears as a Lagrange multiplier, is let free.

[84] This definition makes sense and may be relevant for graphs having an infinite number of edges. Nevertheless, in the general case, this set is not necessarily compact and we will call it $\mathcal{B}$ to recall that it is the set of *bounded* edges of the graph (see in particular Theorems 2.27 and 2.29 in Chapter 2).

[85] If the maximum principle implies that there are no positive solutions of this type, there may exist nodal solutions with $\lambda \leq 0$, see [298, Theorem 4.2 and Remark 4.6] as well as the discussion at the level of Figure 4.1 in Chapter 4.

The article [166] (see also discussions in [255, page 18] and [322, Section 1, Introduction]) details the interest from the physical point of view to study models where differential equations are given by $-u'' + \lambda u = c_e|u|^2u$, where the constant c_e may depend on edges and may vanish on some of them (see [166, Equation (2)]). We may for instance think of a network made of different optical fibers, not all leading to the same nonlinear effects, some of them may not even possess any at all. The richness of the model lies in the interaction between the nonlinearity and the diffusion ("scattering").

Finally, let us mention the overview article [80], devoted to the study of problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) as well as generalizations of this problem for the Dirac equation.

II.8.2 Works in the L^2-subcritical $(2 < p < 6)$ and L^2-critical regimes $(p = 6)$

The existence and non-existence questions concerning energy ground states for ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) have been treated in [322]. Remarkably, the exponent $p = 4$ plays a role of "critical exponent" in those results (see [322, Theorem 3.3 and Theorem 3.4]). It does not have any equivalent for standard problems where the nonlinearity is not localized.

In [297], an arbitrary multiplicity result for normalized solutions of ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) is obtained using a method based on the parity of functional E_{loc} (i.e. on the equality $E_{\mathrm{loc}}(u) = E_{\mathrm{loc}}(-u)$). It uses elements from genus theory (see e.g. [284, Chapters 7 and 8]) and was already used in [124] to obtain a multiplicity result on compact graphs.

The article [298] specifies the role played by the exponent $p = 4$ and by the properties of the graph (both metrical ones and topological ones) in the existence and non-existence results of energy ground states for ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$).

Notably, the works of Serra and Tentarelli imply that (see [298, Theorem 1.1]):

- if $p \in (2, 4)$, then for all $\mu > 0$, the problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) admits an energy ground state of mass μ;
- if $p \in (2, 6)$, then for all μ large enough, the problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) admits many solutions of mass μ;
- if $p \in [4, 6)$, then for all μ large enough, the problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) admits an energy ground state of mass μ;
- if $p \in [4, 6)$, then for all μ small enough, the problem ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) does not admit any energy ground state of mass μ.

Finally, let us mention that the articles [132, 133] treat the critical case $(p = 6)$.

It appears that the problem with a localized nonlinearity is a kind of "hybrid" between the compact case and the noncompact case. In particular, thanks to the localization of the nonlinearity, E_{loc} has compactness properties that the functional

$$E(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \, \mathrm{d}x - \frac{1}{p}\int_{\mathcal{G}} |u|^p \, \mathrm{d}x,$$

does not possess. Thus, in [297, Proposition 4.4], the authors prove that E_{loc} satisfies the Palais-Smale condition for negative energy levels.

II.8.3 Works in the L^2-supercritical regime ($6 < p$)

When $p > 6$, the functional E_{loc} is not bounded from below on the mass constraint, but possesses a *mountain pass geometry* on it. This allows us to search a solution as a *saddle point* of the functional under the constraint.

In [82], the authors adapt their method developed in [102] to the case of a non-compact graph with localized nonlinearity. Thus, they obtain the existence of a positive normalized solution for every mass. To do so, they use a result implying the existence of bounded Palais-Smale sequences with information on the approximate Morse index (see section II.5.7) as well as a *blow-up analysis* devoted to the study of the behaviour of solutions when λ converges to infinity.

In Chapter 4 we show that, for every mass $\mu > 0$, ($\text{NLS}_{\mathcal{G}}^{\text{loc}}$) possesses *infinitely many* normalized solutions (which are not necessarily positive). To do so, we use [81, Theorem 1.12], an existence result of bounded Palais-Smale sequences with information on the approximate Morse index.

II.9 Solutions vanishing identically on edges

As observed in the example of the tadpole graph in section II.3.4, there is in general no *unique continuation principle*[86,87] on metric graphs. This phenomenon also manifests itself in spectral problems where one may encounter *eigenfunctions vanishing identically on edges.* We will come back to this in Chapter 5.

In Chapter 2, we show that nodal ground states may vanish identically on edges of graphs and that their nodal sets may be very rich.

The possible presence of compactly supported solutions to ($\text{NLS}_{\mathcal{G}}$) (which may exist even when $\lambda \leq 0$) on noncompact graphs, will pose a few technical difficulties for us in Chapter 4.

[86] Let us recall that a unique continuation principle for a partial differential equation claims that, if a solution of the equation vanishes identically on a sub-domain, then it vanishes identically on the whole domain. This principle applies to the elliptic equation (NLS), we refer to [286, Theorem XIII.63] for a precise statement.

[87] We thank Matthias Täufer for his seminar on quantitative unique continuation principles given at UPHF and for sending references for control problems on graphs.

Finally, let us mention that there exists a *maximum principle* on graphs (see Appendix D). Thus, even if problems on graphs may hold surprises with regard to nodal solutions, we will be able to use the fact that a nonzero nonnegative solution to ($\mathrm{NLS}_\mathcal{G}$) is positive.

II.10 Solutions of equations versus minimizers of constrained problems

II.10.1 Different types of solutions to (NLS)

In the preceding sections, we have encountered most notions hereunder, relative to the solutions to (NLS) (for instance on a domain Ω with the Dirichlet boundary condition). Now, let us summarize them and examine the relationships between them.

Given a function $u \in H_0^1(\Omega) \setminus \{0\}$, we say that u is:

1. *an action ground state* if u belongs to the Nehari manifold $\mathcal{N}_\lambda$ and minimizes the action functional J_λ among functions of $\mathcal{N}_\lambda$;
2. *a minimal action solution* if u is a solution of (NLS) which minimizes the action functional J_λ among nonzero solutions of (NLS);
3. *a nodal ground state* if u belongs to the nodal Nehari set $\mathcal{N}_\lambda^{nod}$ and minimizes the action functional J_λ among functions of $\mathcal{N}_\lambda^{nod}$;
4. *a minimal action nodal solution* if u is a nodal solution of (NLS) which minimizes the action functional J_λ among nodal solutions of (NLS);
5. *an energy ground state having (prescribed) mass* $\mu \geq 0$ if u has mass μ and minimizes the energy functional E among functions of mass μ;
6. *a minimal energy solution of mass* $\mu \geq 0$ if u is a solution of (NLS) for a certain $\lambda \in \mathbb{R}$, which has mass μ, and which minimizes the energy functional E among solutions of (NLS) having mass μ (where $\lambda \in \mathbb{R}$ is not fixed and may vary between two solutions having same mass);
7. *a minimal energy nodal solution having mass* $\mu \geq 0$ if u is a nodal solution of (NLS) for a certain $\lambda \in \mathbb{R}$, which has mass μ, and which minimizes the energy functional E among nodal solutions of (NLS) having mass μ (where $\lambda \in \mathbb{R}$ is not fixed).

The minimization problems allow to find solutions to (NLS). Thus, when they exist (for instance when Ω is bounded), action ground states, nodal ground states and energy ground states having mass μ are respectively minimal action solutions, minimal action nodal solutions and minimal energy solutions of mass μ.

Several questions arise.

(Q1) What happens when working on noncompact domains, in which case the minimization problems do not necessarily possess solutions?

(Q2) What is the relation between minimal action solutions and minimal energy solutions?

(Q3) How to prove existence of normalized solutions in the mass-supercritical regime, in which case energy ground states do not exist?

(Q4) How to find nodal normalized solutions?

All those questions will be tackled in the following sections.

II.10.2 Action ground states vs minimal action solutions

In this section, we will study question (Q1) by examining the link between action ground states and minimal action solutions. To do so, let us use a noncompact metric graph $\mathcal{G}$ as domain[88].

We define the action level

$$\mathcal{J}_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u).$$

We also define the set

$$\mathcal{S}_\lambda(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \setminus \{0\} \;\middle|\; u \text{ is a solution of } (\mathrm{NLS}_{\mathcal{G}}) \right\}$$

and the action level

$$\sigma_{\mathcal{G}}(\lambda) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u).$$

Let us remark that an action ground state is a function $u \in \mathcal{N}_\lambda(\mathcal{G})$ such that $J_\lambda(u) = \mathcal{J}_{\mathcal{G}}(\lambda)$ and that a minimal action solution is a function $u \in \mathcal{S}_\lambda(\mathcal{G})$ such that $J_\lambda(u) = \sigma_{\mathcal{G}}(\lambda)$.

The reasons leading to consider notions of action ground state and of minimal action solution are different.

[88]The reason for which it is beneficial to work on graphs and not on open sets of $\mathbb{R}^N$ will be clarified later.

In the first case, we are interested in the level $\mathcal{J}_{\mathcal{G}}(\lambda)$ and we try to prove that the infimum is attained by showing that there exists a minimal action function *among all functions of* $\mathcal{N}_\lambda(\mathcal{G})$.

In the second case, one wonders whether *among solutions of* ($\mathrm{NLS}_{\mathcal{G}}$), there is one having minimal action. Let us remark that there may be many functions whose action is lower than $\sigma_{\mathcal{G}}(\lambda)$ in $\mathcal{N}_\lambda(\mathcal{G})$, none of them being a solution of problem ($\mathrm{NLS}_{\mathcal{G}}$) if $\mathcal{J}_{\mathcal{G}}(\lambda)$ is not attained. Even if $\mathcal{S}_\lambda(\mathcal{G})$ is a much smaller set than $\mathcal{N}_\lambda(\mathcal{G})$, there might not exist any function in $\mathcal{S}_\lambda(\mathcal{G})$ attaining $\sigma_{\mathcal{G}}(\lambda)$.

Four situations may occur.

A1) $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\mathcal{G}}(\lambda)$ and both levels are attained;

A2) $\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\mathcal{G}}(\lambda)$ and they are not attained;

B1) $\mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_{\mathcal{G}}(\lambda)$, $\sigma_{\mathcal{G}}(\lambda)$ is attained but $\mathcal{J}_{\mathcal{G}}(\lambda)$ is not;

B2) $\mathcal{J}_{\mathcal{G}}(\lambda) < \sigma_{\mathcal{G}}(\lambda)$ and neither is attained.

One might wonder if those four situations are indeed possible. In order to prove it, we must, for every case among A1–B2, produce an example of graph $\mathcal{G}$ having the prescribed behavior.

Let us remark that in cases A2 and B2, the level $\sigma_\lambda(\mathcal{G})$ is not attained, which requires the problem to admit an infinite number of solutions having different levels.

In Chapter 1, we construct graphs realizing the four cases among A1–B2 for ($\mathrm{NLS}_{\mathcal{G}}$), showing a richness in the possible behaviors.

Working on graphs turns out to be particularly convenient to study the links between the notions of action ground state and of minimal action solution. Indeed, it is possible to construct solutions localized on edges (see section II.7) and to control their action levels using rearrangement arguments (see Appendix B).

Remark. In a certain way, we are doing proof theory here. Indeed, since there exist graphs realizing A1–B2, this shows that it is not possible to exclude those phenomena at an "abstract" level[89].

Moreover, it is expected that there exist domains in $\mathbb{R}^N$ realizing A1–B2 but, to this day, it remains an open problem.

[89]Somewhat like the existence of models of hyperbolic geometry shows that it is not possible to deduce the parallel postulate from the four other Euclid's postulates, see the book [324] (in particular Chapter 7) and the site `https://mathshistory.st-andrews.ac.uk/HistTopics/Non-Euclidean_geometry` thanks to which we discovered the previous reference.

II.10.3 Action ground states vs energy ground states

In [129], Dovetta, Serra and Tilli study the links between action ground states and energy ground states. Even if those two notions of "ground states" are classical to find solutions of (NLS), the relation between them had never been studied under this angle before the publication of their article.

The results of [129] are more general but here we will state them by considering equation (NLS) set on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with the Dirichlet condition.

Let $p \in (2, 2 + \frac{4}{N})$. For all $\lambda \in \mathbb{R}$, we consider the Nehari manifold

$$\mathcal{N}_\lambda := \left\{ u \in H_0^1(\Omega) \,\middle|\, \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \right\}$$

and for all $\mu \geq 0$, we consider the set

$$\mathcal{M}_\mu := \left\{ u \in H_0^1(\Omega) \,\middle|\, \|u\|_{L^2(\Omega)}^2 = 2\mu \right\}$$

associated to the mass constraint[90] μ.

Action ground states are functions $u \in \mathcal{N}_\lambda$ such that $J_\lambda(u) = \mathcal{J}(\lambda)$, where

$$\mathcal{J}(\lambda) := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v).$$

Similarly, energy ground states having mass μ are the functions $u \in \mathcal{M}_\mu$ such that $E(u) = \mathcal{E}(\mu)$, where

$$\mathcal{E}(\mu) := \inf_{v \in \mathcal{M}_\mu} E(v).$$

In [129, Theorem 1.2], the authors show that $-\mathcal{E}(\mu)$ is the Legendre-Fenchel transform of $\mathcal{J}(\lambda)$, which means that for all $\mu \geq 0$, there holds

$$\mathcal{E}(\mu) = \inf_{\lambda \in \mathbb{R}} (\mathcal{J}(\lambda) - \lambda\mu). \tag{II.22}$$

Another striking result of the article claims that, if $u \in \mathcal{M}_\mu$ is an energy ground state having mass μ and Lagrange multiplier λ, then u is also an action ground state in $\mathcal{N}_\lambda$ (see [129, Theorem 1.3] for a more complete statement).

The paper [129] does not aim to obtain new results on the existence of solutions. For that matter, the authors suppose *a priori* that action ground states and energy ground states exist (see [129, Assumption A]).

We will show that *this is not necessary*: to obtain normalized solutions, it suffices to assume that action ground states exist and to use the "bridge" (II.22) between action and energy levels. This approach, which will be carried out in Chapter 3, is precised in the following section.

[90] Let us signal that μ verifies here equality $\|u\|_{L^2(\Omega)}^2 = 2\mu$ and not equality $\|u\|_{L^2(\Omega)}^2 = \mu$ as in section II.5. The use of different normalization conventions aims to avoid the appearance of constants in duality results.

II.10.4 The method of Chapter 3

The philosophy of the method is the following.

> *Finding (nodal) normalized solutions to* (NLS)
> *by studying the masses of (nodal) action ground states.*

On a bounded domain $\Omega \subseteq \mathbb{R}^N$, action ground states exist for all $\lambda > -\gamma_1(\Omega)$ and for all $p \in (2, 2^*)$ where $\gamma_1(\Omega)$ is the first eigenvalue of the Laplacian on Ω with the Dirichlet boundary condition (see e.g. [35, Section 2.3.2]).

Similarly, nodal ground states exist for all $\lambda > -\gamma_2(\Omega)$ and for all $p \in (2, 2^*)$, $\gamma_2(\Omega)$ being the second eigenvalue of the Laplacian on $H_0^1(\Omega)$. In this case, the existence results are more delicate than those of action ground states. When $\lambda > -\gamma_1(\Omega)$, the existence of nodal ground states may be proved using the direct method of calculus of variations (see e.g. [93] or [318, Theorem 18]). Concerning the case $\lambda \in (-\gamma_2(\Omega), -\gamma_1(\Omega)]$, it was treated by T. Bartsch and T. Weth in [53], assuming that the boundary of Ω is $\mathcal{C}^\infty$. We will see in Chapter 3 that we will be able to get rid of regularity assumptions on Ω.

Let us consider the *action levels*[91]

$$\mathcal{J}(\lambda) := \inf_{v \in \mathcal{N}_\lambda} J_\lambda(v) \quad \text{and} \quad \mathcal{J}^{nod}(\lambda) := \inf_{v \in \mathcal{N}_\lambda^{nod}} J_\lambda(v).$$

We then show (under certain hypotheses) that, given $\mu_* \in (0, +\infty)$, the real function $f_{\mu_*} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_{\mu_*}(\lambda) := \mathcal{J}(\lambda) - \mu_* \lambda$$

admits a (local) minimum. It turns out that the corresponding action ground state is a normalized solution having mass μ_*.

Moreover, in the mass-subcritical regime, or assuming that Ω is star-shaped, one may show that the obtained solution has minimal energy.

Analogously, we show that the function $f_{\mu_*}^{nod} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_{\mu_*}^{nod}(\lambda) := \mathcal{J}^{nod}(\lambda) - \mu_* \lambda$$

admits a minimum, corresponding to a normalized nodal solution having mass μ_*.

The big difference between this method and the results presented in II.10.3 is to use the link between the energy and the action as a *starting point* and not as a consequence. In a certain sense, we obtain normalized solutions *by duality*[92].

[91] We show that $\mathcal{J}(\lambda) = 0$ when $\lambda \leq -\gamma_1(\Omega)$ and that $\mathcal{J}^{nod}(\lambda) = 0$ when $\lambda \leq -\gamma_2(\Omega)$ (see Proposition 3.11). Thus, $\mathcal{J}$ and $\mathcal{J}^{nod}$ are continuous functions defined on $\mathbb{R}$.

[92] This is reminiscent of the relationship between the Lagrangian and Hamiltonian approaches in classical mechanics, see e.g. [33, Sections 14 and 15] or [306, Section 3.2].

The approach we have just presented allows us to provide some answers to questions (Q2), (Q3) and (Q4) from section II.10.1, namely:

- clarifying the relationship between minimal action solutions and normalized minimal energy solutions;
- finding normalized solutions in the mass-supercritical regime, even when the domain Ω is not star-shaped. In the star-shaped case, it is further shown that the obtained normalized solutions have minimal energy;
- find normalized nodal solutions having two nodal zones[93].

To the best of our knowledge, there does not exist any minimization problem corresponding to minimal energy normalized nodal solutions[94]. The method of Chapter 3 is thus the first variational method which allows to find them.

In this thesis, we apply this approach to the study of equation (NLS) with the Dirichlet boundary condition on bounded domains of $\mathbb{R}^N$.

II.10.5 Minimizing sequences, Palais-Smale sequences or sequences of solutions?

In our proofs, we will use different types of sequences in order to obtain solutions by taking weak limits. Thus, we will consider:

- minimizing sequences, for instance in the proof of the result showing existence of concentrated solutions (Theorem 1.12);
- sequences of solutions on truncated domains, for instance in the proof of the "abstract" theorem of Chapter 2 (Theorem 2.3);
- Palais-Smale sequences, after using the "monotonicity trick", see section 4.5;
- sequences of solutions for perturbed problems when we let the parameter "ρ" of the monotonicity trick approach 1, see section 4.6.

In practice, the kind of sequence to consider depends greatly on the situation, the various notions having their advantages and drawbacks.

For instance, if $(u_n)_n \subseteq \mathcal{N}_\lambda$ is a minimizing sequence for $\inf_{v\in\mathcal{N}_\lambda} J_\lambda(v)$, then so is the sequence of absolute values $(|u_n|)_n \subseteq \mathcal{N}_\lambda$ (since $|u_n|$ belongs to $\mathcal{N}_\lambda$ and we have $J_\lambda(|u_n|) = J_\lambda(u_n)$). In other words, we may assume without loss of generality that we work with a sequence of positive functions.

The situation is different for a sequence of solutions: if u is a nodal solution, then the equality $J_\lambda(|u|) = J_\lambda(u)$ still holds, but $|u|$ in general is *not* a solution of the problem.

[93] A nodal zone being a connected component of the set $\{x \in \Omega \mid u(x) \neq 0\}$.
[94] In other words, there is no notion of "*nodal energy ground state*".

II.11 Weakly superlinear regime ($p \approx 2$)

II.11.1 Presentation

When $p > 2$ is close to 2, the differential equation in ($\text{NLS}_{\mathcal{G}}$) becomes "almost linear". Thus, we expect to be able to link its solutions to the eigenfunctions of the spectral problem ($\text{Spec}_{\mathcal{G}}$):

$$\begin{cases} -u'' + \lambda u = \gamma u & \text{on every edge } e \text{ of the graph } \mathcal{G}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}. \end{cases}$$

The study of this "*weakly superlinear*" regime will be realized in Chapter 5.

One of the main motivations to study the weakly superlinear regime is to obtain uniqueness results for certain classes of solutions (such as positive solutions, action ground states, nodal ground states). In order to put the results we will obtain on graphs into perspective, we believe it is useful to present the works devoted to the study of the *uniqueness of solutions* to (NLS).

II.11.2 Uniqueness of solutions to (NLS) on $\mathbb{R}^N$ and on balls

Let us present the existing works on $\mathbb{R}^N$ and on balls $B(0, R)$ with the Dirichlet boundary condition.

Positive solutions on radially symmetric domains

The typical strategy to prove the uniqueness of the positive solution to (NLS) on a ball or on $\mathbb{R}^N$ consists in two steps.

1. Showing that the solution is *radially symmetric*[95], which means that there exists a real function U such that, for all x, $u(x) = U(|x|)$. Nowadays, this step is rather well understood thanks to the so-called *moving plane* argument, see the seminal paper [161] by B. Gidas, W.M. Ni and L. Nirenberg.
2. Showing that the differential equation

$$-\partial_{rr}U - \tfrac{N-1}{r}\partial_r U + \lambda U = |U|^{p-2}U \qquad (\text{ODE}_U)$$

obtained by writing (NLS) in polar coordinates[96] admits a *unique* positive solution (satisfying suitable boundary conditions, for instance $U'(0) = 0$ and $U(R) = 0$ if the domain is the ball $B(0, R)$).

[95] In the case of $\mathbb{R}^N$, one shows that the solution is radial *up to translation*. The positive solution will be unique only up to translation as well.

[96] That is, by taking $r := |x|$.

In summary, here is the approach to follow.

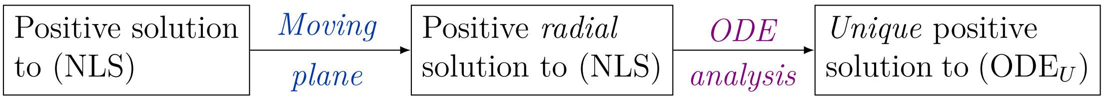


Figure II.33: Approach to prove the uniqueness of the positive solution to (NLS) on a radially symmetric domain.

Regarding the analysis of (ODE_U), let us mention:

- the pioneering work of C.V. Coffman [106];
- the famous paper by M.K. Kwong [213] proving the uniqueness of the positive solution for all dimensions and for all values of the exponent p;
- the paper [236] of K. McLeod which generalizes and simplifies Kwong's proof;
- the references [319, Appendix B] and [157, Section 5] where one may learn the proof techniques used in the aforementioned works;
- the paper [105] by C.B. Clemons and C.K.R.T. Jones for a more geometrical approach as well as the more recent works by B.L. Maultsby [232].

Let us signal that knowing that solutions are radial is an essential starting point of all those methods.

Nodal radial solutions

Regarding nodal solutions, the moving plane argument does not apply anymore and some solutions may be nonradial[97].

Nevertheless, let us focus on radial solutions whose classification is given by the following conjecture (see [177, Sections 19.3 and 19.6]).

Conjecture. *For every nonnegative integer k, there exists a **unique** solution U to* (ODE_U) *such that $U(0) > 0$ and such that U has exactly k roots and converges to* 0 *as $r \to +\infty$.*

Regarding the existence of those solutions, it is known that for all nonnegative integers k, there exists *at least one* solution to (ODE_U) having k roots and which converges to 0 as $r \to +\infty$ (see [237] for a proof using differential equations and [54] for a variational proof).

Regarding their uniqueness, the conjecture is open for most values of p and N (even when $k = 1$). A *computer-assisted proof*[98] was carried out by A. Cohen, Z. Li and W. Schlag [107] when $p = 4$, $N = 3$ and $k \leq 20$. It requires to perform computations of solutions to differential equations using *interval arithmetic*, for fixed p, N and k.

[97] There even exist solutions to (NLS) *without any symmetry*, see for instance [30] for such a construction on $\mathbb{R}^2$.

[98] We will come back to computer-assisted proofs in section II.11.5.

Thus, uniqueness results for nonlinear problems are generally quite delicate to obtain. Moreover, contrarily to the existence results, the uniqueness ones typically do not use variational methods.[99]

II.11.3 Uniqueness and symmetries in the $p \approx 2$ regime

One of the goals of Chapter 5 is the following.

Studying uniqueness and symmetries of solutions to (NLS$_{\mathcal{G}}$) *when* $p \approx 2$.

We will carry out this approach on *compact* graphs.

To this end, we will use a *Lyapunov-Schmidt reduction* method adapted to the asymptotic regime $p \to 2$.

In particular, we will obtain *the uniqueness of the positive solution to* (NLS$_{\mathcal{G}}$) *when p is close enough to* 2 *(Theorem 5.9).* Let us signal that this result is not specific to graphs. Thus, it is shown on arbitrary domains[100] $\Omega \subset \mathbb{R}^N$ bounded and with a $\mathcal{C}^\infty$ boundary (see [117, Lemma 1]).

Regarding nodal solutions, we will study the asymptotic behavior of nodal ground states when $p \to 2$. An analogous study on bounded domains of $\mathbb{R}^N$ was performed by D. Bonheure, V. Bouchez, C. Grumiau and J. Van Schaftingen in [74]. It also applies on graphs. It shows that eigenfunctions corresponding to limit points to sequences of nodal ground states when $p \to 2$ belong to the second eigenspace E_2 and minimize a *reduced functional* on a *reduced Nehari manifold*. As for the uniqueness of the positive solution when $p \approx 2$, we will not observe any difference between domains and graphs at the level of the "abstract" theory.

A phenomenon specific to graphs will nevertheless manifest itself during our study: the possible presence of solutions vanishing identically on edges, presented in section II.9. This will cause some "technical" problems (caused by a lack of regularity) and we will not be able to study those solutions thanks to the Lyapunov-Schmidt method.

We are thus led to investigate this phenomenon further.

Thus, for *compact star graphs*, we will present conditions on the lengths of edges for which nodal ground states vanish identically on edges.

---

[99] It is nevertheless natural to try to obtain uniqueness results by keeping the variational point of view. A work following this philosophy is [75].

[100] Let us remark that the uniqueness of positive solutions is not necessarily verified *for all* $p \in (2, 2^*)$ on any domain Ω, see e.g. [118].

II.11.4 The tetrahedron graph, a rich example

The last section of Chapter 5 is devoted to the study of the "tetrahedron graph".

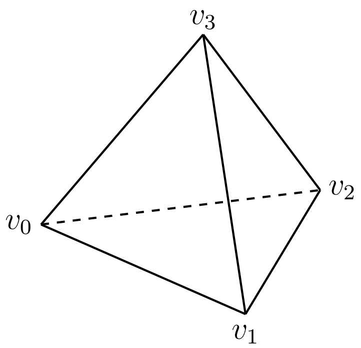


Figure II.34: The tetrahedron graph, a graph made of four vertices and six edges having same length.

This example is interesting in several respects. Thus, even though it only possesses four vertices, its *symmetry group* is particularly rich. This will provide eigenspaces of dimension higher than 1 as well as the presence of critical points associated to the symmetries (through the *principle of symmetric criticality*[101]).

Moreover, we will see that, even though there exist eigenfunctions vanishing identically on some edges in the eigenspace E_2, nodal ground states do not vanish identically on any edges when p is close enough to 2. In order to prove this, we will need to study the minimization problem of the reduced functional on the reduced Nehari manifold (see section 5.2.5 for definitions). This problem turns out to be quite difficult to treat[102] "by hand". Thus, we have resorted to a *computer-assisted proof.*

II.11.5 Computer-assisted proofs

Numerical errors

Numerical computations involving floating point numbers[103] lead to errors ("round-off errors") due to the fact that machines can only manipulate a finite number of digits.

Thus, in the language Python3, the result returned by `math.sin(math.pi)` is `1.2246467991473532e-16` and not `0`. Those errors may cause serious problems, sometimes spectacular (see [248, Section 1.3]).

[101] See the paper [261] by R.S. Palais.

[102] This is also the case for equivalent problems on domains of $\mathbb{R}^N$, see [294] for the study of the reduced functional associated to the $p \to 2$ limit on the square $(0,1)^2$.

[103] The most commonly used implementation for representing digitally "continuous" values, see [248, Section 2.1] for precise definitions.

Consequently, a question naturally arises: *how to obtain rigorous proofs relying on numerical computations performed by a computer?*

Interval arithmetic[104]

The basic idea of interval arithmetic is very simple: *replacing numbers by intervals in such a way that the result of an operation belongs to the returned interval.*

Let us illustrate this by going back to the computation of $\sin(\pi)$.
If we use the "`mpmath`" library[105] in Python3, the result of `iv.pi` is

```
mpi('3.1415926535897931','3.1415926535897936')
```

which means that

$$3.1415926535897931 \leq \pi \leq 3.1415926535897936.$$

Moreover, the result of `iv.sin(iv.pi)` is

```
mpi('-3.2162452993532732e-16','1.2246467991473532e-16')
```

which means that

$$-3.2162452993532732 \cdot 10^{-16} \leq \sin(\pi) \leq 1.2246467991473532 \cdot 10^{-16}.$$

Such rigorous controls on values may be obtained using computations involving only "a finite number of digits", thus implementable on a computer. Nevertheless, interval arithmetic has its limits.

- It may allow to prove that some values are nonzero, but it cannot prove that some values are equal to zero. For instance, evaluating `iv.sin(1.)` returns `mpi('0.8414709848078965','0.84147098480789662')`, which implies that $\sin(1) \neq 0$. However, the results give $|\sin(\pi)| \leq 10^{-15}$, but we cannot guarantee that $\sin(\pi) = 0$.
- If a returned interval is "too big", it is valid but useless. In this way, `iv.sin(x)` could return `[-1, 1]` regardless of the value of `x`, but this bound is useless. In practice, we will seek to obtain "sufficiently small" intervals.

Among applications of interval arithmetic in analysis, let us mention the study of the ground state of a quantum system involving a Thomas-Fermi potential by C.L. Fefferman and L.A. Seco (see [148, 149]) or the one of the Lorenz strange attractor by W. Tucker [107]. We may also refer to the book [249] devoted to the use of computer-assisted proofs in analysis.

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[104]Many thanks to Prof. Christophe Troestler for his course on interval arithmetic given at UPHF and for its application to the study of the tetrahedron graph.

[105]See in particular the module `iv`, devoted to interval arithmetic at `https://www.mpmath.org/doc/1.0.0/contexts.html`.

II.12 Main results of the thesis

The goal of this section is twofold. One the one hand, we will underline the *interest in studying problems on metric graphs*, that we could summarize in the following way.

> *Metric graphs allow to study problems in dimension one in a much richer class of domains that the one of real intervals.*

On the other hand, we will highlight the main results of the chapters, from the author's point of view.

Chapter 1 corresponds to an article that has been published in the journal "Calculus of Variations and Partial Differential Equations", written jointly with Colette De Coster, Simone Dovetta and Enrico Serra (see [120]).

Therein, we prove a result showing *existence of concentrated solutions to* (NLS$_\mathcal{G}$) (see Theorems 1.12 and 1.13). Thanks to it, we are able to answer the questions posed in section II.10.2.

Thus, we compare the two action levels $\mathcal{J}_\mathcal{G}(\lambda)$ and $\sigma_\mathcal{G}(\lambda)$ defined by

$$\mathcal{J}_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) \quad \text{and} \quad \sigma_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u)$$

where $\mathcal{S}_\lambda(\mathcal{G})$ is the set of nonzero solutions to (NLS$_\mathcal{G}$) and $\mathcal{N}_\lambda(\mathcal{G})$ is the Nehari manifold. We say that one of those levels is *attained* if the corresponding infimum is a minimum. If $\mathcal{J}_\mathcal{G}(\lambda)$ is attained, then action ground states exist. If $\sigma_\mathcal{G}(\lambda)$ is attained, then minimal action solutions exist.

The following result shows that many relations may exist between those two levels (see Theorem 1.3).

Theorem. *Given two real numbers $p > 2$ and $\lambda > 0$, there exist four metric graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ (depending on p and λ) such that:*

A1) $\mathcal{J}_{\mathcal{G}_1}(\lambda) = \sigma_{\mathcal{G}_1}(\lambda)$ and both levels are attained;

A2) $\mathcal{J}_{\mathcal{G}_2}(\lambda) = \sigma_{\mathcal{G}_2}(\lambda)$ and they are not attained;

B1) $\mathcal{J}_{\mathcal{G}_3}(\lambda) < \sigma_{\mathcal{G}_3}(\lambda)$, $\sigma_{\mathcal{G}_3}(\lambda)$ is attained but not $\mathcal{J}_{\mathcal{G}_3}(\lambda)$;

B2) $\mathcal{J}_{\mathcal{G}_4}(\lambda) < \sigma_{\mathcal{G}_4}(\lambda)$ and none of those levels is attained.

The *precise version of the Polyá-Szegő inequality*[106]*, specific to dimension one*, allows to prove the existence result of localized solutions and to control action levels of the different solutions.

[106] See Theorem B.23.

Chapter 2 corresponds to a joint preprint with Colette De Coster, Simone Dovetta, Enrico Serra and Christophe Troestler (see [121]). Therein:

- we prove *"abstract" existence results for action ground states and nodal ground states*, in function of a "level at infinity" (Theorem 2.3);
- we introduce *topological conditions* generalizing "assumption (H)" and we show that they imply *non-existence of action ground states and of nodal ground states* (Theorem 2.6);
- we characterize the cases where ground states and nodal ground states exist on *periodic graphs* and *infinite trees* (Theorems 2.7 and 2.8);
- we show a big richness in the *nodal zones* of nodal ground states on graphs (Theorem 2.9).

Let us precise the last point thanks to the following statement.

Theorem. *For all integers $k \geq 0$, $m \geq 2$ and $n \geq 0$, there exists a metric graph $\mathcal{G}$ and a nodal ground state u on $\mathcal{G}$ such that the nodal set $u^{-1}(\{0\})$ is the union of k isolated points, m half-lines and n bounded edges.*

During our study, metric graphs permitted to *experiment with various sources of non-compactness* (graphs with finitely many edges and half-lines, periodic graphs, infinite trees, etc.). As in Chapter 1, the *precised Polyá-Szegő inequality* plays a big role, notably in the use of topological hypotheses on the domain. The existence of nodal ground states for ($\text{NLS}_{\mathcal{G}}$) which *vanish identically on edges* is a phenomenon specific to the graph setting.

Results of Chapter 3 come as well from a collaboration with Colette De Coster, Simone Dovetta and Enrico Serra.

We develop a *new method* to study normalized solutions of (NLS) on bounded domains of $\mathbb{R}^N$, with the Dirichlet boundary condition. This technique allows to show existence of *normalized nodal solutions* and to treat the *L^2-supercritical regime*. Here is the outcome of our research (see Theorems 3.1 and 3.4, where more precise statements can be found).

Theorem. *If Ω is a bounded domain in $\mathbb{R}^N$ and if $p \in (2, 2^*)$, then equation* (NLS) *possesses a positive solution of mass μ and a solution with two nodal zones of mass μ for every small enough value of the mass μ.*

If moreover Ω has a $\mathcal{C}^\infty$ boundary and is star-shaped, then if the mass $\mu > 0$ is small enough, there exist (nodal) normalized solutions of minimal energy among the solutions of the problem.

Our method is "abstract" and does not require to work only in dimension one. We thus presented it in the setting of bounded open sets in $\mathbb{R}^N$.

Chapter 4 corresponds to a joint preprint with Pablo Carrillo, Louis Jeanjean and Christophe Troestler (see [91]), devoted to the proof of the following theorem.

Theorem. *Let $\mathcal{G}$ be a metric graph with finitely many edges including at least one bounded edge and at least one half-line. Given two real numbers $p > 6$ and $\mu > 0$, the problem* $(\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}})$ *possesses a sequence of normalized solutions of mass μ whose energy levels converge to $+\infty$.*

In the proof of this theorem (Theorem 4.2), we use during certain passages methods based on the theory of *ordinary differential equations*. Furthermore, the *localization of the nonlinearity*, essential in the results, is formulated in a particularly convenient way on graphs with finitely many edges, where one may distinguish the compact core with the half-lines.

As for Chapter 5, it was developed jointly with my PhD advisors Colette De Coster and Christophe Troestler and does not yet correspond to a preprint. It contains more details in the proofs than the previous chapters.

Therein, we study the behaviour of $(\mathrm{NLS}_{\mathcal{G}})$ on compact graphs in the regime $p \approx 2$ and we study the branches of solutions emanating from eigenvalues. To this end, we use a *Lyapunov-Schmidt redution*. We will see that dimension one will be used in an essential way in order to *prove regularity*[107] *of certain applications between functional spaces*.

A general consequence of the theory is the following theorem (Theorem 5.9).

Theorem. *If $\mathcal{G}$ is a compact graph and if the real $p > 2$ is close enough to 2, then the problem* $(\mathrm{NLS}_{\mathcal{G}})$ *possesses a* <u>*unique*</u> *positive solution.*

We also study the behavior of *nodal ground states* when $p \approx 2$.

The *eigenfunctions vanishing identically on edges* have a particular status when studying the regime $p \approx 2$. This led us to study the existence of *solutions of* $(\mathrm{NLS}_{\mathcal{G}})$ *vanishing identically on edges* on compact star graphs, thus completing results from Chapter 2.

Finally, we apply the method based on the Lyapunov-Schmidt reduction to an *example rich in symmetries: the tetrahedron graph*. Thanks to a *computer-assisted proof*, we obtain the uniqueness, up to symmetries, of nodal ground states for this graph in the regime $p \approx 2$.

[107] At the technical level, we will use the fact that roots of eigenfunctions which do not vanish identically on any edges have necessarily multiplicity one. This argument is based on the *existence and uniqueness theorem for the Cauchy problem associated to an ODE*.

II.13 Presentation of the appendices

Six appendices follow the introduction and the chapters.

Appendix A presents the *notion of metric graph* and details the *metric space* and *measured space* structures with which the graphs are equipped. We also present there the notion of *weak derivative*, the *Sobolev space* $H^1(\mathcal{G})$ and some *embedding results*. Finally, we prove a *coarea formula*, used in Appendix B. Its content is extracted from documents written with the help of Colette De Coster.

Appendix B is devoted to the study of the *decreasing rearrangement* process, in particular to a "self-contained" proof of the *Polyá-Szegő inequality, refined* thanks to the *number of preimages* of functions (Theorem B.23). We will systematically refer to it when using those arguments in the document. If those results are not new, the approach we follow is original. The proofs have been detailed with the help of Colette De Coster during our study of equations set on graphs.

Appendix C centralizes various pieces of information about the *differential equation* $-u'' + \lambda u = |u|^{p-2}u$. Its content will be frequently used in the manuscript and was influenced by discussions with my PhD supervisors.

Appendix D presents a *maximum principle* on metric graphs, which stems from the collaboration with Pablo Carillo, Colette De Coster, Louis Jeanjean and Christophe Troestler.

Appendix E presents two statements of *implict function theorems* (one at the "topological" level, the other at the "differentiable" level), assuming rather weak regularity hypotheses. It is based on notes written by Christophe Troestler and reworked in collaboration with Colette De Coster.

Appendix F comments a few results about the nonlinear Schrödinger *evolution equation* on $\mathbb{R}^N$. This allows us to pursue discussions started in this introduction. Its content is is largely based on my master's thesis, supervised by Christophe Troestler and which benefited from numerous discussions with Colette De Coster during its writing.

Final remarks

- We wanted to make the chapters as "self-contained" as possible. Thus, we will not hesitate to reintroduce therein some concepts already encountered in the previous chapters.
- Chapters 1, 2 and 4 contain different assumptions on the graphs under study. We decided to call $\mathbf{G_1}$, $\mathbf{G_2}$ and $\mathbf{G_4}$ the classes of graphs we consider, in reference to the numbering of chapters, in order to remove any possible ambiguity.

Chapter 1

On the notion of ground state for nonlinear Schrödinger equations on metric graphs

1.1 Presentation of the chapter

Nonlinear Schrödinger equations on metric graphs have attracted the interest of a large – and increasing – number of researchers in the last few years. As the literature on the subject witnessed a massive growth, we refrain from overviewing it here, redirecting e.g. to [4, 9, 23, 70, 71, 72, 77, 78, 81, 82, 102, 134, 167, 198, 201, 257, 271, 272] for some of the most recent developments and to the reviews [5, 199] for more comprehensive discussions. Within the whole theory, prominent efforts have been devoted to the analysis of existence of positive standing wave solutions, with a particular focus on *ground states*.

The notion of ground state however, albeit often suggested unequivocally by the specific problem under study, is by no means univocally defined. This aspect is of course not specific to Schrödinger equations on metric graphs, but is a general feature appearing in the study of various types of scalar field equations set on a variety of domains, from open subsets of $\mathbb{R}^N$ to Riemannian manifolds.

To describe it more concretely, we consider a metric graph $\mathcal{G}$ and the NLS equation $-u'' + \lambda u = |u|^{p-2}u$ on $\mathcal{G}$, where $\lambda > 0$ and $p > 2$ are two real numbers. As usual, it is required that the differential equation be satisfied pointwise on every edge of $\mathcal{G}$, while additional matching conditions have to be imposed at the vertices of the structure. In this chapter, this differential equation is coupled with the so–called natural, or Kirchhoff, boundary conditions, prescribing that on every vertex v of $\mathcal{G}$ the sum of the outgoing derivatives of u along every edge incident at v is zero.

Thus, defining a coordinate $x_e \in (0, |e|)$ on every edge e of $\mathcal{G}$ (where $|e|$ is the length of edge e), the problem we are addressing reads

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on every edge } e \text{ of } \mathcal{G}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}. \end{cases} \tag{NLS$_\mathcal{G}$}$$

In the previous problem, $\lambda > 0$ and $p > 2$ are real numbers, the symbol $e \succ v$ means that the sum is extended to all edges emanating from v and $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ is the outward derivative of u at v (i.e. $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ is equal to $u'_{|e}(0)$ or $-u'_{|e}(|e|)$, according to whether the vertex v is identified with 0 or with $|e|$).

In this framework, the solutions to (NLS$_\mathcal{G}$) can be characterized variationally as the critical points of the standard action functional $J_\lambda : H^1(\mathcal{G}) \to \mathbb{R}$

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}, \tag{1.1}$$

where

$$H^1(\mathcal{G}) := \Big\{u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous and } u, u' \in L^2(\mathcal{G})\Big\}.$$

In the search for ground states, as the functional J_λ is not bounded from below, one may impose extra constraints to recover boundedness and make a minimization procedure meaningful. For instance, one could restrict J_λ to the unit sphere of $L^p(\mathcal{G})$ or to the Nehari manifold associated with J_λ (this second approach has the advantage that the nonlinearity needs not be homogeneous). Both procedures make J_λ bounded from below on these sets and one then defines ground states as the functions that achieve the infimum of J_λ on the constraint. Specifically, for our problem, the Nehari manifold is the set

$$\begin{aligned} \mathcal{N}_\lambda(\mathcal{G}) &:= \Big\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid J'_\lambda(u)u = 0\Big\} \\ &= \Big\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})}\Big\}. \end{aligned} \tag{1.2}$$

As is well known, the Nehari manifold contains all nonzero critical points of J_λ and is a natural constraint, in the sense that constrained critical points are in fact true critical points of J_λ. This leads to a first definition of ground state. Defining

$$\mathcal{J}_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u),$$

it is customary to set the following definition.

Definition 1.1. An *action ground state* for the problem (NLS$_\mathcal{G}$) is a function $u \in \mathcal{N}_\lambda(\mathcal{G})$ such that

$$J_\lambda(u) = \mathcal{J}_\mathcal{G}(\lambda). \tag{1.3}$$

Clearly, any action ground state is a constant sign solution to the problem (NLS$_\mathcal{G}$), often referred to also as a “solution of minimal action”, though we adopt here a different terminology (see Definition 1.2). In applications, action ground states play a prominent role for various reasons, which fully justifies the preceding definition. In practice, however, one is very often confronted with the following inconvenience: there need not exist any function u in $\mathcal{N}_\lambda(\mathcal{G})$ satisfying condition (1.3) (a frequent fact in many noncompact settings). Thus, the existence of a ground state is not guaranteed in general.

To overcome this obstacle, sometimes one assumes a different definition of ground state. Let

$$\mathcal{S}_\lambda(\mathcal{G}) := \Big\{u \in H^1(\mathcal{G}) \setminus \{0\} \;\Big|\; u \text{ solves } (\mathrm{NLS}_\mathcal{G})\Big\}$$

be the set of nonzero solutions to ($\mathrm{NLS}_\mathcal{G}$) and let us set

$$\sigma_\mathcal{G}(\lambda) := \inf_{u \in \mathcal{S}_\lambda(\mathcal{G})} J_\lambda(u). \tag{1.4}$$

Although it is common use to call ground states too the functions described in the following definition, we prefer to label them with a different name, to avoid misunderstandings.

Definition 1.2. A *least action solution* for the problem ($\mathrm{NLS}_\mathcal{G}$) is a function $u \in \mathcal{S}_\lambda(\mathcal{G})$ such that

$$J_\lambda(u) = \sigma_\mathcal{G}(\lambda). \tag{1.5}$$

The two points of view motivating the above definitions are clearly different. In the former case, one fixes the attention on the level $\mathcal{J}_\mathcal{G}(\lambda)$ and tries to prove that it is attained, obtaining in this way a solution that has minimal action among all *functions* in $\mathcal{N}_\lambda(\mathcal{G})$. In the latter, the aim is to ascertain if, *among solutions* of ($\mathrm{NLS}_\mathcal{G}$), there is one of least action. In this case, there might be plenty of functions with lesser action (and none of them will solve problem ($\mathrm{NLS}_\mathcal{G}$) if $\mathcal{J}_\mathcal{G}(\lambda)$ is not attained).

Although $\mathcal{S}_\lambda(\mathcal{G})$ is a much smaller set than $\mathcal{N}_\lambda(\mathcal{G})$, it is not at all clear in general that there exist any functions in $\mathcal{S}_\lambda(\mathcal{G})$ that achieve $\sigma_\mathcal{G}(\lambda)$. Thus, even with the second definition, the existence of "ground states" is not guaranteed *a priori.*

The aim of this chapter is to analyze the relations among $\mathcal{J}_\mathcal{G}(\lambda)$ and $\sigma_\mathcal{G}(\lambda)$ and, in particular, to investigate if all the theoretical cases can actually take place.

To make this point clear we start by observing that, trivially,

$$\mathcal{J}_\mathcal{G}(\lambda) \leq \sigma_\mathcal{G}(\lambda), \tag{1.6}$$

without any further assumption.

Moreover, if $\mathcal{J}_\mathcal{G}(\lambda)$ is attained by a function $u \in \mathcal{N}_\lambda(\mathcal{G})$, then u belongs to $\mathcal{S}_\lambda(\mathcal{G})$, the equality $\mathcal{J}_\mathcal{G}(\lambda) = \sigma_\mathcal{G}(\lambda)$ holds and $\sigma_\mathcal{G}(\lambda)$ is attained too.

In view of these preliminary considerations, the possibilities to consider are exactly the following four:

A1) $\mathcal{J}_\mathcal{G}(\lambda) = \sigma_\mathcal{G}(\lambda)$ and they are attained;

A2) $\mathcal{J}_\mathcal{G}(\lambda) = \sigma_\mathcal{G}(\lambda)$ and they are not attained;

B1) $\mathcal{J}_\mathcal{G}(\lambda) < \sigma_\mathcal{G}(\lambda)$, $\sigma_\mathcal{G}(\lambda)$ is attained and $\mathcal{J}_\mathcal{G}(\lambda)$ is not;

B2) $\mathcal{J}_\mathcal{G}(\lambda) < \sigma_\mathcal{G}(\lambda)$ and neither is attained.

To prove or disprove the actual occurrence of these situations then amounts, for each case between A1 and B2, to produce an example of a graph $\mathcal{G}$ where the behavior is exactly the one prescribed by the alternative in question.

We wish to make clear that the previous discussion is completely independent of the domain on which the NLS equation is set. For instance, replacing $\mathcal{G}$ by an open subset Ω of $\mathbb{R}^N$ (and the second derivative by the Laplacian, and so on, e.g. $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)\dots$), every issue described so far can be stated in the new context without any modification except notation. In particular, the four alternatives listed above remain and it would be extremely interesting to understand if they really occur. While this can be easily achieved in some of the alternatives (A1, for instance), some of them appear rather difficult to deal with when considering the NLS equation on subsets of $\mathbb{R}^N$ and are currently out of reach.

Our present contribution aims at shedding some light on this problem starting from the context of metric graphs, where one can profitably use the advantage of the local one–dimensional nature of the ambient space to obtain sharper results. Even in this setting, though, the constructions we will provide are quite involved, very specific to the metric graph environment and do not seem to be extendable to other frameworks.

Stated in a compact form, our main result is the following. In a nutshell, as far as action ground states are concerned, “anything can happen” on metric graphs.

Theorem 1.3. *For every $p > 2$, every $\lambda > 0$ and every choice of an alternative among A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where that alternative takes place.*

As the proof of Theorem 1.3 will be carried out case by case, let us briefly comment here on A1–B2, postponing the technical issues to subsequent sections.

Case A1 corresponds to problems where the infimum $\mathcal{J}_{\mathcal{G}}(\lambda)$ is achieved. This is what one usually tries to obtain in the existence results and it is the case for all compact graphs and for some noncompact ones. One can e.g. refer to [18, 19, 21, 124, 127, 132, 211, 256, 257, 322], works that deal with L^2-normalized solutions (thus a slightly different notion of ground state) but whose techniques adapt easily to the present setting. The theory in this framework is rather well developed, not only on metric graphs of course and there is not much to add.

In case A2, the graph is necessarily noncompact. There are plenty of examples where it is known that $\mathcal{J}_{\mathcal{G}}(\lambda)$ is not attained. This is due to topological or metric obstructions on the graph that have been widely described in the literature (see again e.g. [18, 19, 21, 132]). Nonetheless, this leaves open the question of the existence of a least action solution in the sense of Definition 1.2. As far as we know, examples of this kind, where $\mathcal{J}_{\mathcal{G}}(\lambda)$ coincides with $\sigma_{\mathcal{G}}(\lambda)$ but the level is not achieved, have never been described before. The construction of a graph with this property will be the object of Section 1.4.2 (see Theorem 1.19) and is one of the principal proofs of the chapter.

Due to existing results, it is easy to produce an example where alternative B1 occurs. We will briefly describe it anyway for completeness in Section 1.4.3, since it has never been considered under this perspective.

Case B2 is the hardest one and will be treated in Section 1.4.4 (Theorem 1.20). It is well known that, typically, the lack of a function attaining $\mathcal{J}_{\mathcal{G}}(\lambda)$ is due to the presence of a "problem at infinity" that attracts nonconvergent minimizing sequences. This is a standard phenomenon in problems with lack of compactness and is essentially what takes places in cases A2 and B1. The main novelty in B2, which is what makes this case rather delicate, is that the infimum over the set of solutions is not attained due to the presence of a *second* problem at infinity, at level $\sigma_{\mathcal{G}}(\lambda) > \mathcal{J}_{\mathcal{G}}(\lambda)$. By this we do not mean a problem at infinity with loss of compactness at different levels, but rather the presence of two *distinct* problems: the first, as we said, attracting nonconvergent minimizing sequences for J_λ and preventing at the same time the existence of solutions with action arbitrarily close to such infimum level; the second attracting nonconvergent sequences *of solutions* of lower and lower level. In fact, the gap between $\mathcal{J}_{\mathcal{G}}(\lambda)$ and $\sigma_{\mathcal{G}}(\lambda)$ reflects the coexistence of this pair of different problems at infinity. This seems to be a new phenomenon and it is what makes quite involved the construction of a graph exhibiting it.

As a byproduct, we notice that in cases A2 and B2, the fact that $\sigma_{\mathcal{G}}(\lambda)$ is not attained immediately implies the existence of infinitely many positive solutions for ($\mathrm{NLS}_{\mathcal{G}}$), with accumulating levels, which we believe to be a remarkable fact.

From the technical point of view, the proofs of the aforementioned results exploit deeply the role of both the topology and the metric of graphs to determine existence/non–existence of solutions to specific variational problems.

On the one hand, the results about non–existence of ground states are obtained via by–now standard arguments in the theory of NLS on metric graphs.

On the other hand, the construction of noncompact sequences of solutions in $\mathcal{S}_\lambda(\mathcal{G})$ with specific action level is achieved, in both cases A2 and B2, through a careful analysis of doubly–constrained minimization problems in the form

$$\inf_{u\in\mathcal{N}_\lambda(\mathcal{G})\cap X_e} J_\lambda(u) \tag{1.7}$$

where

$$X_e := \left\{u\in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)}\right\}$$

is the subset of H^1 functions attaining their L^∞ norm on a given bounded edge e of $\mathcal{G}$. A rather general existence result of independent interest is derived for this kind of problems in Section 1.3. Precisely, for a wide class of noncompact graphs, given $\lambda > 0$ we show that the infimum in (1.7) is attained whenever the length of e is larger than a threshold depending only on λ and p (Theorem 1.12). Moreover, such a minimizer is a solution of problem (NLS$_\mathcal{G}$) provided e is sufficiently long (Theorem 1.13), the threshold depending this time also on $\inf_{e\in\mathbb{E}}|e|$, where $\mathbb{E}$ is the set of all edges of $\mathcal{G}$. This approach was originally introduced for mass–constrained critical points of the energy functional in [17], where it proved suitable to obtain multiplicity results. However, in that article, a crucial assumption is that the (prescribed) mass be sufficiently large. By scaling properties, this is equivalent to the assumption that the length of all bounded edges is large. Here, on the contrary, it is sufficient that a *single* edge is long enough. A direct consequence of these results is the existence of multiple positive solutions, each attaining its maximum on one of the edges longer than the threshold (Theorem 1.15).

To conclude, let us point out that, as is well–known, to look at solutions of (NLS$_\mathcal{G}$) as critical points of the action functional (1.1) is not the unique variational characterization at disposal. In particular, moving from the seminal articles[1] [99, 183], in the past decade a lot of attention has been devoted, both on metric graphs and on domains in $\mathbb{R}^N$, to L^2-normalized solutions, i.e. critical points of the energy functional $E : H^1(\mathcal{G}) \to \mathbb{R}$

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}$$

constrained to the space of functions with prescribed mass μ

$$H^1_\mu(\mathcal{G}) := \left\{u\in H^1(\mathcal{G}) \mid \|u\|^2_{L^2(\mathcal{G})} = \mu\right\}.$$

In this setting, the parameter λ appearing in (NLS$_\mathcal{G}$) is not known a priori and pops up as a Lagrange multiplier associated to the mass constraint.

[1] See Section II.5 from the introduction for more information on the literature concerning normalized solutions, in particular the early contributions by C.A. Stuart.

One may consider definitions of energy ground states and least energy solutions analogous to those given above when dealing with J_λ. Precisely, letting

$$\mathcal{E}_{\mathcal{G}}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E(u)$$

and

$$\widehat{\sigma}_{\mathcal{G}}(\mu) := \inf_{u \in \widehat{\mathcal{S}}_\mu} E(u),$$

where

$$\widehat{\mathcal{S}}_\mu(\mathcal{G}) := \left\{ u \in H^1_\mu(\mathcal{G}) \mid u \text{ solves } (\mathrm{NLS}_{\mathcal{G}}) \text{ for some } \lambda \in \mathbb{R} \right\},$$

one has the following mutually exclusive four alternatives (the analogue of A1–B2):

$\widehat{\mathrm{A1}}$) $\mathcal{E}_{\mathcal{G}}(\mu) = \widehat{\sigma}_{\mathcal{G}}(\mu)$ and they are attained;

$\widehat{\mathrm{A2}}$) $\mathcal{E}_{\mathcal{G}}(\mu) = \widehat{\sigma}_{\mathcal{G}}(\mu)$ and they are not attained;

$\widehat{\mathrm{B1}}$) $\mathcal{E}_{\mathcal{G}}(\mu) < \widehat{\sigma}_{\mathcal{G}}(\mu)$, $\widehat{\sigma}_{\mathcal{G}}(\mu)$ is attained and $\mathcal{E}_{\mathcal{G}}(\mu)$ is not;

$\widehat{\mathrm{B2}}$) $\mathcal{E}_{\mathcal{G}}(\mu) < \widehat{\sigma}_{\mathcal{G}}(\mu)$ and neither is attained.

As a matter of fact, the analysis we develop here for J_λ can be naturally adapted to prove (with the very same constructions) that also in the context of normalized critical points of E all cases $\widehat{\mathrm{A1}}-\widehat{\mathrm{B2}}$ do actually occur. Note that, even though both the approaches have been widely exploited in the literature, a detailed discussion of the relation between ground states (and more generally local minima) of J_λ and E was started only recently in [129, 190]. Further contributions about the relation between those two approaches will be presented in Chapter 3.

This chapter is organized as follows. Section 1.2 recalls several preliminary results, whereas Section 1.3 deals with doubly–constrained variational problems for the action functional in a general setting, proving the existence results in Theorems 1.12–1.13-1.15. Section 1.4 is devoted to the proof of Theorem 1.3.

Notation. In what follows, we will write e.g. $\|u\|_p$ in place of $\|u\|_{L^p(\mathcal{G})}$ whenever possible. When needed, the full notation will be used to indicate explicitly the domain of integration.

1.2 Preliminaries

In this chapter, we will use a number of properties and results that have been established (mostly) in the recent literature. For the ease of the reader, we collect them in the present section, referring to the original articles or to the appendices where proofs can be found.

We assume that the reader is familiar with the concept of metric graph (see Appendix A). However, we make precise that in this chapter we consider metric graphs $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ satisfying the following definition.

Definition 1.4. We denote by $\mathbf{G_1}$ the class of metric graphs $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ such that

- $\mathcal{G}$ is connected and has an at most countable number of edges;
- $\mathcal{G}$ has at least one unbounded edge (i.e. a half-line);
- $\deg(v) < \infty$ for every $v \in \mathbb{V}$, where $\deg(v)$ denotes the degree of the vertex v, i.e. the number of edges emanating from it;
- $\forall v \in \mathbb{V}$, $\deg(v) \neq 2$;
- $\inf_{e\in\mathbb{E}} |e| > 0$, where $|e|$ denotes the length of e.

There is no loss of generality in assuming that $\deg(v) \neq 2$ since every vertex of degree two can a priori be eliminated from any metric graph, by melting the two edges incident at v into a single edge (see the discussion about degree two nodes in Section II.2.4 of the introduction). Note that every $\mathcal{G} \in \mathbf{G_1}$ is noncompact. Further assumptions will be made when needed in the course of the chapter.

In the study of the NLS equation on a metric graph in class $\mathbf{G_1}$, a fundamental tool is provided by the Gagliardo–Nirenberg inequalities (see [21])

$$\|u\|_q^q \leq 2^{\frac{q}{2}-1}\|u\|_2^{\frac{q}{2}+1}\|u'\|_2^{\frac{q}{2}-1}. \tag{1.8}$$

that hold for every $u \in H^1(\mathcal{G})$ and every $q \geq 2$ as well as their L^∞ version

$$\|u\|_\infty^2 \leq 2\|u\|_2\|u'\|_2. \tag{1.9}$$

A second fundamental tool that we will use very frequently in the next sections is provided by the rearrangement techniques of H^1 functions on a generic graph $\mathcal{G}$, for the details of which we refer to Appendix B. For the reader's convenience we recall that, given a nonnegative function $u \in H^1(\mathcal{G})$, the *decreasing* rearrangement of u is the unique nonincreasing function $u^* \in H^1(0, |\mathcal{G}|)$ equimeasurable with u.

The equimeasurability property entails that

$$\|u^*\|_{L^q(0,|\mathcal{G}|)} = \|u\|_{L^q(\mathcal{G})}, \qquad \text{for every } q \in [1, +\infty]. \tag{1.10}$$

Moreover, by the classical Pólya–Szegő inequality, we have

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}. \tag{1.11}$$

Similarly, the *symmetric* rearrangement $\widehat{u} \in H^1(-|\mathcal{G}|/2, |\mathcal{G}|/2)$ of u is given by $\widehat{u}(x) := u^*(2|x|)$. By definition, $\widehat{u}$ is symmetric, nonincreasing on $[0, -|\mathcal{G}|/2)$ and equimeasurable with u. Furthermore, it is well known (see e.g. [19]) that if $\#u^{-1}(t) \geq 2$ for almost every $t \in (0, \|u\|_\infty)$, then

$$\|\widehat{u}'\|_{L^2(-|\mathcal{G}|/2,|\mathcal{G}|/2)} \leq \|u'\|_{L^2(\mathcal{G})}, \tag{1.12}$$

where equality implies that $\#u^{-1}(t) = 2$ for almost every $t \in (0, \|u\|_\infty)$.

The following result, that will be used in Section 1.4, is essentially a refinement of the Pólya–Szegő inequality. Its proof follows combining results of [135] and [158], more details may be found in Appendix B.

Proposition 1.5. *Let $\mathcal{G}$ be a metric graph and let u be a nonnegative function in $H^1(\mathcal{G})$. Let $T \subseteq [0,+\infty[$ be a set of positive measure. Let us assume that, for some integer $K \geq 1$,*

$$\#u^{-1}(s) \geq K \qquad \text{for a.e. } s \in T.$$

Then the decreasing rearrangement u^* of u satisfies

$$\|u^*\|_{L^q((u^*)^{-1}(T))} = \|u\|_{L^q(u^{-1}(T))}, \text{ for every } q \in [1,+\infty]$$

and

$$\|(u^*)'\|_{L^2((u^*)^{-1}(T))} \leq \frac{1}{K}\|u'\|_{L^2(u^{-1}(T))}.$$

Next, if $\mathcal{G}$ is a metric graph in class $\mathbf{G_1}$ and $\lambda > 0$, we define the *action functional* $J_\lambda \in C^1(H^1(\mathcal{G}))$ by

$$J_\lambda(u) := \frac{1}{2}\|u'\|_2^2 + \frac{\lambda}{2}\|u\|_2^2 - \frac{1}{p}\|u\|_p^p$$

and the *Nehari manifold* associated to J_λ by

$$\begin{aligned}\mathcal{N}_\lambda(\mathcal{G}) &:= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid J_\lambda'(u)u = 0\right\} \\ &= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid \|u'\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p\right\}.\end{aligned}$$

It is well known that there exists a natural projection of $H^1(\mathcal{G}) \setminus \{0\}$ on $\mathcal{N}_\lambda$. Defining $n_\lambda : H^1(\mathcal{G}) \setminus \{0\} \to \mathbb{R}$ by

$$n_\lambda(u) := \left(\frac{\|u'\|_2^2 + \lambda\|u\|_2^2}{\|u\|_p^p}\right)^{\frac{1}{p-2}}, \tag{1.13}$$

we have $u \in \mathcal{N}_\lambda(\mathcal{G})$ if and only if $n_\lambda(u) = 1$. We will also need the functional $L : H^1(\mathcal{G}) \setminus \{0\} \to \mathbb{R}$ defined by

$$L(u) := \frac{\|u\|_p^p - \|u'\|_2^2}{\|u\|_2^2}. \tag{1.14}$$

If u solves problem (NLS$_\mathcal{G}$), then $L(u) = \lambda$ and, more generally, $L(u) = \lambda$ if and only if $u \in \mathcal{N}_\lambda(\mathcal{G})$.

When $u \in \mathcal{N}_\lambda(\mathcal{G})$, the functional J_λ takes the simple form

$$J_\lambda(u) = \varkappa\|u\|_p^p = \varkappa\Big(\|u'\|_2^2 + \lambda\|u\|_2^2\Big), \quad \text{where } \varkappa := \frac{1}{2} - \frac{1}{p}, \tag{1.15}$$

so that J_λ is positive on $\mathcal{N}_\lambda(\mathcal{G})$.

Actually more can be said and we summarize it in the next proposition.

Proposition 1.6. *There exists a constant $C > 0$ depending only on λ and p such that*

$$\inf_{u\in\mathcal{N}_\lambda(\mathcal{G})} \|u\|_p \geq C > 0. \tag{1.16}$$

Moreover, if $(u_n)_n \subseteq \mathcal{N}_\lambda(\mathcal{G})$ satisfies $\sup_n J_\lambda(u_n) < \infty$, then $(u_n)_n$ is bounded in $H^1(\mathcal{G})$ and

$$\inf_n \|u_n\|_2 > 0, \quad \inf_n \|u_n\|_\infty > 0.$$

Proof. Since $u \mapsto \sqrt{\|u'\|_2^2 + \lambda\|u\|_2^2}$ is equivalent to the usual $H^1(\mathcal{G})$ norm, Sobolev inequalities imply the existence of $C = C(p, \lambda) > 0$ such that, for all $u \in \mathcal{N}_\lambda$,

$$\|u\|_p \leq C(\|u'\|_2^2 + \lambda\|u\|_2^2)^{1/2} = C\|u\|_p^{p/2},$$

whence

$$\inf_{u\in\mathcal{N}_\lambda} \|u\|_p \geq C^{\frac{2}{2-p}} > 0$$

which proves (1.16).

From (1.15), as $\sup_n J_\lambda(u_n) < \infty$, we see that $(u_n)_n$ is bounded in $H^1(\mathcal{G})$, hence in $L^2(\mathcal{G})$ and $L^\infty(\mathcal{G})$. Observing that

$$\|u_n\|_p^p \leq \|u_n\|_\infty^{p-2}\|u_n\|_2^2,$$

we see that $\|u_n\|_2$ and $\|u_n\|_\infty$ are also uniformly bounded away from zero. □

When $\mathcal{G} = \mathbb{R}$, the non-trivial solutions to (NLS$_\mathcal{G}$) are called *solitons*, they are unique up to translations and sign (see Appendix C) and they are the action ground states of J_λ over $\mathcal{N}_\lambda(\mathbb{R})$ (see e.g. [214, Proposition 3.12]). Denoting by ϕ_λ the unique positive and even soliton, letting $s_1 := J_1(\phi_1)$ and

$$s_\lambda := s_1\lambda^\alpha, \qquad \alpha := \frac{p+2}{2(p-2)}, \tag{1.17}$$

for $\lambda > 0$, the action level of the soliton is given by

$$J_\lambda(\phi_\lambda) = \inf_{u\in\mathcal{N}_\lambda(\mathbb{R})} J_\lambda(u) = s_\lambda. \tag{1.18}$$

When $\mathcal{G} = [0, +\infty)$, for every $\lambda > 0$ there is a unique positive action ground state, given by the restriction of ϕ_λ to $[0, +\infty)$ and

$$\inf_{u\in\mathcal{N}_\lambda(0,+\infty)} J_\lambda(u) = J_\lambda\Big(\phi_\lambda{}_{|[0,+\infty)}\Big) = \frac{1}{2}s_\lambda. \tag{1.19}$$

The level s_λ plays a fundamental role in many works. It will also be the case in this chapter.

Lemma 1.7. *Let $\mathcal{G}$ be a metric graph. Let us assume that, for every $\ell > 0$, the graph $\mathcal{G}$ has an edge e_ℓ of length at least ℓ. Then, there exists a sequence of functions $(u_\ell)_{\ell\in\mathbb{Z}^{\geq 1}} \subseteq \mathcal{N}_\lambda$, equal to zero outside e_ℓ and such that*

$$\lim_{\ell\to\infty} J_\lambda(u_\ell) = s_\lambda.$$

In particular, the inequality

$$\inf_{u\in\mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) \leq s_\lambda \tag{1.20}$$

holds.

Proof. Let ϕ_λ be the soliton in $\mathcal{N}_\lambda(\mathbb{R})$ and let us set $\delta_\ell := \phi_\lambda(\ell/2) = o(1)$ as $\ell \to \infty$. For every $\ell > 0$, we identify the interval $[-\ell/2, \ell/2]$ with a subset of the edge e_ℓ and we define $v_\ell \in H^1(\mathcal{G})$ as

$$v_\ell(x) := \begin{cases} (\phi_\lambda(x) - \delta_\ell)^+ & \text{if } x \in e_\ell, \\ 0 & \text{elsewhere on } \mathcal{G}. \end{cases}$$

Since $\|v_\ell\|_{H^1(\mathcal{G})} = \|v_\ell\|_{H^1(e_\ell)} = \|\phi_\lambda\|_{H^1(\mathbb{R})} + o(1)$ as $\ell \to \infty$ and likewise for all the L^q norms, we see that $n_\lambda(v_\ell) \to 1$ as $\ell \to \infty$. Therefore, as $n_\lambda(v_\ell)v_\ell \in \mathcal{N}_\lambda(\mathcal{G})$,

$$J_\lambda(n_\lambda(v_\ell)v_\ell) = J_\lambda(v_\ell) + o(1) = J_\lambda(\phi_\lambda) + o(1) = s_\lambda + o(1)$$

as $\ell \to \infty$ and we conclude. □

In [19], the authors introduced a topological condition on the graph $\mathcal{G}$ under which inequality (1.20) is reversed. In our setting, this condition, that we call assumption (H) as in [19], takes the following form.

Definition 1.8. A metric graph $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty) \to \mathcal{G}$ which are parameterized by arclength, have disjoint images except for an at most countable number of points and satisfy $\gamma_1(0) = \gamma_2(0) = x_0$.

If a graph $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H), it is easy to see (and proved in [19]) that for every nonnegative $u \in H^1(\mathcal{G})$,

$$\# u^{-1}(t) \geq 2 \qquad \text{for almost every } t \in (0, \|u\|_\infty).$$

Therefore, letting $\widehat{u}$ be the symmetric rearrangement of u, if $u \in \mathcal{N}_\lambda(\mathcal{G})$, we have $n_\lambda(\widehat{u}) \leq 1$ by (1.12). As $n_\lambda(\widehat{u})\widehat{u} \in \mathcal{N}_\lambda(\mathbb{R})$, we conclude by (1.18) and (1.15) that

$$s_\lambda \leq J_\lambda(n_\lambda(\widehat{u})\widehat{u}) = \varkappa\, n_\lambda(\widehat{u})^p \|\widehat{u}\|^p_{L^p(\mathbb{R})} \leq \varkappa \|u\|^p_{L^p(\mathcal{G})} = J_\lambda(u)$$

and since this holds for every $u \in \mathcal{N}_\lambda(\mathcal{G})$, inequality (1.20) is in fact an equality.

As a consequence, with the same techniques as in [19], it is easy to prove the following result.

Theorem 1.9. *If $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H), then*

$$\inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) = s_\lambda$$

but it is never achieved, unless the graph $\mathcal{G}$ is isometric to $\mathbb{R}$ or to the same exceptional graphs as in Theorem II.2 (see Figures II.28, II.29 and II.30 on pages 123 and 124).

If more is known on the number of preimages of a function $u \in \mathcal{N}_\lambda(\mathcal{G})$, one can obtain sharper estimates. The following result will be used in Section 1.4.

Proposition 1.10. *Let $\mathcal{G} \in \mathbf{G_1}$ and let $u \in \mathcal{N}_\lambda(\mathcal{G})$ be a nonnegative function satisfying*

$$\#u^{-1}(t) \geq K \qquad \text{for a.e. } t \in \left(\inf_{\mathcal{G}} u, \sup_{\mathcal{G}} u\right)$$

for some integer $K \geq 1$. Then,

$$J_\lambda(u) \geq K \frac{s_\lambda}{2}.$$

Proof. Since $\mathcal{G} \in \mathbf{G_1}$, we have $|\mathcal{G}| = +\infty$. By Proposition 1.5 applied to the set $T = (\inf_{\mathcal{G}} u, \sup_{\mathcal{G}} u) = (0, \sup_{\mathcal{G}} u)$, the decreasing rearrangement u^* of u satisfies $u^* \in H^1(0,+\infty)$ and

$$\|u^*\|_{L^q(0,+\infty)} = \|u\|_{L^q(\mathcal{G})}, \qquad \text{for every } q \in [1,+\infty],$$

$$\|(u^*)'\|_{L^2(0,+\infty)} \leq \frac{1}{K}\|u'\|_{L^2(\mathcal{G})}.$$

Setting $u_K(x) := u^*(Kx)$, a standard change of variable shows that

$$\begin{aligned}
n_\lambda(u_K)^{p-2} &= \frac{\|u_K'\|^2_{L^2(0,+\infty)} + \lambda\|u_K\|^2_{L^2(0,+\infty)}}{\|u_K\|^p_{L^p(0,+\infty)}} \\
&= \frac{K\|(u^*)'\|^2_{L^2(0,+\infty)} + \frac{\lambda}{K}\|u^*\|^2_{L^2(0,+\infty)}}{\frac{1}{K}\|u^*\|^p_{L^p(0,+\infty)}} \\
&\leq \frac{\|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})}}{\|u\|^p_{L^p(\mathcal{G})}} \\
&= 1
\end{aligned}$$

and therefore, as $n_\lambda(u_K)u_K$ belongs to $\mathcal{N}_\lambda(0,+\infty)$, by (1.19)

$$\frac{s_\lambda}{2} \leq J_\lambda(n_\lambda(u_K)u_K) = \varkappa\, n_\lambda(u_K)^p \|u_K\|^p_{L^p(0,+\infty)} \leq \frac{\varkappa}{K}\|u\|^p_{L^p(\mathcal{G})} = \frac{1}{K}J_\lambda(u). \qquad \square$$

1.3 A general existence result

In this section, we prove a general existence result for positive solutions to (NLS$_\mathcal{G}$) attaining their maximum in the interior of a prescribed sufficiently long edge. Throughout this section, we work with metric graphs in class $\mathbf{G_1}$, most of the time requiring also that the graph satisfies assumption (H). However, the arguments described in this section can be adapted with minor modifications to cover even broader classes of graphs.

Let $\mathcal{G} \in \mathbf{G}$ be a graph satisfying assumption (H) and let e be one of its bounded edges. We set

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}$$

and we consider the doubly–constrained minimization problem

$$\mathcal{J}_{\mathcal{G},e}(\lambda) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e} J_\lambda(u). \tag{1.21}$$

Remark 1.11. The set X_e is closed in the weak topology of $H^1(\mathcal{G})$. Indeed, if $u_n \in X_e$ and $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, then by semicontinuity and uniform convergence of u_n to u on e, we have

$$\|u\|_{L^\infty(\mathcal{G})} \le \liminf_n \|u_n\|_{L^\infty(\mathcal{G})} = \liminf_n \|u_n\|_{L^\infty(e)} = \|u\|_{L^\infty(e)}.$$

The next two theorems state the main results of this section.

Theorem 1.12. *There exists $\overline{R} > 0$ depending only on λ and p such that, if $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H) and has a bounded edge e of length $R \ge \overline{R}$, then $\mathcal{J}_{\mathcal{G},e}(\lambda)$ is attained.*

Theorem 1.13. *Let $\mathcal{G} \in \mathbf{G_1}$ satisfy assumption (H) and have a bounded edge e of length R. Let $\ell_0 \le \inf_{e \in \mathbb{E}} |e|$. Then there exists $\widetilde{R}$ depending only on ℓ_0, λ and p such that if $R \ge \widetilde{R}$ and u is a minimizer for $\mathcal{J}_{\mathcal{G},e}(\lambda)$, then $u \in \mathcal{S}_\lambda(\mathcal{G})$ and $u > 0$ or $u < 0$ on $\mathcal{G}$. Moreover, $\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G} \setminus e)}$.*

Remark 1.14. Interestingly, the thresholds in the two previous results have different dependances. In the first one, there is no dependence on the rest of the graph while in the second one, there is a dependence on the infimum of the length of all edges.

A consequence of the preceding theorems is the following multiplicity result.

Theorem 1.15. *Under the assumptions of Theorem 1.13, there exists $\widetilde{R} > 0$ depending only on ℓ_0, λ and p such that for every bounded edge e of length larger than $\widetilde{R}$, problem* ($\mathrm{NLS}_{\mathcal{G}}$) *has a positive solution attaining its absolute maximum on e only. Hence, if $\mathcal{G}$ has n bounded edges of length greater than $\widetilde{R}$, then problem* ($\mathrm{NLS}_{\mathcal{G}}$) *has at least n distinct positive solutions.*

We need several lemmas for the proof of Theorems 1.12–1.13.

Lemma 1.16. *Let $\mathcal{G} \in \mathbf{G_1}$ satisfy assumption (H). If $(u_n)_n \subseteq H^1(\mathcal{G})$ is a bounded sequence such that $\liminf_n \|u_n\|_p > 0$ and $\lim_n L(u_n) = \theta > 0$, then*

$$\liminf_{n\to\infty} \|u_n\|_p^p \geq \frac{s_\theta}{\varkappa},$$

where s_θ is defined in (1.17)*.*

Proof. Let $\theta_n := L(u_n)$, so that $u_n \in \mathcal{N}_{\theta_n}(\mathcal{G})$ and $\theta = \lim_n \theta_n$. Then, since $(u_n)_n$ is bounded in $L^2(\mathcal{G})$, $\liminf_n \|u_n\|_p > 0$ and $\lim_n \theta_n = \theta$, we have

$$\begin{aligned}
\pi_\theta(u_n)^{p-2} &= \frac{\|u_n'\|_2^2 + \theta\|u_n\|_2^2}{\|u_n\|_p^p} \\
&= \frac{\|u_n'\|_2^2 + \theta_n\|u_n\|_2^2}{\|u_n\|_p^p} + (\theta - \theta_n)\frac{\|u_n\|_2^2}{\|u_n\|_p^p} \\
&= 1 + (\theta - \theta_n)\frac{\|u_n\|_2^2}{\|u_n\|_p^p},
\end{aligned}$$

whence $\lim_n \pi_\theta(u_n) = 1$. By Theorem 1.9, we conclude that

$$\begin{aligned}
s_\theta &= \inf_{v \in \mathcal{N}_\theta(\mathcal{G})} J_\theta(v) \\
&\leq \liminf_n J_\theta(\pi_\theta(u_n)u_n) \\
&= \liminf_n \varkappa \pi_\theta(u_n)^p \|u_n\|_p^p \\
&= \liminf_n \varkappa \|u_n\|_p^p
\end{aligned}$$

and the result follows. □

The next result describes how minimizing sequences for problem (1.21) behave.

Lemma 1.17. *Let $\mathcal{G} \in \mathbf{G_1}$ be a metric graph and let $e \in \mathcal{G}$ be a bounded edge. Let $(u_n)_n \subseteq \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ be a minimizing sequence for J_λ in $\mathcal{N}_\lambda(\mathcal{G}) \cap X_e$. Then, $(u_n)_n$ admits a subsequence (not relabeled) such that*

1) $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, $u_n \to u$ in $L^q_{loc}(\mathcal{G})$ for every $q \in [1, +\infty]$, $\inf \|u_n\|_p > 0$;

2) $\lim_n \|u_n\|_2^2 = \mu > 0$;

3) $u \in X_e \setminus \{0\}$;

4) $L(u) \leq \lambda$;

5) if $L(u) = \lambda$, then u is a minimizer for J_λ on $\mathcal{N}_\lambda(\mathcal{G}) \cap X_e$;

6) if $L(u) < \lambda$, then $m := \|u\|_2^2 < \mu$ and

$$\lim_n L(u_n - u) = \lambda + \frac{m}{\mu - m}(\lambda - L(u)) \,. \tag{1.22}$$

Proof. Since $(u_n)_n$ is a minimizing sequence, up to subsequences, *1)* and *2)* can be deduced from Proposition 1.6.

Let us notice that $u \not\equiv 0$ since if this were not the case, by L^∞_{loc} convergence, thus convergence in $L^\infty(e)$, one has

$$\begin{aligned} \|u_n\|^p_{L^p(\mathcal{G})} &\leq \|u_n\|^{p-2}_{L^\infty(\mathcal{G})} \|u_n\|^2_{L^2(\mathcal{G})} \\ &= \|u_n\|^{p-2}_{L^\infty(e)} \|u_n\|^2_{L^2(\mathcal{G})} \\ &\xrightarrow[n\to\infty]{} 0, \end{aligned}$$

violating *1)*. Moreover, since X_e is weakly closed (see Remark 1.11), u belongs to X_e and *3)* is proved.

To prove *4)*, we observe that by lower semicontinuity, since $n_\lambda(u)u$ belongs to $\mathcal{N}_\lambda(\mathcal{G}) \cap X_e$, the estimate

$$\begin{aligned} \varkappa \|u\|_p^p &\leq \liminf_n \varkappa \|u_n\|_p^p \\ &= \liminf_n J_\lambda(u_n) \\ &= \mathcal{J}_{\mathcal{G},e}(\lambda) \\ &\leq J_\lambda(\pi_\lambda(u)u) \\ &= \varkappa\, \pi_\lambda(u)^p \|u\|_p^p \end{aligned}$$

yields $\pi_\lambda(u) \geq 1$, which is equivalent to $L(u) \leq \lambda$.

Now if $L(u) = \lambda$, then $u \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ and the last estimate shows that u is a minimizer, which is *5)*.

Finally, let us assume that $L(u) < \lambda$, so that $u_n - u \not\equiv 0$ for all n large. By weak lower semicontinuity, $m \leq \mu$.

To prove that the inequality is strict, we first observe that, by the Gagliardo–Nirenberg inequality (1.8),

$$\begin{aligned} L(u_n - u) &= \frac{\|u_n - u\|_p^p - \|u_n' - u'\|_2^2}{\|u_n - u\|_2^2} \\ &\leq \frac{\|u_n - u\|_p^p}{\|u_n - u\|_2^2} \\ &\leq \frac{K\|u_n - u\|_2^{\frac{p}{2}+1}\|u_n' - u'\|_2^{\frac{p}{2}-1}}{\|u_n - u\|_2^2} \\ &= K\|u_n - u\|_2^{\frac{p}{2}-1}\|u_n' - u'\|_2^{\frac{p}{2}-1} \\ &\leq C, \end{aligned} \tag{1.23}$$

for every n, since $(u_n)_n$ is bounded in $H^1(\mathcal{G})$.

Now, by the Brezis–Lieb Lemma [86], as $n \to \infty$,

$$\begin{aligned} L(u_n - u) &= \frac{\|u_n - u\|_p^p - \|u_n' - u'\|_2^2}{\|u_n - u\|_2^2} \\ &= \frac{\|u_n\|_p^p - \|u_n'\|_2^2 - \|u\|_p^p + \|u'\|_2^2 + o(1)}{\|u_n\|_2^2 - \|u\|_2^2 + o(1)} \\ &= \frac{\lambda\|u_n\|_2^2 - L(u)\|u\|_2^2 + o(1)}{\|u_n\|_2^2 - \|u\|_2^2 + o(1)} \\ &= \lambda + \frac{(\lambda - L(u))\|u\|_2^2 + o(1)}{\|u_n\|_2^2 - \|u\|_2^2 + o(1)}. \end{aligned}$$

Since $L(u) < \lambda$ and $u \not\equiv 0$, we see that $\mu = \lim_n \|u_n\|_2^2 > \|u\|_2^2 = m$, since otherwise (1.23) is violated. Letting $n \to \infty$ in the preceding equality, we obtain (1.22) and the proof is complete. □

Remark 1.18. As a consequence of Lemma 1.7, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that, if $\mathcal{G}$ is any metric graph containing a bounded edge e of length greater than R_ε, then $\mathcal{J}_{\mathcal{G},e}(\lambda) \leq s_\lambda + \varepsilon$.

We are now ready to prove the main results of this section.

Proof of Theorem 1.12. Let us fix $\varepsilon > 0$ such that

$$\begin{cases} s_{\lambda+\varepsilon} = s_1(\lambda + \varepsilon)^\alpha \leq 2s_1\lambda^\alpha = 2s_\lambda, \\ \lambda - C\varepsilon > 0, \\ (\lambda + \varepsilon)^\alpha < \lambda^\alpha + (\lambda - C\varepsilon)^\alpha, \end{cases} \tag{1.24}$$

where

$$C := 16s_1^2/\varkappa^2. \tag{1.25}$$

We observe that the constant C depends only on p. Therefore, ε depends only on p and λ but not on $\mathcal{G}$.

Let $\overline{R}$ be large enough so that for every $R \geq \overline{R}$,

$$\mathcal{J}_{\mathcal{G},e}(\lambda) < s_{\lambda+\varepsilon},$$

which is possible by Remark 1.18. Again, we observe that $\overline{R}$ depends only on p and λ but not on $\mathcal{G}$.

Let $(u_n)_n \subseteq \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ be a minimizing sequence for J_λ such that

$$J_\lambda(u_n) \leq s_{\lambda+\varepsilon}$$

for every n. Applying Lemma 1.17, $(u_n)_n$ has (up to subsequences) a weak limit $u \in X_e \setminus \{0\}$ such that $L(u) \leq \lambda$ and, in case equality holds, u is the required minimizer.

Let us now show that $L(u) < \lambda$ cannot happen. This will end the proof. Indeed, if $L(u) < \lambda$, then by (1.22),

$$\lim_n L(u_n - u) = \lambda + \frac{m}{\mu - m}(\lambda - L(u)) > \lambda > 0,$$

with $0 < m = \|u\|_2^2 < \lim_n \|u_n\|_2^2 = \mu$. We note that

$$\liminf_n \|u_n - u\|_p > 0, \tag{1.26}$$

as otherwise, up to a subsequence, $(u_n)_n$ converges to u in $L^p(\mathcal{G})$ and by lower semicontinuity we would have

$$\lambda > L(u) = \frac{\|u\|_p^p - \|u'\|_2^2}{\|u\|_2^2} \geq \liminf_n \frac{\|u_n\|_p^p - \|u_n'\|_2^2}{\|u_n\|_2^2} = \lambda,$$

a contradiction.

As $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H), (1.26), (1.22) and Lemma 1.16 give

$$\liminf_n \|u_n - u\|_p^p \geq \frac{s_1}{\varkappa}\left(\lambda + \frac{m}{\mu - m}(\lambda - L(u))\right)^\alpha. \tag{1.27}$$

Let us notice that

$$\lim_n \left(\|u_n'\|_2^2 + \lambda\|u_n\|_2^2\right) = \frac{1}{\varkappa}\lim_n J_\lambda(u_n) \leq \frac{s_{\lambda+\varepsilon}}{\varkappa} \leq \frac{2s_1}{\varkappa}\lambda^\alpha, \tag{1.28}$$

hence

$$\mu = \lim_n \|u_n\|_2^2 \leq \lim_n \frac{1}{\lambda}\left(\|u_n'\|_2^2 + \lambda\|u_n\|_2^2\right) \leq \frac{2s_1}{\varkappa}\lambda^{\alpha-1}. \tag{1.29}$$

Furthermore, as

$$\lambda\|u_n\|_2^2 \le \|u_n'\|_2^2 + \lambda\|u_n\|_2^2 = \|u_n\|_p^p \le \|u_n\|_\infty^{p-2}\|u_n\|_2^2,$$

we see that

$$\|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} = \lim_n \|u_n\|_{L^\infty(e)} = \lim_n \|u_n\|_{L^\infty(\mathcal{G})} \ge \lambda^{\frac{1}{p-2}}$$

and therefore, by the Gagliardo-Nirenberg inequality (1.9) and (1.28), recalling that $m = \|u\|_2^2$, we have

$$\lambda^{\frac{4}{p-2}} \le \|u\|_\infty^4 \le 4m\|u'\|_2^2 \le 4m \liminf_n \|u_n'\|_2^2 \le \frac{4m}{\varkappa} s_{\lambda+\varepsilon} \le \frac{8s_1 m}{\varkappa}\lambda^\alpha. \tag{1.30}$$

Thus, recalling from (1.17) the value of α, we see from (1.29) and (1.30) that

$$\frac{\mu}{m} \le \frac{16 s_1^2}{\varkappa^2}\lambda^{2\alpha-1-\frac{4}{p-2}} = C, \tag{1.31}$$

where C is given by (1.25).

In conclusion, by the Brezis–Lieb Lemma and (1.27), we have

$$\begin{aligned} s_1(\lambda+\varepsilon)^\alpha \ge \mathcal{J}_{\mathcal{G},e}(\lambda) = \lim_n J_\lambda(u_n) = \lim_n \varkappa\|u_n\|_p^p = \lim_n \varkappa\left(\|u_n - u\|_p^p + \|u\|_p^p\right) \\ \ge s_1\left(\lambda + \frac{m}{\mu - m}(\lambda - L(u))\right)^\alpha + \varkappa\|u\|_p^p. \end{aligned} \tag{1.32}$$

Neglecting the last term, we obtain

$$\varepsilon \ge \frac{m}{\mu - m}(\lambda - L(u)),$$

or, rearranging terms and using (1.31),

$$L(u) \ge \lambda - \frac{\mu - m}{m}\varepsilon > \lambda - C\varepsilon > 0.$$

Using this, we see from Lemma 1.16 that

$$\varkappa\|u\|_p^p \ge s_1 L(u)^\alpha \ge s_1(\lambda - C\varepsilon)^\alpha.$$

Hence, by (1.32),

$$(\lambda+\varepsilon)^\alpha \ge \lambda^\alpha + (\lambda - C\varepsilon)^\alpha$$

which contradicts (1.24). □

Proof of Theorem 1.13. First, let us recall that, letting

$$\mathcal{D} := \left\{ u \in H^1(\mathbb{R}) \;\middle|\; u \geq 0, u \text{ is even}, u \text{ is nonincreasing on } [0, +\infty) \right\}, \tag{1.33}$$

for every $\xi > 0$ there exists $\delta > 0$ such that, for every $u \in \mathcal{N}_\lambda(\mathbb{R}) \cap \mathcal{D}$ with $J_\lambda(u) \leq s_\lambda + \delta$, the inequality

$$\|u - \phi_\lambda\|_{H^1} \leq \xi$$

holds. This can be readily seen by standard compactness arguments and the uniqueness of the minimizer of J_λ in $\mathcal{N}_\lambda(\mathbb{R}) \cap \mathcal{D}$ (see e.g [98, 214]).

In particular, there exists $\delta > 0$ such that, for every $u \in \mathcal{N}_\lambda(\mathbb{R}) \cap \mathcal{D}$ with $J_\lambda(u) \leq s_\lambda + \delta$, the inequality

$$\|u - \phi_\lambda\|_{H^1} \leq \frac{1}{3}\|\phi'_\lambda\|_{L^2(-\ell_0/2,\ell_0/2)} \tag{1.34}$$

holds. Let $0 < \varepsilon \leq \delta$ satisfy

$$\left[\frac{4}{5}\left(1 - \left(\frac{s_\lambda}{s_\lambda + \varepsilon}\right)^{\frac{p-2}{p}}\right)\frac{s_\lambda + \varepsilon}{\varkappa}\right]^{1/2} \leq \frac{1}{3}\|\phi'_\lambda\|_{L^2(-\ell_0/2,\ell_0/2)} \tag{1.35}$$

and, accordingly to Remark 1.18, let $\widetilde{R} > 0$ be such that

$$\mathcal{J}_{\mathcal{G},e}(\lambda) \leq s_\lambda + \varepsilon \tag{1.36}$$

for every edge e with length $R \geq \widetilde{R}$. Let us observe that $\widetilde{R}$ depends only on λ, p and ℓ_0.

Let $u_R \in \mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ be a function satisfying $J_\lambda(u_R) = \mathcal{J}_{\mathcal{G},e}(\lambda)$. To show that $u_R \in \mathcal{S}_\lambda(\mathcal{G})$, it is enough to prove that

$$\|u_R\|_{L^\infty(e)} > \|u_R\|_{L^\infty(\mathcal{G}\setminus e)}. \tag{1.37}$$

Indeed, (1.37) implies that u_R belongs to the relative interior of $\mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ and therefore it is not only a global minimizer of J_λ in the double constraint space, but also a local minimizer in $\mathcal{N}_\lambda(\mathcal{G})$ and, as such, solves (NLS$_\mathcal{G}$). Since $|u_R|$ belongs to $\mathcal{N}_\lambda(\mathcal{G}) \cap X_e$ and that $J_\lambda(u_R) = J_\lambda(|u_R|)$, we will assume that $u_R \geq 0$ on $\mathcal{G}$.

We proceed by contradiction and assume that

$$\|u_R\|_{L^\infty(e)} = \|u_R\|_{L^\infty(\mathcal{G}\setminus e)}.$$

Let $M_R := \|u_R\|_{L^\infty(e)}$. We denote by $\mathcal{B}$ the set of all bounded edges of the graph $\mathcal{G}$ and we set

$$\delta_R := \max_{h\in\mathcal{B}} \min_{x\in h} u_R(x) \qquad (0 \leq \delta_R \leq M_R). \tag{1.38}$$

The definition of δ_R as a maximum is correct even if $\mathcal{B}$ contains infinitely many edges. Indeed, as $\|u_R\|_2$ is finite and the length of the edges is bounded from below by ℓ_0, there is only a finite number of edges h where $\min_h u_R \geq t$, for every $t > 0$.

Since $\mathcal{G}$ satisfies assumption (H), by Step 1 of the proof of [17, Lemma 4.2],

$$\# u_R^{-1}(t) \geq 3 \quad \text{for almost every } t \in [\delta_R, M_R]. \tag{1.39}$$

Let us notice that the set

$$A_R := \Big\{ x \in \mathcal{G} \mid u_R(x) \in [\delta_R, M_R] \Big\}$$

contains at least one bounded edge of $\mathcal{B}$ (the one where the minimum of u_R is exactly δ_R) and therefore, we have

$$\Big| A_R \Big| \geq \ell_0.$$

Let now $\widehat{u}_R$ be the symmetric rearrangement of u_R. By Proposition 1.5 applied to $T_R = [\delta_R, M_R]$ and (1.39), defining $\ell_R = \Big| A_R \Big| / 2$, we have

$$\|\widehat{u}_R'\|^2_{L^2(-\ell_R, \ell_R)} = 4 \|(u_R^*)'\|^2_{L^2(0, 2\ell_R)} \leq \frac{4}{9} \|u_R'\|^2_{L^2(A_R)}. \tag{1.40}$$

Next, denoting[2] $B_R := \Big\{ x \in \mathcal{G} \mid u_R(x) \in [0, \delta_R) \Big\}$, since by assumption (H)

$$\# u_R^{-1}(t) \geq 2 \quad \text{for almost every } t \in (0, \delta_R),$$

we obtain

$$\|\widehat{u}_R'\|_{L^2(\mathbb{R} \setminus (-\ell_R, \ell_R))} \leq \|u_R'\|_{L^2(B_R)}.$$

From these relations it follows

$$\|\widehat{u}_R'\|^2_{L^2(\mathbb{R})} \leq \frac{4}{9} \|u_R'\|^2_{L^2(A_R)} + \|u_R'\|^2_{L^2(B_R)} = \|u_R'\|^2_{L^2(\mathcal{G})} - \frac{5}{9} \|u_R'\|^2_{L^2(A_R)}$$

so that

$$n_\lambda(\widehat{u}_R)^{p-2} \leq \frac{\|u_R'\|^2_{L^2(\mathcal{G})} + \lambda \|u_R\|^2_{L^2(\mathcal{G})} - \frac{5}{9} \|u_R'\|^2_{L^2(A_R)}}{\|u_R\|^p_{L^p(\mathcal{G})}} = 1 - \frac{5}{9} \frac{\|u_R'\|^2_{L^2(A_R)}}{\|u_R\|^p_{L^p(\mathcal{G})}}. \tag{1.41}$$

Since $n_\lambda(\widehat{u}_R)\widehat{u}_R$ belongs to $\mathcal{N}_\lambda(\mathbb{R})$, by (1.18) we obtain

$$s_\lambda \leq J_\lambda(n_\lambda(\widehat{u}_R)\widehat{u}_R) = \varkappa n_\lambda(\widehat{u}_R)^p \|\widehat{u}_R\|_p^p = \varkappa n_\lambda(\widehat{u}_R)^p \|u_R\|_p^p = n_\lambda(\widehat{u}_R)^p J_\lambda(u_R). \tag{1.42}$$

Using (1.42) and recalling that $J_\lambda(u_R) = \mathcal{J}_{\mathcal{G},e}(\lambda)$, (1.36) and (1.41) imply that

$$s_\lambda \leq J_\lambda(n_\lambda(\widehat{u}_R)\widehat{u}_R) \leq s_\lambda + \varepsilon. \tag{1.43}$$

Since, by definition $n_\lambda(\widehat{u}_R)\widehat{u}_R$ belongs to $\mathcal{D}$ (with $\mathcal{D}$ defined by (1.33)), (1.43), (1.34) and the fact that $\varepsilon \leq \delta$ imply that

$$\|\phi_\lambda - n_\lambda(\widehat{u}_R)\widehat{u}_R\|_{H^1(\mathbb{R})} \leq \frac{1}{3} \|\phi_\lambda'\|_{L^2(-\ell_0/2, \ell_0/2)}.$$

---

[2]If $\delta_R = 0$, the set B_R is empty, $A_R = \mathcal{G}$ and the proof is simpler, working with A_R only.

This last inequality implies that

$$\|\phi_\lambda'\|_{L^2(-\ell_0/2,\ell_0/2)} \leq \frac{\|\phi_\lambda'\|_{L^2(-\ell_0/2,\ell_0/2)}}{3} + \|n_\lambda(\widehat{u}_R)\widehat{u}_R'\|_{L^2(-\ell_0/2,\ell_0/2)}. \tag{1.44}$$

In order to obtain a contradiction, we now prove that

$$\|n_\lambda(\widehat{u}_R)\widehat{u}_R'\|_{L^2(-\ell_0/2,\ell_0/2)} \leq \frac{\|\phi_\lambda'\|_{L^2(-\ell_0/2,\ell_0/2)}}{3}. \tag{1.45}$$

By (1.42), we have

$$\frac{s_\lambda}{J_\lambda(u_R)} \leq n_\lambda(\widehat{u}_R)^p.$$

Using the previous inequality, (1.36) and (1.41), we deduce that

$$\left(\frac{s_\lambda}{s_\lambda+\varepsilon}\right)^{1/p} \leq n_\lambda(\widehat{u}_R) \leq 1. \tag{1.46}$$

Using (1.41), (1.46) and $\|u_R\|^p_{L^p(\mathcal{G})} = \frac{\mathcal{J}_{\mathcal{G},e}(\lambda)}{\varkappa} \leq \frac{s_\lambda+\varepsilon}{\varkappa}$, we obtain

$$\|u_R'\|^2_{L^2(A_R)} \leq \left(1-\left(\frac{s_\lambda}{s_\lambda+\varepsilon}\right)^{\frac{p-2}{p}}\right)\frac{9}{5}\|u_R\|^p_{L^p(\mathcal{G})} \leq \frac{9}{5}\left(1-\left(\frac{s_\lambda}{s_\lambda+\varepsilon}\right)^{\frac{p-2}{p}}\right)\frac{s_\lambda+\varepsilon}{\varkappa}.$$

Combining this inequality with (1.35), (1.40), (1.46), and the fact that $2\ell_R \geq \ell_0$ for every n, we obtain

$$\|n_\lambda(\widehat{u}_R)\widehat{u}_R'\|_{L^2(-\ell_0/2,\ell_0/2)} \leq \left[\frac{4}{5}\left(1-\left(\frac{s_\lambda}{s_\lambda+\varepsilon}\right)^{\frac{p-2}{p}}\right)\frac{s_\lambda+\varepsilon}{\varkappa}\right]^{1/2} \leq \frac{\|\phi_\lambda'\|_{L^2(-\ell_0/2,\ell_0/2)}}{3}$$

which concludes the proof of (1.37).

Since $u_R \geq 0$ on $\mathcal{G}$, we deduce that $u_R > 0$ in $\mathcal{G}$ using the strong maximum principle (see Proposition D.2). □

1.4 Proof of Theorem 1.3

This section is devoted to the proof of the main result of the chapter. In fact, alternatives A1 and B1 are straightforward, whereas A2 will follow as a direct application of the results proved in Section 1.3. Conversely, case B2 is the most involved and will occupy the largest part of the section.

For the sake of completeness, each case A1–B2 is presented here independently from the others.

1.4.1 Case A1: $\mathcal{J}(\lambda) = \sigma(\lambda)$, both attained

As already pointed out in Section 1.1, this is the easiest case and there is essentially nothing to say. It is the case where $\mathcal{J}_{\mathcal{G}}(\lambda)$ is attained by an action ground state, which is of course also a least action solution. Straightforward examples for this are compact graphs, where ground states exist for every value of λ and p (see for example [124]), but many graphs realizing alternative A1 can be identified also in the noncompact setting. We will study in more detail examples of noncompact graphs admitting action ground states in Chapter 2 (see e.g. Theorem 2.26).

1.4.2 Case A2: $\mathcal{J}(\lambda) = \sigma(\lambda)$, neither attained

The proof of this alternative is one of the main results of the chapter and it relies heavily on Theorems 1.12–1.13 above. To exhibit a graph where A2 occurs, we will use the following construction.

On a real line we insert, for each integer $k \geq 1$, a node v_k at the point of coordinate k. At each v_k we attach a self-loop $\mathcal{L}_k$ of length k, by identifying v_k with the only vertex of $\mathcal{L}_k$. We obtain in this way the graph shown in Figure 1.1.

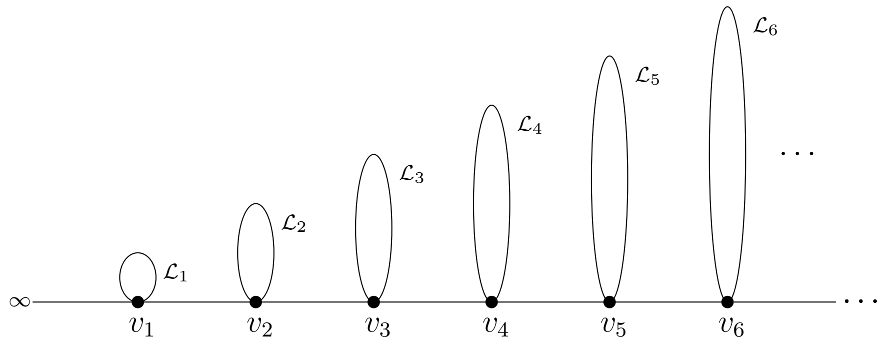


Figure 1.1: The graph $\mathcal{G}$ of Theorem 1.19

Theorem 1.19. *Let $\mathcal{G}$ be the graph depicted in Figure 1.1. For every $\lambda > 0$,*

$$\mathcal{J}_{\mathcal{G}}(\lambda) = \sigma_{\mathcal{G}}(\lambda) = s_\lambda$$

and neither $\mathcal{J}_{\mathcal{G}}(\lambda)$ nor $\sigma_{\mathcal{G}}(\lambda)$ is attained.

Proof. On the one hand, $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H) by construction, so that $\mathcal{J}_{\mathcal{G}}(\lambda) = s_\lambda$ and $\mathcal{J}_{\mathcal{G}}(\lambda)$ is not attained by Theorem 1.9. On the other hand, Theorems 1.12–1.13 ensure that, for sufficiently large k, there exists $u_k \in \mathcal{S}_\lambda(\mathcal{G})$ such that $J_\lambda(u_k) = \mathcal{J}_{\mathcal{G},\mathcal{L}_k}(\lambda)$. Hence, by Remark 1.18,

$$\sigma_{\mathcal{G}}(\lambda) \leq \liminf_{k\to\infty} J_\lambda(u_k) \leq s_\lambda,$$

in turn implying $\sigma_{\mathcal{G}}(\lambda) = \mathcal{J}_{\mathcal{G}}(\lambda)$ and ending the proof. □

1.4.3 Case B1: $\mathcal{J}_\mathcal{G}(\lambda) < \sigma_\mathcal{G}(\lambda)$, $\sigma_\mathcal{G}(\lambda)$ attained

As anticipated in Section 1.1, in view of the existing literature, it is easy to produce graphs realizing alternative B1. Indeed, it is for instance enough to let $\mathcal{G}$ be a star graph, i.e. a graph made of a finite number $N \geq 3$ of half-lines, glued together at their common origin (see the pictures in section II.2).

Star graphs have been widely investigated, as they provide the simplest example of noncompact graphs with half-lines. On the one hand, since star graphs satisfy assumption (H), it is immediate to see that $\mathcal{J}_\mathcal{G}(\lambda) = s_\lambda$ and that the level is not attained by Theorem 1.9. On the other hand, one can exploit the simple structure of these graphs to characterize explicitly the set of all solutions $\mathcal{S}_\lambda(\mathcal{G})$ (see e.g. [10, Section 2], this was also explained in section II.3.1):

- if N is odd, the problem (NLS$_\mathcal{G}$) only admits two nonzero solutions $\pm u$, the positive one given by a copy of the restriction of ϕ_λ to $[0,+\infty)$ on each half-line of the graph;
- if N is even, the set of non-zero solutions of (NLS$_\mathcal{G}$) is given by

$$\Big\{\pm u_{I,a} \mid I \subset \{1,\cdots,N\}, \#I = N/2, a \in [0,+\infty)\Big\},$$

where

$$u_{I,a}(x) := \begin{cases} \phi_\lambda(x+a) & \text{if } x \in \mathcal{H}_i, \text{ for some } i \in I, \\ \phi_\lambda(x-a) & \text{otherwise,} \end{cases}$$

with $\mathcal{H}_i$ being the i–th half-line of the graph.

Hence, we easily deduce that $\sigma_\mathcal{G}(\lambda) = \frac{N}{2}s_\lambda > s_\lambda$ and it is attained for instance by a function $u \in \mathcal{S}_\lambda(\mathcal{G})$ whose restriction to each half-line of the graph coincides with the restriction of the soliton ϕ_λ to $[0,+\infty)$.

1.4.4 Case B2: $\mathcal{J}_\mathcal{G}(\lambda) < \sigma_\mathcal{G}(\lambda)$, neither attained

The discussion of alternative B2 requires a deeper analysis with respect to the other cases. To prove that this alternative actually occurs, we will consider the one–parameter family of noncompact graphs constructed as follows.

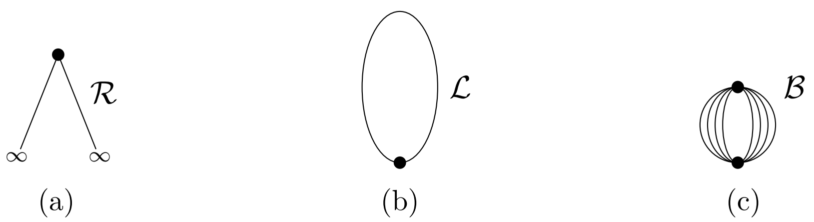


Figure 1.2: The building blocks of the graph $\mathcal{G}_N$. Two half-lines emanating from a vertex (a); a self-loop of length N (b); N edges of length 1 between two vertices (c).

Let $N \in \mathbb{Z}^{\geq 1}$ be fixed. Keeping in mind Figure 1.2, we consider a straight line on which we insert, for every $k \in \mathbb{Z}$, a vertex v_k at the point of coordinate k. At each v_k we attach a copy of the graph $\mathcal{R}$, called $\mathcal{R}_k$, by identifying v_k with the vertex of $\mathcal{R}_k$. Next, denoting by $\mathcal{L}$ the self-loop of length N in Figure 1.2(b), we attach at each v_k except v_0 a copy of the graph $\mathcal{L}$, called $\mathcal{L}_k$, by identifying v_k with the only vertex of $\mathcal{L}_k$. Finally, we attach at v_0 the graph $\mathcal{B}$ with N edges of length 1 in Figure 1.2(c), by identifying one of its two vertices with v_0. We call $\mathcal{G}_N$ the resulting graph, shown in Figure 1.3.

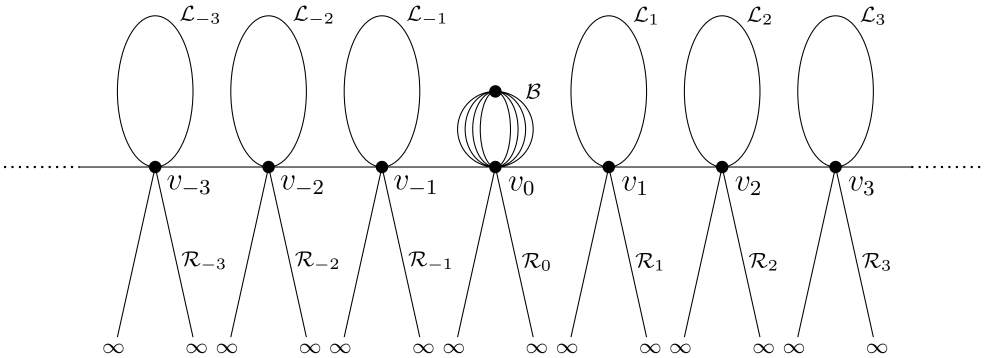


Figure 1.3: The graph $\mathcal{G}_N$

A second graph we will use below, that plays the role of "limit graph" with respect to $\mathcal{G}_N$, is depicted in Figure 1.4. It is exactly equal to $\mathcal{G}_N$ except that the subgraph $\mathcal{B}$ is replaced by a loop $\mathcal{L}_0$ identical to all other loops. Note that this graph is $\mathbb{Z}$–periodic. We call it $\widetilde{\mathcal{G}}_N$ and we label all its vertices, edges and subgraphs with the same letters as for those of $\mathcal{G}_N$, superposed by a tilde.

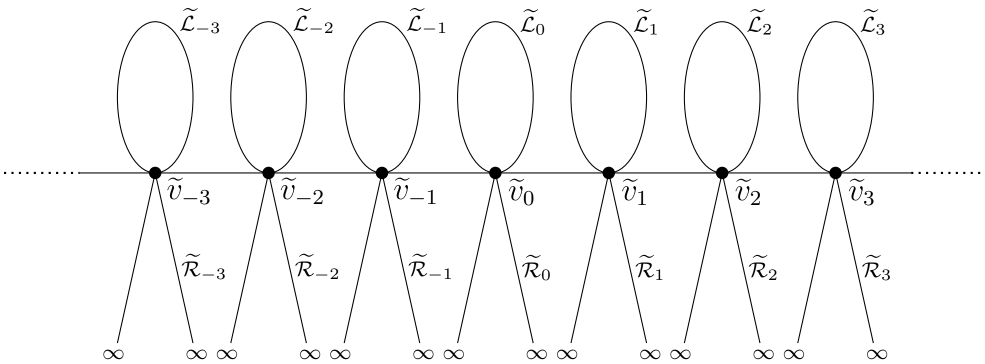


Figure 1.4: The graph $\widetilde{\mathcal{G}}_N$

Let us note that the graph $\mathcal{G}_N$ is made of edges of length 1 (the horizontal edges and the N edges of $\mathcal{B}$), of self-loops of length N and of half-lines. Let us also notice, as it will be important later, that the total length of $\mathcal{B}$ equals the length of the loop $\mathcal{L}$.

Exploiting the dependence of $\mathcal{G}_N$ on the parameter N, we have the following result, which proves the actual occurrence of case B2.

Theorem 1.20. *For every $N \in \mathbb{Z}^{\geq 1}$, let $\mathcal{G}_N$ be the graph in Figure 1.3. There exists $\overline{N} \in \mathbb{Z}^{\geq 1}$ such that for every $N \geq \overline{N}$,*

$$s_\lambda = \mathcal{J}_{\mathcal{G}_N}(\lambda) < \sigma_\lambda(\mathcal{G}_N)$$

and neither $\mathcal{J}_{\mathcal{G}_N}(\lambda)$ nor $\sigma_\lambda(\mathcal{G}_N)$ is attained.

Before turning to the proof of Theorem 1.20, let us introduce some notation.

Since λ is fixed, we omit to write it in quantities that depend on it, except for the levels s_λ, $\mathcal{J}_{\mathcal{G}_N}(\lambda)$ and $\sigma_\lambda(\mathcal{G}_N)$ (and the same for quantities relative to $\widetilde{\mathcal{G}}_N$).

Next, in order to have lighter notation, we omit to write the dependence from N in various quantities (such as the loops $\mathcal{L}_k$) and we keep it only in the names of the graphs. It is understood anyway that N is a parameter that we will tune in various proofs.

In what follows, we split the set of solutions to problem (NLS$_\mathcal{G}$), as

$$\mathcal{S}(\mathcal{G}_N) = \mathcal{S}_1(\mathcal{G}_N) \cup \mathcal{S}_2(\mathcal{G}_N) \cup \mathcal{S}_3(\mathcal{G}_N),$$

where

$$\begin{aligned}
\mathcal{S}_1(\mathcal{G}_N) &:= \Big\{u \in \mathcal{S}(\mathcal{G}_N) \ \big|\ \|u\|_{L^\infty(\mathcal{G}_N)} = \|u\|_{L^\infty(\mathcal{R}_k)} \text{ for some } k \in \mathbb{Z}\Big\},\\
\mathcal{S}_2(\mathcal{G}_N) &:= \Big\{u \in \mathcal{S}(\mathcal{G}_N) \ \big|\ \|u\|_{L^\infty(\mathcal{G}_N)} = \|u\|_{L^\infty(e)}\\
&\qquad\qquad\qquad \text{for some edge } e \in \mathcal{G}_N \text{ of length } 1\Big\},\\
\mathcal{S}_3(\mathcal{G}_N) &:= \Big\{u \in \mathcal{S}(\mathcal{G}_N) \ \big|\ \|u\|_{L^\infty(\mathcal{G}_N)} = \|u\|_{L^\infty(\mathcal{L}_k)}\\
&\qquad\qquad\qquad \text{for some self-loop } \mathcal{L}_k \in \mathcal{G}_N \text{ of length } N\Big\}.
\end{aligned}$$

We define analogously the sets $\mathcal{S}_1(\widetilde{\mathcal{G}}_N)$, $\mathcal{S}_2(\widetilde{\mathcal{G}}_N)$, $\mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ so that we also have

$$\mathcal{S}(\widetilde{\mathcal{G}}_N) = \mathcal{S}_1(\widetilde{\mathcal{G}}_N) \cup \mathcal{S}_2(\widetilde{\mathcal{G}}_N) \cup \mathcal{S}_3(\widetilde{\mathcal{G}}_N)\,. \tag{1.47}$$

Remark 1.21. By construction, both $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$ satisfy assumption (H), so we have immediately that for every $N \in \mathbb{Z}^{\geq 1}$,

$$\mathcal{J}_{\mathcal{G}_N}(\lambda) = \mathcal{J}_{\widetilde{\mathcal{G}}_N}(\lambda) = s_\lambda$$

and neither infimum is attained as a consequence of Theorem 1.9.

Remark 1.22. Let us observe that, if $\mathcal{G} \in \mathbf{G_1}$ satisfies assumption (H) and if u is a sign-changing solution, then u^+ and u^- belong to $\mathcal{N}_\lambda(\mathcal{G})$. Thus, by Theorem 1.9,

$$J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-) \geq 2 \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) = 2s_\lambda.$$

The proof of Theorem 1.20 relies on the next series of lemmas. As

$$J_\lambda(u) = J_\lambda(|u|),$$

we will assume without loss of generality in the rest of this section that all the functions $u \in \mathcal{N}_\lambda(\mathcal{G}_N)$ that we will consider will be non-negative.

Lemma 1.23. *Let $e \in \mathcal{G}_N$, $N \geq 2$, be a bounded edge. We identify $\mathcal{G}_N \setminus \mathcal{B}$ with $\widetilde{\mathcal{G}}_N \setminus \widetilde{\mathcal{L}}_0$ and we define $\widetilde{e} \in \widetilde{\mathcal{G}}_N$ by*

$$\widetilde{e} := \begin{cases} e & \text{if } e \notin \mathcal{B}, \\ \widetilde{\mathcal{L}}_0 & \text{if } e \in \mathcal{B}. \end{cases}$$

Then, for every $u \in \mathcal{N}_\lambda(\mathcal{G}_N) \cap X_e$, there exists $\widetilde{v} \in \mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{e}}$ such that

$$J_\lambda(\widetilde{v}) \leq J_\lambda(u). \tag{1.48}$$

Moreover, if for t in a set of positive measure one has

$$\#\{x \in \mathcal{B} \mid u(x) = t\} \geq 3, \tag{1.49}$$

then the inequality (1.48) *is strict.*

Proof. Since $\mathcal{B}$ contains a loop,

$$\#\{x \in \mathcal{B} \mid u(x) = t\} \geq 2 \quad \text{for every } t \in \Big(\min_{\mathcal{B}} u, \max_{\mathcal{B}} u\Big).$$

Letting $\widehat{u}(x)$ be the symmetric rearrangement of the restriction of u to $\mathcal{B}$, then $\widehat{u}$ belongs to $H^1(-N/2, N/2)$ and we have

$$\|\widehat{u}\|_{L^q(-N/2,N/2)} = \|u\|_{L^q(\mathcal{B})} \quad \forall q \in [1, +\infty], \quad \|\widehat{u}'\|_{L^2(-N/2,N/2)} \leq \|u'\|_{L^2(\mathcal{B})}. \tag{1.50}$$

Furthermore, as $\widehat{u}(p) = u(v_0)$ for some p in $[-N/2, N/2]$, we can view $\widehat{u}$ as a function on $\widetilde{\mathcal{L}}_0$ (after identifying $\widetilde{\mathcal{L}}_0$ with $(-N/2, N/2)$ and p with v_0). We can then define $v \in H^1(\widetilde{\mathcal{G}}_N)$ as

$$v(x) := \begin{cases} u(x) & \text{if } x \in \widetilde{\mathcal{G}}_N \setminus \widetilde{\mathcal{L}}_0 = \mathcal{G}_N \setminus \mathcal{B}, \\ \widehat{u}(x) & \text{if } x \in \widetilde{\mathcal{L}}_0. \end{cases}$$

The continuity of v is guaranteed since $\widehat{u}(p) = u(v_0)$. By construction, we have $\|v'\|_{L^2(\widetilde{\mathcal{G}}_N)} \leq \|u'\|_{L^2(\mathcal{G}_N)}$ and $\|v\|_{L^q(\widetilde{\mathcal{G}}_N)} = \|u\|_{L^q(\mathcal{G}_N)}$ for every $q \in [1, +\infty]$, leading to $n_\lambda(v) \leq 1$.

Moreover, by construction, if u attains its L^∞ norm on some edge $e \in \mathcal{G}_N \setminus \mathcal{B}$, then v attains its L^∞ norm on the corresponding edge of $\widetilde{\mathcal{G}}_N \setminus \widetilde{\mathcal{L}}_0$; if, instead, u attains its L^∞ norm on some edge of $\mathcal{B}$, then v attains it on $\widetilde{\mathcal{L}}_0$. This shows that $\widetilde{v} := n_\lambda(v)v$ belongs to $\mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{e}}$ and that

$$J_\lambda(\widetilde{v}) = J_\lambda(n_\lambda(v)v) = \varkappa n_\lambda(v)^p \|v\|_p^p \leq \varkappa \|u\|_p^p = J_\lambda(u).$$

Finally, in case (1.49) holds, the inequality in (1.50) is strict by Proposition 1.5, resulting in the strict inequality in (1.48). □

The next lemma shows that the action of solutions attaining their maximum on a half-line cannot be too close to s_λ.

Lemma 1.24. *There exists $\delta_1 > 0$ such that for every $N \in \mathbb{Z}^{\geq 1}$,*

$$\inf_{u \in \mathcal{S}_1(\mathcal{G}_N)} J_\lambda(u) \geq s_\lambda + \delta_1 \qquad \text{and} \qquad \inf_{u \in \mathcal{S}_1(\widetilde{\mathcal{G}}_N)} J_\lambda(u) \geq s_\lambda + \delta_1 \,.$$

Proof. We prove the statement explicitly for $\mathcal{G}_N$ only, as the argument works exactly in the same way for $\widetilde{\mathcal{G}}_N$.

Let $u \in \mathcal{S}_1(\mathcal{G}_N)$. If u is sign-changing then, by Remark 1.22, we know that $J_\lambda(u) \geq 2s_\lambda$. Thus we consider the case where u does not change sign. We may assume that $u \geq 0$ and, by the strong maximum principle, we have that $u > 0$ on $\mathcal{G}$.

Let us call $\mathcal{H}_1 \subset \mathcal{R}_k$, for some $k \in \mathbb{Z}$, the half-line where u attains its L^∞ norm and $\mathcal{H}_2$ the other half-line emanating from v_k. Let us assume that u attains its maximum at v_k. Then,

$$\#u^{-1}(t) \geq 3 \qquad \forall t \in (0, \|u\|_\infty), \tag{1.51}$$

since every value $t \in (0, \|u\|_\infty)$ is attained at least once in $\mathcal{H}_1$, in $\mathcal{H}_2$ and in $\mathcal{G}_N \setminus \mathcal{R}_k$.

Then, Proposition 1.10 applied with $K = 3$ yields

$$J_\lambda(u) \geq \frac{3}{2} s_\lambda. \tag{1.52}$$

Let us assume now that u attains its maximum in the interior of $\mathcal{H}_1$. Since u solves (NLS$_\mathcal{G}$), its restrictions to $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide with suitable parts of the soliton ϕ_λ (see Proposition C.2). In particular, since u attains its maximum in the interior of $\mathcal{H}_1$, we see that $u(x) = \phi_\lambda(x - a)$, for some $a > 0$, on $\mathcal{H}_1$. Therefore, by continuity at v_k, on $\mathcal{H}_2$ either $u(x) = \phi_\lambda(x - a)$ as well, or $u(x) = \phi_\lambda(x + a)$.

In the first case, we have a copy of $\phi_\lambda(x-a)$ on each of the two half-lines and the maximum of u is attained on both half-lines. But then, (1.51) holds again for u, since every value in $(u(v_k), \|u\|_\infty)$ is attained twice on each half-line and every value in $(0, u(v_k))$ is attained once on each half-line and at least once in $\mathcal{G}_N \setminus \mathcal{R}_k$. Thus we conclude, exactly as above, that (1.52) holds.

In the second case, the restriction of u to $\mathcal{R}_k$ is the whole soliton ϕ_λ, which is smooth. In particular, the derivatives of ϕ_λ at v_k, being opposed, do not contribute to the Kirchhoff condition. Therefore, u solves problem (NLS$_\mathcal{G}$) on $\mathcal{G}_N \setminus \mathcal{R}_k$ and, as such, u belongs to $\mathcal{N}_\lambda(\mathcal{G}_N \setminus \mathcal{R}_k)$. Then, by (1.16), we have

$$\|u\|^p_{L^p(\mathcal{G}_N\setminus\mathcal{R}_k)} \geq C,$$

with C depending only on λ and p. In conclusion,

$$J_\lambda(u) = \varkappa\left(\|u\|^p_{L^p(\mathcal{R}_k)} + \|u\|^p_{L^p(\mathcal{G}_N\setminus\mathcal{R}_k)}\right) \geq \varkappa\left(\|\phi_\lambda\|^p_{L^p(\mathbb{R})} + C\right) = s_\lambda + \varkappa C,$$

and the lemma is proved choosing $\delta_1 = \min\left\{\frac{1}{2}s_\lambda, \varkappa C\right\}$. □

Now we prove that an estimate similar to that of the previous lemma holds also for solutions attaining their maximum on an edge of length 1.

Lemma 1.25. *There exists $\delta_2 > 0$ such that for every $N \geq 2$,*

$$\inf_{u\in\mathcal{S}_2(\mathcal{G}_N)} J_\lambda(u) \geq s_\lambda + \delta_2 \qquad \text{and} \qquad \inf_{u\in\mathcal{S}_2(\widetilde{\mathcal{G}}_N)} J_\lambda(u) \geq s_\lambda + \delta_2\,. \tag{1.53}$$

Proof. We will prove the existence of $\delta_2 > 0$ such that for every $N \geq 2$,

$$\inf_{u\in\mathcal{N}_2(\widetilde{\mathcal{G}}_N)} J_\lambda(u) \geq s_\lambda + \delta_2$$

where

$$\mathcal{N}_2(\widetilde{\mathcal{G}}_N) := \Big\{u \in \mathcal{N}(\widetilde{\mathcal{G}}_N) \ \mid\ \|u\|_{L^\infty(\widetilde{\mathcal{G}}_N)} = \|u\|_{L^\infty(e)} \text{ for some edge } e \in \widetilde{\mathcal{G}}_N \text{ of length } 1\Big\}.$$

Then, the result will follow via Lemma 1.23 and (1.6).

Let $u \in \mathcal{N}_2(\widetilde{\mathcal{G}}_N)$. Let us assume that u attains its maximum on an edge e of length 1.

We first construct a new graph and we rearrange u on it. Let $\overline{e}$ be an edge of length 1. Let us attach a pair of half-lines $\mathcal{H}_1, \mathcal{H}_2$ to one of its vertices and another pair $\mathcal{H}_3, \mathcal{H}_4$ to the other one. We obtain in this way an H-shaped graph denoted by $\overline{\mathcal{G}}$. We now claim that there exists $v \in \mathcal{N}_\lambda(\overline{\mathcal{G}}) \cap X_{\overline{e}}$ such that

$$J_\lambda(v) \leq J_\lambda(u). \tag{1.54}$$

To see this, let v_k, v_{k+1} be the vertices of e and let $\widetilde{\mathcal{G}}_{v_k}, \widetilde{\mathcal{G}}_{v_{k+1}}$ be the connected components of $\widetilde{\mathcal{G}}_N \setminus e$ containing v_k and v_{k+1} respectively.

Let us note that both $\widetilde{\mathcal{G}}_{v_k}$ and $\widetilde{\mathcal{G}}_{v_{k+1}}$ satisfy assumption (H), hence

$$\begin{aligned}\#\{x \in \widetilde{\mathcal{G}}_{v_k} \mid u(x) = t\} \geq 2 \qquad &\text{for almost every } t \in \Big(0, \|u\|_{L^\infty(\widetilde{\mathcal{G}}_{v_k})}\Big),\\ \#\{x \in \widetilde{\mathcal{G}}_{v_{k+1}} \mid u(x) = t\} \geq 2 \qquad &\text{for almost every } t \in \Big(0, \|u\|_{L^\infty(\widetilde{\mathcal{G}}_{v_{k+1}})}\Big).\end{aligned}$$

Let u_1 and u_2 be the symmetric rearrangements on $\mathbb{R}$ of the restrictions of u to $\widetilde{\mathcal{G}}_{v_k}$, $\widetilde{\mathcal{G}}_{v_{k+1}}$ respectively, so that $u_1, u_2 \in H^1(\mathbb{R})$ and

$$\begin{gathered}\|u_1\|_{L^q(\mathbb{R})} = \|u\|_{L^q(\widetilde{\mathcal{G}}_{v_k})}, \quad \|u_2\|_{L^q(\mathbb{R})} = \|u\|_{L^q(\widetilde{\mathcal{G}}_{v_{k+1}})} \text{ for every } q \in [1,\infty],\\ \|u_1'\|_{L^2(\mathbb{R})} \leq \|u'\|_{L^2(\widetilde{\mathcal{G}}_{v_k})}, \quad \|u_2'\|_{L^2(\mathbb{R})} \leq \|u'\|_{L^2(\widetilde{\mathcal{G}}_{v_{k+1}})}.\end{gathered}$$

Furthermore, there exist $x_1, x_2 \in \mathbb{R}$ such that $u_1(x_1) = u(v_k)$ and $u_2(x_2) = u(v_{k+1})$.

Let us then define $\overline{u} \in H^1(\overline{\mathcal{G}})$ by

$$\overline{u}(x) := \begin{cases} u(x) & \text{if } x \in \overline{e},\\ u_1(x + x_1) & \text{if } x \in \mathcal{H}_1 \cup \mathcal{H}_2,\\ u_2(x + x_2) & \text{if } x \in \mathcal{H}_3 \cup \mathcal{H}_4,\end{cases}$$

where, with a slight abuse of notation, we identified $\overline{e} \in \overline{\mathcal{G}}$ with $e \in \widetilde{\mathcal{G}}_N$.

By construction, $\overline{u}$ attains its L^∞ norm on $\overline{e}$, $\|\overline{u}\|_{L^q(\overline{\mathcal{G}})} = \|u\|_{L^q(\widetilde{\mathcal{G}}_N)}$ for every q and $\|\overline{u}'\|_{L^2(\overline{\mathcal{G}})} \leq \|u'\|_{L^2(\widetilde{\mathcal{G}}_N)}$, so that $n_\lambda(\overline{u}) \leq 1$. Hence, setting $v := n_\lambda(\overline{u})\overline{u}$, we obtain $v \in \mathcal{N}_\lambda(\overline{\mathcal{G}}) \cap X_{\overline{e}}$ fulfilling (1.54). Thus,

$$J_\lambda(u) \geq \inf_{v \in \mathcal{N}_\lambda(\overline{\mathcal{G}}) \cap X_{\overline{e}}} J_\lambda(v) = \mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda)$$

and it suffices to show that $\mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda) > s_\lambda$.

Now, if $\mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda)$ is attained this is trivial by Theorem 1.9, since $\overline{\mathcal{G}} \in \mathbf{G}$ satisfies assumption (H). If $\mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda)$ is not attained and $(w_n)_n \subseteq \mathcal{N}_\lambda(\overline{\mathcal{G}}) \cap X_{\overline{e}}$ is a minimizing sequence, then Lemma 1.17 applies, yielding a weak limit $w \in X_{\overline{e}} \setminus \{0\}$ with $m := \|w\|_2^2 < \lim_n \|w_n\|_2^2 =: \mu$, $L(w) < \lambda$ and

$$\lim_n L(w_n - w) = \lambda + \frac{m}{\mu - m}(\lambda - L(w)) \geq \lambda.$$

In conclusion, as usual by the Brezis–Lieb Lemma and Lemma 1.16, we obtain

$$\begin{aligned}\mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda) &= \lim_n J_\lambda(w_n) = \lim_n \varkappa \|w_n\|_p^p = \varkappa \lim_n \Big(\|w_n - w\|_p^p + \|w\|_p^p\Big)\\ &\geq s_1 \lambda^\alpha + \varkappa \|w\|_p^p = s_\lambda + \varkappa \|w\|_p^p\end{aligned}$$

and the proof is complete also when $\mathcal{J}_{\overline{\mathcal{G}},\overline{e}}(\lambda)$ is not attained as $w \neq 0$. □

In the next lemma, we study the properties of the graph $\widetilde{\mathcal{G}}_N$.

Lemma 1.26. *For every $\varepsilon > 0$, there exists= $N_\varepsilon := N_\varepsilon(\varepsilon, \lambda, p) \in \mathbb{Z}^{\geq 1}$ such that, for every $N \geq N_\varepsilon$,*

$$\inf_{u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)} J_\lambda(u) < s_\lambda + \varepsilon$$

and the infimum is attained.

Proof. By Remark 1.18, for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{Z}^{\geq 1}$ such that for every $N \geq N_\varepsilon$, the inequality

$$\mathcal{J}_{\widetilde{\mathcal{G}}_N, \widetilde{\mathcal{L}}_0}(\lambda) < s_\lambda + \varepsilon$$

holds. By taking N_ε even larger if necessary, Theorem 1.12 guarantees that $\mathcal{J}_{\widetilde{\mathcal{G}}_N, \widetilde{\mathcal{L}}_0}(\lambda)$ is attained by some u and, by Theorem 1.13, $u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$. If v is any other element of $\mathcal{S}_3(\widetilde{\mathcal{G}}_N)$, by the periodicity of $\widetilde{\mathcal{G}}_N$, there exists $\widetilde{v} \in \mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{\mathcal{L}}_0}$ (a translation of v) such that $J_\lambda(\widetilde{v}) = J_\lambda(v)$. Therefore, we obtain

$$J_\lambda(v) = J_\lambda(\widetilde{v}) \geq \mathcal{J}_{\widetilde{\mathcal{G}}_N, \widetilde{\mathcal{L}}_0}(\lambda) = J_\lambda(u),$$

which shows that $\inf_{u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)} J_\lambda(u) = \mathcal{J}_{\widetilde{\mathcal{G}}_N, \widetilde{\mathcal{L}}_0}(\lambda)$ is attained (by u). □

Corollary 1.27. *Let $\varepsilon \leq \frac{1}{2}\min\{\delta_1, \delta_2\}$, where δ_1, δ_2 are given by Lemmas 1.24–1.25 and N_ε be the corresponding number given by Lemma 1.26. For every $N \geq N_\varepsilon$, one has*

$$\sigma_\lambda(\widetilde{\mathcal{G}}_N) = \inf_{u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)} J_\lambda(u) < s_\lambda + \varepsilon$$

and $\sigma_\lambda(\widetilde{\mathcal{G}}_N)$ is attained.

Proof. By Lemmas 1.24–1.25, for every $N \geq 2$, the inequalities

$$\inf_{u \in \mathcal{S}_1(\widetilde{\mathcal{G}}_N)} J_\lambda(u) \geq s_\lambda + \varepsilon, \qquad \inf_{u \in \mathcal{S}_2(\widetilde{\mathcal{G}}_N)} J_\lambda(u) \geq s_\lambda + \varepsilon,$$

hold, while by Lemma 1.26

$$\inf_{u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)} J_\lambda(u) < s_\lambda + \varepsilon.$$

Therefore, in view of (1.47), we have

$$\sigma_\lambda(\widetilde{\mathcal{G}}_N) = \inf_{u \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)} J_\lambda(u)$$

and it is attained, again by Lemma 1.26. □

To proceed, we prove a further preliminary result, in the spirit of Lemma 1.23, that establishes a *strict* inequality when passing from $\mathcal{S}_3(\mathcal{G}_N)$ to $\mathcal{S}_3(\widetilde{\mathcal{G}}_N)$.

Lemma 1.28. *For every integer N large enough and for every $u \in \mathcal{S}_3(\mathcal{G}_N)$ such that $J_\lambda(u) < 2s_\lambda$, there exists $v \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ such that $J_\lambda(v) < J_\lambda(u)$.*

Proof. Let $u \in \mathcal{S}_3(\mathcal{G}_N)$ be such that $J_\lambda(u) < 2s_\lambda$. By Remark 1.22, u does not change sign. We assume that $u \geq 0$ and by the strong maximum principle, we have that $u > 0$ on $\mathcal{G}$.

If u attains its maximum in a loop $\mathcal{L}_k$, by Lemma 1.23 there exists a function $\widetilde{v} \in \mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{\mathcal{L}}_k}$ such that $J_\lambda(\widetilde{v}) \leq J_\lambda(u)$, where $\widetilde{\mathcal{L}}_k$ corresponds to $\mathcal{L}_k$ after the identification of $\mathcal{G}_N \setminus \mathcal{B}$ with $\widetilde{\mathcal{G}}_N \setminus \widetilde{\mathcal{L}}_0$.

If u is constant on m edges of $\mathcal{B}$, then it necessarily equals $\lambda^{\frac{1}{p-2}}$. Moreover,

$$2s_\lambda \geq J_\lambda(u) \geq m\varkappa\lambda^{\frac{p}{p-2}}$$

shows that m is bounded by a constant depending only on λ and p. Since N can be assumed as large as we wish, there are at least $N - m \geq 6$ edges of $\mathcal{B}$ on which u is not constant. Any pair of these edges in $\mathcal{B}$ forms a loop on which u is not constant. Since there are at least 3 such loops, necessarily

$$\#\{x \in \mathcal{B} \mid u(x) = t\} \geq 3$$

for t in a set of positive measure, so that by Lemma 1.23, the element $\widetilde{v}$ found above belongs to $\mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{\mathcal{L}}_k}$ and satisfies the strict inequality $J_\lambda(\widetilde{v}) < J_\lambda(u)$.

Finally, invoking again Theorems 1.12 and 1.13, provided N is sufficiently large, there exists $v \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ such that $J_\lambda(v) = \inf_{w \in \mathcal{N}_\lambda(\widetilde{\mathcal{G}}_N) \cap X_{\widetilde{\mathcal{L}}_k}} J_\lambda(w)$, hence

$$J_\lambda(v) \leq J_\lambda(\widetilde{v}) < J_\lambda(u),$$

and the proof is complete. □

Lemma 1.29. *There exists $\overline{N} \in \mathbb{Z}^{\geq 1}$ such that, for every $N \geq \overline{N}$, we have $\sigma_\lambda(\widetilde{\mathcal{G}}_N) > s_\lambda$,*

$$\sigma_\lambda(\mathcal{G}_N) = \sigma_\lambda(\widetilde{\mathcal{G}}_N) \tag{1.55}$$

and $\sigma_\lambda(\mathcal{G}_N)$ is not attained.

Proof. Let ε and N_ε be as in Corollary 1.27, so that there exists $w \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ satisfying

$$J_\lambda(w) = \sigma_\lambda(\widetilde{\mathcal{G}}_N) < s_\lambda + \varepsilon.$$

Let us note that $\sigma_\lambda(\widetilde{\mathcal{G}}_N) = J_\lambda(w) > s_\lambda$ by Remark 1.21.

We assume, in accordance with Theorem 1.13, that $w > 0$ and that w attains its maximum in the loop $\widetilde{\mathcal{L}}_0$, which is possible as usual by the periodicity of $\widetilde{\mathcal{G}}_N$. For every $\delta > 0$, we define $w_\delta \in H^1(\widetilde{\mathcal{G}}_N)$ by

$$w_\delta(x) := (w(x) - \delta)^+$$

and we notice that $w_\delta \to w$ strongly in $H^1(\widetilde{\mathcal{G}}_N)$ as $\delta \to 0$. The support of w_δ is, by construction, a bounded subgraph $\mathcal{G}_\delta$ of $\widetilde{\mathcal{G}}_N$ that can also be considered as a subgraph of $\mathcal{G}_N$: it suffices to embed it, for every δ, into $\mathcal{G}_N$ in such a way that it does not contain v_0. Thus, after extending w_δ to 0 in $\mathcal{G}_N \setminus \mathcal{G}_\delta$, we can view it as a function in $H^1(\mathcal{G}_N)$ attaining its maximum on some loop $\mathcal{L}_{k_\delta}$. Let us also note that, by strong convergence,

$$n_\lambda(w_\delta)^{p-2} = \frac{\|w_\delta'\|^2_{L^2(\mathcal{G}_N)} + \lambda\|w_\delta\|^2_{L^2(\mathcal{G}_N)}}{\|w_\delta\|^p_{L^p(\mathcal{G}_N)}} = \frac{\|w_\delta'\|^2_{L^2(\widetilde{\mathcal{G}}_N)} + \lambda\|w_\delta\|^2_{L^2(\widetilde{\mathcal{G}}_N)}}{\|w_\delta\|^p_{L^p(\widetilde{\mathcal{G}}_N)}} \to 1$$

as $\delta \to 0$.

By Theorems 1.12–1.13, for every $\delta > 0$ there exists $v_\delta \in \mathcal{S}(\mathcal{G}_N)$ such that

$$J_\lambda(v_\delta) = \inf_{u \in \mathcal{N}_\lambda(\mathcal{G}_N) \cap X_{\mathcal{L}_{k_\delta}}} J_\lambda(u).$$

Therefore, as $\delta \to 0$, we obtain

$$\sigma_\lambda(\mathcal{G}_N) \le J_\lambda(v_\delta) \le J_\lambda(n_\lambda(w_\delta)w_\delta) \to J_\lambda(w) = \sigma_\lambda(\widetilde{\mathcal{G}}_N),$$

showing that

$$\sigma_\lambda(\mathcal{G}_N) \le \sigma_\lambda(\widetilde{\mathcal{G}}_N). \tag{1.56}$$

Hence, by Lemmas 1.24–1.25, we have

$$\sigma_\lambda(\mathcal{G}_N) = \inf_{u \in \mathcal{S}_3(\mathcal{G}_N)} J_\lambda(u) < s_\lambda + \varepsilon.$$

Without loss of generality, let us assume that $\varepsilon < s_\lambda$. Next, for every $u \in \mathcal{S}_3(\mathcal{G}_N)$ with $J_\lambda(u) < s_\lambda + \varepsilon$, let $v \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ be the function provided by Lemma 1.28. Then, the inequalities

$$\sigma_\lambda(\widetilde{\mathcal{G}}_N) \le J_\lambda(v) < J_\lambda(u)$$

hold. Taking the infimum over u, we obtain $\sigma_\lambda(\widetilde{\mathcal{G}}_N) \le \sigma_\lambda(\mathcal{G}_N)$ which, coupled with (1.56), establishes (1.55).

Finally, to prove that $\sigma_\lambda(\mathcal{G}_N)$ is not attained, let us assume instead that there exists $u \in \mathcal{S}_3(\mathcal{G}_N)$ such that $J(u) = \sigma_\lambda(\mathcal{G}_N)$. By Lemma 1.28 again, let $v \in \mathcal{S}_3(\widetilde{\mathcal{G}}_N)$ satisfy $J_\lambda(v) < J_\lambda(u)$. Then, we obtain

$$\sigma_\lambda(\widetilde{\mathcal{G}}_N) \le J_\lambda(v) < J_\lambda(u) = \sigma_\lambda(\mathcal{G}_N),$$

contradicting (1.55). □

Proof of Theorem 1.20. It is enough to take $\overline{N}$ as in Lemma 1.29, the result is then a straightforward consequence of Remark 1.21 and Lemma 1.29. □

Chapter 2

Constant sign and sign changing NLS ground states on noncompact metric graphs

2.1 Presentation of the chapter

In this chapter we investigate the existence of constant sign and sign changing solutions of the nonlinear Schrödinger equation

$$-u'' + \lambda u = |u|^{p-2}u, \tag{2.1}$$

where $p > 2$ and λ are real numbers, on noncompact metric graphs under various assumptions.

Throughout this chapter we consider the class $\mathbf{G_2}$ of connected metric graphs $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ where the sets $\mathbb{V}$ and $\mathbb{E}$ are at most countable, every vertex $v \in \mathbb{V}$ has a finite degree, and the lengths of the edges $e \in \mathbb{E}$ are bounded away from zero (see Definition 2.10 below). A graph of this type is noncompact if at least one edge is unbounded (i.e. it is a half-line) or if the number of edges is infinite, giving rise to two classes of graphs that behave quite differently and that we will treat separately. Let us note that every half-line is considered to end at a "vertex at infinity" of degree one. The set of all such vertices of $\mathbb{V}$ is denoted by $\mathbb{V}_\infty$.

The analysis of differential equations on metric graphs experienced a massive growth in recent years, in particular motivated by the potential of graphs to serve as simple models for signal propagation in branched structures.

In this context, stationary nonlinear Schrödinger equations as (2.1) gained a prominent interest, as it is well-known that to any couple (u, λ) satisfying (2.1) there corresponds a standing wave solution $\psi(t, x) := \mathrm{e}^{\boldsymbol{i}\lambda t}\, u(x)$ of the time-dependent nonlinear Schrödinger equation

$$-\boldsymbol{i}\partial_t \psi(t, x) = \partial^2_{xx}\psi(t, x) + |\psi(t, x)|^{p-2}\psi(t, x)\,. \tag{2.2}$$

Nonlinear dispersive equations as (2.2) are largely studied in view of the role they play in many applications, as e.g. in the modeling of quantum mechanical systems in Bose–Einstein condensation or in the modeling of optical fibers.

It is clear however that, when considering any differential model on graphs, it is not enough to prescribe a differential equation that describes the behaviour of the system in the interior of each edge. The equation has indeed to be complemented with suitable matching conditions at the vertices, specifying how the interaction among edges behaves at the junctions.

In the case of nonlinear Schrödinger equations, there is a wide class of vertex conditions that can be considered. In the present chapter, we couple equation (2.1) with a specific choice of boundary conditions. Precisely, given a (not necessarily finite) set $Z \subseteq \mathbb{V} \setminus \mathbb{V}_\infty$ of degree 1 vertices, we are interested in solutions to the problem

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on every edge of } \mathcal{G}, \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{for every } v \in \mathbb{V} \setminus (Z \cup \mathbb{V}_\infty), \\ u(v) = 0 & \text{for every } v \in Z, \end{cases} \tag{NLS$_{\mathcal{G},Z}$}$$

where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ is the outgoing derivative along the edge e incident at the vertex v and $e \succ v$ means that the sum is extended to all such edges. The boundary condition for $v \notin Z$ (together with the continuity of u) is the homogeneous Kirchhoff condition, by far the most used in the literature. It is a natural analogue of the Neumann boundary condition to metric graphs (see e.g. [68, Section 1.4]). The boundary condition for $v \in Z$ is the homogeneous Dirichlet condition, which by contrast has been discussed only by few papers (see for instance [142]). Here we choose to include mixed conditions to highlight their role in the existence (or nonexistence) of various types of solutions to (NLS$_{\mathcal{G},Z}$). The requirement that all nodes of Z have degree 1 prevents that the graph be disconnected by the Dirichlet conditions.

Coupling the operator $-\mathrm{d}^2/\mathrm{d}x^2$ with our boundary conditions guarantees its self-adjointness.

This is a natural requirement in the analysis of quantum mechanical problems (see e.g. [5] for an overview on boundary conditions ensuring self-adjointness on metric graphs and [68, Section 1.4] for a thorough discussion).

Solutions to (NLS$_{\mathcal{G},Z}$) can be found by a variational approach that has been employed very frequently to deal with this kind of problem or with its variants. In our setting, the appropriate function space to set problem (NLS$_{\mathcal{G},Z}$) is

$$H^1_Z(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \,\middle|\, u(v) = 0 \text{ for every } v \in Z \right\}.$$

Standard arguments show that the $H^1(\mathcal{G})$ solutions of (NLS$_{\mathcal{G},Z}$) are exactly the critical points of the *action* functional $J_\lambda : H^1_Z(\mathcal{G}) \to \mathbb{R}$ defined by

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}, \tag{2.3}$$

that is of class C^2 on $H^1_Z(\mathcal{G})$.

Hereafter the parameter λ satisfies[1] $\lambda > -\omega_Z(\mathcal{G})$, where

$$\omega_Z(\mathcal{G}) := \inf_{v \in H^1_Z(\mathcal{G}) \setminus \{0\}} \frac{\|v'\|^2_{L^2(\mathcal{G})}}{\|v\|^2_{L^2(\mathcal{G})}}$$

is the bottom of the spectrum of the Laplacian on $\mathcal{G}$ associated to the boundary conditions in ($\mathrm{NLS}_{\mathcal{G},Z}$). This assumption is standard when working with this problem and is justified by the fact that, under it, the quadratic part in (2.3) provides a norm on $H^1(\mathcal{G})$ equivalent to the usual H^1 norm.

When looking at solutions to ($\mathrm{NLS}_{\mathcal{G},Z}$) from the variational perspective, one has to take into account that the functional J_λ is not bounded from below on $H^1_Z(\mathcal{G})$. A standard way to recover the notion of minimality is to introduce the Nehari manifold associated to J_λ, defined by

$$\begin{aligned}\mathcal{N}_{\lambda,Z}(\mathcal{G}) &:= \left\{ u \in H^1_Z(\mathcal{G}) \;\middle|\; u \neq 0,\ J'_\lambda(u)u = 0 \right\} \\ &= \left\{ u \in H^1_Z(\mathcal{G}) \;\middle|\; u \neq 0,\ \|u'\|^2_{L^2(\mathcal{G})} + \lambda \|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})} \right\}.\end{aligned}$$

Clearly, $\mathcal{N}_{\lambda,Z}(\mathcal{G})$ contains all solutions to the problem ($\mathrm{NLS}_{\mathcal{G},Z}$). It is a C^1-manifold diffeomorphic to the unit sphere of $H^1_Z(\mathcal{G})$ and is a natural constraint for J_λ, in the sense that constrained critical points of J_λ are in fact true critical points. Other approaches are possible (for instance one could constrain J_λ on the unit sphere of $L^p(\mathcal{G})$), but the Nehari approach has the advantage that it works also in cases where the nonlinearity is not homogeneous, thus providing a framework suitable to be generalized to a wider class of nonlinear terms.

Definition 2.1. We say that a function $u \in \mathcal{N}_{\lambda,Z}(\mathcal{G})$ is a *ground state* for problem ($\mathrm{NLS}_{\mathcal{G},Z}$) if

$$J_\lambda(u) = \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v).$$

Since we want to discuss both one-signed and sign changing solutions, we will also consider *nodal ground states*, roughly the analogue of ground states among sign changing functions. To define rigorously this notion, we let

$$u^+ := \max(u, 0), \qquad u^- := \min(u, 0)$$

and we define the *nodal Nehari set* as

$$\mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G}) := \left\{ u \in H^1_Z(\mathcal{G}) \;\middle|\; u^\pm \in \mathcal{N}_{\lambda,Z}(\mathcal{G}) \right\} = \left\{ u \in H^1_Z(\mathcal{G}) \;\middle|\; u^\pm \neq 0,\ J'_\lambda(u)u^\pm = 0 \right\}.$$

[1]This is a common assumption in the literature. Let us already point out that in the next chapter, concerned with bounded domains, it will be important to also consider cases where $-\lambda$ lies between the first and the second eigenvalue of the domain. In the current chapter, we will mostly deal with noncompact graphs, and in this setting, we will stick to the usual assumption $\lambda > -\omega_Z(\mathcal{G})$.

The nodal Nehari set contains all nodal solutions of ($\mathrm{NLS}_{\mathcal{G},Z}$) but, contrary to $\mathcal{N}_{\lambda,Z}(\mathcal{G})$, in general it is neither a manifold (see e.g. [53, 93, 318]) nor a natural constraint for J_λ, which causes some difficulties when proving existence results.

Definition 2.2. We say that a function $u \in \mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})$ is a *nodal ground state* for problem ($\mathrm{NLS}_{\mathcal{G},Z}$) if

$$J_\lambda(u) = \inf_{v \in \mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})} J_\lambda(v).$$

Let us recall that whenever they exist, ground states (respectively nodal ground states) provide constant sign solutions (respectively sign changing solutions) of (2.1) of minimal action.

As we already saw, to look for one-signed solutions Schrödinger equations as minimizers of suitable functionals is a standard strategy, that has been widely exploited on graphs in the mass constrained setting, in which case ground states are defined as minimizers of the energy functional $u \mapsto \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}$ restricted to an L^2-sphere (see e.g. [4, 9, 14, 19, 21, 70, 71, 72, 76, 77, 78, 134, 198, 199, 271, 272, 322] and references therein). In particular, these works have shown that the existence of ground states for the prescribed-mass problem on noncompact graphs is a rather unlikely event. Obstructions to existence are provided mostly by the topology of the graph, and sometimes also by its metrical properties.

Conversely, the action approach has not received much attention so far (some results in this direction can be found e.g. in [11, 120, 211, 263]). Let us stress that, though clearly critical points of the action on Nehari sets are also critical points of the energy on a suitable L^2-sphere and viceversa, the general relation between the action and the energy approach is not fully understood.[2]

First investigations in this direction started only recently in [129, 190] for NLS equations posed on domains of $\mathbb{R}^N$, but many of the results obtained therein extend with no difference to metric graphs. Specifically, if on the one hand those analyses proved that mass constrained ground states of the energy are always also ground states of the action, on the other hand they showed that the converse is in general not true. Hence, there may well exist action ground states that are not energy ground states among functions with the same mass, and the actual occurrence of this phenomenon on metric graphs has been proved e.g. in [22, Theorem 1.4] and [126, Proposition 2.4].

This somehow further motivates the independent study of the action ground state problem even in a context, as that of graphs, in which a well-developed theory of energy ground states is already available.

[2]We will give new results relating the "action approach" and the "energy approach" on bounded domains of $\mathbb{R}^N$ in the next chapter.

To the best of our knowledge, nodal ground states, and more generally sign changing solutions, on general metric graphs have never been investigated up to now, neither for the action nor for the energy problem. In the nodal setting, actually, the asymmetry between the action and the energy point of views is more pronounced, as it is not even clear how to define a general variational framework to deal with sign changing solutions with prescribed mass (that is, no analogue of the nodal Nehari set is known for the energy).[3]

The main concern of the present chapter is thus to begin a systematic study of action ground states and nodal ground states for ($\mathrm{NLS}_{\mathcal{G},Z}$). In particular, we are interested in characterizing the dependence of the problem on topological and metrical properties of the graph. On one side, as it is reasonable to expect, with purely Kirchhoff vertex conditions (i.e. $Z = \varnothing$), we will find that sometimes the same topological conditions that rule out mass constrained ground states do prevent existence of action ground states too (this is the case for graphs with half-lines, as in Theorem 2.6 below), and analogous conditions for nodal ground states will be identified. On the other side, we will also show that there are graphs (periodic ones and trees) for which action ground states exist for every admissible value of λ, whereas existence of mass constrained ground states depends on the value of the mass.

To begin the discussion of our results, let us observe that if $\mathcal{G}$ is compact, the existence of a ground state and of a nodal ground state can be proved thanks to standard compactness arguments.

Indeed, the compactness of the domain guarantees compactness for Sobolev embeddings $H^1(\mathcal{G}) \hookrightarrow L^p(\mathcal{G})$ (see Proposition A.9), which is all is needed to obtain strong convergence of minimizing sequences for the above variational problems (see e.g. [124, Proposition 3.1] for further details on such compactness results in the context of mass constrained critical points of the energy).

On the contrary, if $\mathcal{G}$ is noncompact, since as we said above the existence of ground states is unlikely, it is not surprising that the analogous eventuality for nodal ground states is even more so. As for ground states, we are going to derive sufficient conditions for both existence and nonexistence of nodal ground states involving topological features, metrical ones and combinations of the two.

Our analysis is based on a rather abstract procedure, typical of problems with lack of compactness, consisting in locating thresholds on the levels of J_λ, involving the so–called *level at infinity*

$$J_\lambda^\infty(\mathcal{G}; Z) := \inf\left\{\liminf_n J_\lambda(u_n) \;\middle|\; (u_n)_n \subseteq \mathcal{N}_{\lambda,Z}(\mathcal{G}),\ \lim_n u_n = 0 \text{ weakly in } H^1_Z(\mathcal{G})\right\}.$$

[3] Nevertheless, we will provide a strategy to find *normalized nodal solutions*, on bounded domains, in the next chapter.

Theorem 2.3. *Let $\mathcal{G} \in \mathbf{G_2}$ be a noncompact graph and $\lambda > -\omega_Z(\mathcal{G})$.*

(i) If

$$\inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) < J_\lambda^\infty(\mathcal{G}; Z), \tag{2.4}$$

then there exists a ground state of J_λ in $\mathcal{N}_{\lambda,Z}(\mathcal{G})$. Moreover, ground states have constant sign on $\mathcal{G} \setminus Z$.

(ii) If

$$\inf_{v \in \mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})} J_\lambda(v) < J_\lambda^\infty(\mathcal{G}; Z) + \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v), \tag{2.5}$$

then there exists a nodal ground state of J_λ in $\mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})$.

Theorem 2.4. *For every noncompact graph $\mathcal{G} \in \mathbf{G_2}$ and $\lambda > -\omega_Z(\mathcal{G})$,*

$$\inf_{v \in \mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})} J_\lambda(v) \geq 2 \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v). \tag{2.6}$$

If equality holds, then $\mathcal{G}$ admits no nodal ground states of J_λ in $\mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})$.

Remark 2.5. Inequality (2.6) also holds when $\mathcal{G}$ is a compact metric graph, and is then strict as nodal ground states always exist in this case.

This abstract strategy, though general, is insufficient to obtain existence results if one is not able to compute the level $J_\lambda^\infty(\mathcal{G}; Z)$ in concrete cases. Here we will detect specific properties of the graph that permit such computation and make sure that, in certain cases, the ground state level or the nodal ground state level lie below the level at infinity. Since this is where the topology and the metric of the graph become crucial, the analysis of such questions is carried out separately according to the class of graphs under study.

Let us first discuss the case of graphs with at least one half-line. For every such graph, $\omega_Z(\mathcal{G}) = 0$ and so the following results hold for every $\lambda > 0$.

We identify topological conditions on $\mathcal{G}$ that prevent the existence of ground states and nodal ground states. We describe them here in a concise way, referring to section 2.4 for a more detailed discussion. To begin with, let us recall that the set of *vertices at infinity* of $\mathcal{G}$ is

$$\mathbb{V}_\infty := \Big\{ v \in \mathbb{V} \;\Big|\; v \text{ is the vertex of degree 1 of some half-line} \Big\}$$

and let us define the set

$$F(\mathcal{G}) := \Big\{ e \in \mathbb{E} \;\Big|\; \text{at least one connected component of } (\mathbb{V}, \mathbb{E} \setminus \{e\}) \text{ has no vertices in } \mathbb{V}_\infty \cup Z \Big\}.$$

The set $F(\mathcal{G})$ is thus the set of edges of $\mathcal{G}$ (if any) whose removal disconnects $\mathcal{G}$ creating a connected component without vertices in $\mathbb{V}_\infty \cup Z$. Basically, if $F(\mathcal{G})$ is non-empty, there exists at least one "bridging" edge in the graph that, once removed, creates at least one connected component separated from all the half-lines and all the vertices with the homogeneous Dirichlet condition (see Figure 2.1 for a concrete illustration of $F(\mathcal{G})$ on different graph structures).

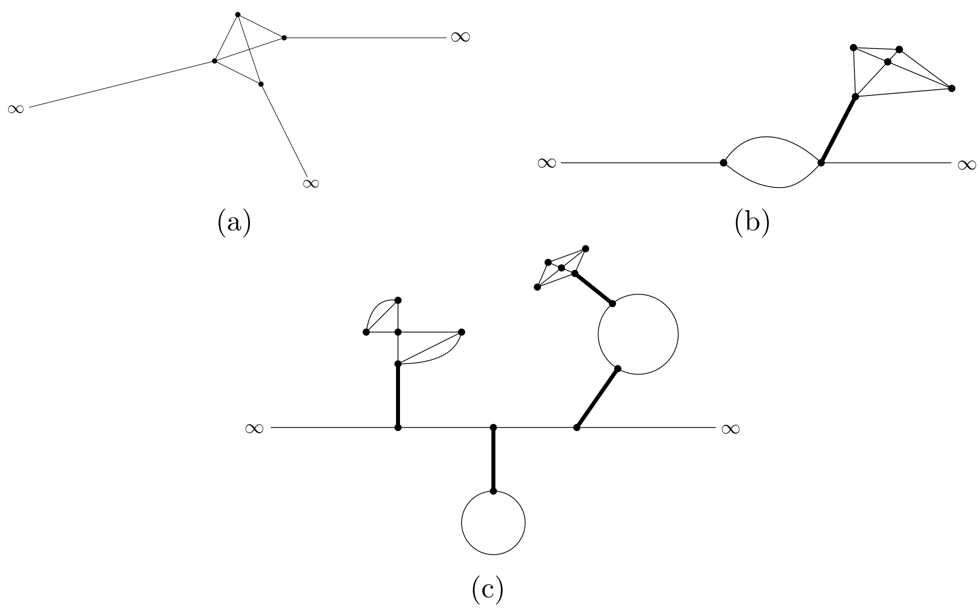


Figure 2.1: Examples of graphs $\mathcal{G}$ with corresponding set $F(\mathcal{G})$ containing 0 (A), 1 (B), and 4 (C) edges, respectively. Here, Kirchhoff conditions are assumed at every vertex, and edges in $F(\mathcal{G})$ are drawn thicker.

The cardinality of $F(\mathcal{G})$ is a purely topological notion and plays a fundamental role in the nonexistence of ground states and nodal ground states. Roughly, we will see that the presence of bridging edges in $F(\mathcal{G})$ may facilitate existence of such states, especially if the connected components without vertices in $\mathbb{V}_\infty \cup Z$ they isolate have a very simple structure. On the contrary, a low cardinality of $F(\mathcal{G})$ somehow corresponds to a too intricated graph structure not compatible with existence. This is stated rigorously in the next theorem, where for every $\lambda > 0$ we denote by

$$s_\lambda := \inf_{v \in \mathcal{N}_\lambda(\mathbb{R})} J_\lambda(v)$$

the ground state level of J_λ on $\mathbb{R}$.

Theorem 2.6. *Let $\mathcal{G} \in \mathbf{G_2}$ be a noncompact graph with at least one half-line and $\lambda > 0$, then*

$$\inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) \leq s_\lambda \tag{2.7}$$

and

$$\inf_{v \in \mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) \leq s_\lambda + \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v). \tag{2.8}$$

Moreover,

(i) if $\#F(\mathcal{G}) = 0$, then

$$\inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) = s_\lambda \tag{2.9}$$

and it is not achieved, unless $\mathcal{G}$ is isometric to $\mathbb{R}$ or to a "tower of bubbles" depicted in Figure 2.2;

(ii) if $\#F(\mathcal{G}) \leq 1$, then

$$\inf_{v \in \mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) = s_\lambda + \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) \tag{2.10}$$

and it is never achieved.

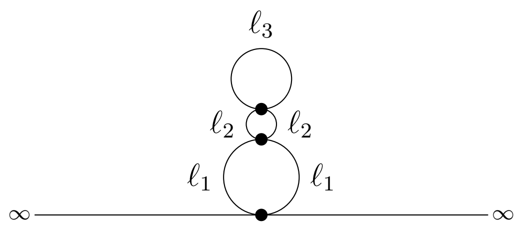
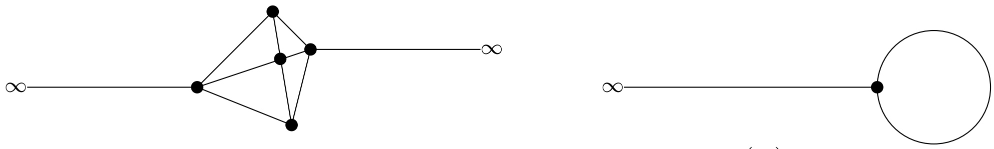


Figure 2.2: An example of a "tower of bubbles" graph. Each of these graphs, identified in Example 2.4 of [19], is built of a real line and a finite sequence of two-by-two tangent circles.

The preceding results are the main examples where the existence of ground states or nodal ground states is ruled out by a purely topological assumption on the graph. The families of graphs fulfilling each of the conditions of Theorem 2.6 are rather large and it is not difficult to exhibit examples of structures with these properties (see Figure 2.1(A)–(B) and Figure 2.3).

In the case of ground states, the condition $\#F(\mathcal{G}) = 0$ was already shown to prevent existence of mass constrained ground states of the energy in [19, Theorem 2.5], where it was named assumption (H). In contrast, the analogous condition for nodal ground states is established here for the first time. We underline that both assumptions on the cardinality of $F(\mathcal{G})$ are sharp for nonexistence. Indeed, in Section 2.4 we will show that there exist graphs satisfying $\#F(\mathcal{G}) \geq 1$ that admit a ground state, and graphs satisfying $\#F(\mathcal{G}) \geq 2$ that admit a nodal ground state.



(A) (B)

Figure 2.3: Further examples of graphs with half-lines satisfying $\#F(\mathcal{G}) = 0$ (A) and $\#F(\mathcal{G}) = 1$ (B).

These nonexistence results are complemented in Section 2.4 by a number of sufficient conditions to ensure existence.

Relying on techniques developed for energy ground states in [19, Section 6], [21, Section 3], it is easy to construct graphs where existence of action ground states is guaranteed by purely *topological* properties whenever $Z = \varnothing$ (see Theorem 2.26 and Figure 2.6 below). Notably, this turns out to be impossible as soon as $Z \neq \varnothing$. In this case, a necessary condition of *metrical* nature for the existence of ground states arises: the diameter of the set of all bounded edges of the graph must exceed a threshold depending on λ but not on the graph itself (Theorem 2.27). The same constraint holds true for nodal ground states, where it is not even needed to have a nonempty set Z (Theorem 2.29).

In addition to provide purely metrical nonexistence results, these theorems also imply that the interplay between topology and metric must be further exploited if one hopes to recover existence. We give examples of this fact by describing two general procedures to construct graphs where existence is granted (Theorems 2.30–2.34). The former relies on the metric only, and shows that one and two sufficiently long edges with vertices of degree 1 not in Z are enough to have ground states and nodal ground states, respectively. The latter basically consists in a suitable gluing of graphs hosting ground states to obtain structures supporting nodal ground states (see e.g. Figure 2.7).

The previous results highlight how the presence of at least one vertex with the homogeneous Dirichlet condition affects the existence of ground states and nodal ground states. Indeed, the fact that vertices in Z play the same role as those at infinity in the definition of $F(\mathcal{G})$ suggests the idea that an edge ending at a vertex with zero Dirichlet conditions behaves as an half-line. This heuristic comparison makes some sense, since H^1 functions tend to zero at infinity along each half-line. However, Theorem 2.27 unravels that the analogy with half-lines is not complete: edges with endpoints in Z are somehow "worse", as they make metrical assumptions necessary to obtain existence. We may be tempted to say that existence of ground states simply requires edges ending in Z to be long enough (and thus sufficiently close to half-lines), but this is not true in general, as ground states can exist even on graphs where all edges with vertices in Z are arbitrarily short (see Remark 2.28 below).

Section 2.5 of the chapter deals with noncompact graphs in the class $\mathbf{G_2}$ with an infinite number of edges, whose length is uniformly bounded from above. Given the huge variety of structures in this class, we confine ourselves to two subclasses of major relevance, that have already been studied extensively in the literature in many contexts (see e.g. [14, 71, 72, 130, 134, 162, 263, 267] for results related to those we discuss here): *periodic graphs* and *regular trees*.

Without entering the details of the definition of periodic graphs (for which we refer to [68, Definition 4.1.1]), let us mention here that we always work assuming that the set Z shares the same type of periodicity as the graph itself. Our main result in this respect completely describes the phenomenology from the point of view of ground states and nodal ground states (results in this direction for ground states on periodic graphs were already given in [263]).

As for graphs with half-lines, if $\mathcal{G}$ is a periodic graph then $\omega_Z(\mathcal{G}) = 0$, so that the next theorem holds again for every $\lambda > 0$.

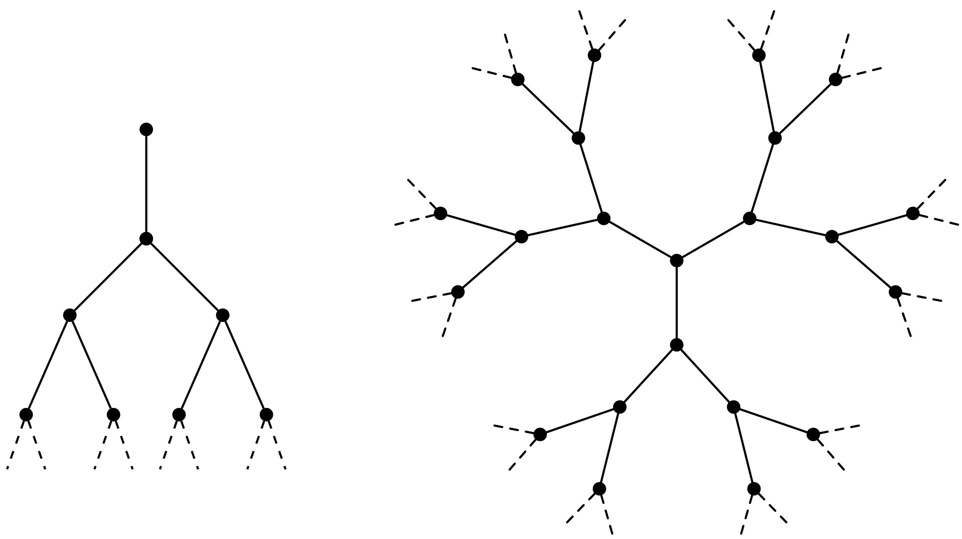
Theorem 2.7. *Let $\mathcal{G} \in \mathbf{G_2}$ be a periodic graph and $\lambda > 0$. Then $\mathcal{G}$ admits a ground state. Furthermore, one has*

$$\inf_{v \in \mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})} J_\lambda(v) = 2 \inf_{v \in \mathcal{N}_{\lambda,Z}(\mathcal{G})} J_\lambda(v)$$

and there are no nodal ground states.

It is interesting to note that the above results are insensitive of the degree of periodicity, i.e. the dimension n of the group $\mathbb{Z}^n$ whose action leaves the graph invariant. This is particularly relevant when put in relation with the available results for prescribed-mass energy ground states (compare e.g. [14, Theorems 1.1–1.2], where $n = 2$, with [125, Theorem 1.1], where $n = 1$), whose existence has been shown to depend strongly on the value of n.

The last results of Section 2.5 concern regular trees, i.e. acyclic, noncompact metric graphs with infinitely many bounded edges, all of the same length, and where all the vertices have the same degree, with the possible exception of a unique root vertex of degree 1. If such a vertex with degree 1 is present, we speak of a rooted tree (see Figure 2.4(A)), otherwise we speak of an unrooted tree (see Figure 2.4(B)). Note that regular trees are well-known examples of noncompact graphs satisfying $\omega_Z(\mathcal{G}) > 0$ (see e.g. [130] and references therein). Hence, in this setting our results involve also negative values of λ.



(A) (B)

Figure 2.4: Examples of a rooted tree (A) and an unrooted tree (B).

Theorem 2.8. *Let $\mathcal{G}$ be a regular tree and $\lambda > -\omega_Z(\mathcal{G})$. Then*

(i) if $\mathcal{G}$ is unrooted or, if $\mathcal{G}$ is rooted and Z is empty, $\mathcal{G}$ admits a ground state;

(ii) if $\mathcal{G}$ is rooted and Z is not empty, there are no ground states on $\mathcal{G}$;

(iii) independently of Z, there are no nodal ground states on $\mathcal{G}$.

As in the case of periodic graphs, the preceding theorem provides a complete description of the problem for regular trees. Note that, as one may expect, the role of the set Z is crucial to discriminate between existence and nonexistence on rooted trees.

Moreover, as already seen for periodic graphs, observe that Theorem 2.8(i) establishes the existence of ground states of the action on trees for every admissible value of the parameter λ.

It is interesting to compare this result with [130, Theorem 1.2], where it is shown that existence of mass constrained ground states of the energy on trees does depend on the mass. In particular, when $p \in [4, 6)$, energy ground states do not exist if the mass is smaller than a positive threshold. Note, however, that this seeming asymmetry is not sufficient to guarantee that, on trees, the action approach is more general than the energy one. Indeed, [130] does not provide any information on the values of the parameter λ associated to energy ground states, and nothing is known about the mass of action ground states identified in this paper. Hence, it is not clear whether the ground states of the action given by Theorem 2.8(i) for $\lambda > -\omega_Z(\mathcal{G})$ coincide with the mass constrained ground states of the energy found in [130, Theorem 1.2].

We observe that the discussion developed here requires no restrictions on the nonlinearity power p, so that all our results apply for every $p > 2$. In particular, the existence statements listed above provide constant sign and sign changing solutions to ($\mathrm{NLS}_{\mathcal{G},Z}$) also when $p > 6$, the so–called L^2–supercritical regime, whose analysis is much harder in the context of fixed mass critical points of the energy (first investigations in this direction have been recently initiated in [82, 102]).

To conclude, let us study nodal domains (i.e. the connected components of[4] $\mathcal{G} \setminus u^{-1}(0)$) and the nodal set (i.e. the set $u^{-1}(0)$) of nodal ground states u. As one may expect, nodal ground states have exactly two nodal domains (Theorem 2.38). We also show that the nodal set can have an arbitrary number of components and an arbitrary measure. This is in contrast with the case of open domains of $\mathbb{R}^N$, where unique continuation principles forbid nonzero solutions to vanish on nonempty open subsets.

Theorem 2.9. *For every nonnegative integers k, m, n with $m \geq 2$, there exists a graph $\mathcal{G}$ and a nodal ground state u on $\mathcal{G}$ such that $u^{-1}(0)$ is the union of k isolated points, m half-lines and n line segments.*

The remainder of the chapter is organized as follows:

- section 2.2 collects some preliminary facts useful for the subsequent analysis;
- section 2.3 provides the proof of the abstract results contained in Theorems 2.3–2.4;
- section 2.4 analyzes the case of graphs with half-lines;
- section 2.5 deals with periodic graphs and trees;
- section 2.6 is devoted to the study of qualitative properties of nodal ground states.

Notation. Throughout, we will drop the dependence of $\mathcal{N}_{\lambda,Z}(\mathcal{G})$, $\mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})$, J_λ on λ and $\mathcal{G}$, writing $\mathcal{N}_Z$, $\mathcal{N}_Z^{nod}$ and J whenever possible, keeping the complete notation only if necessary. Similarly, when the context permits it, we will not explicitly indicate, in norms, the dependence on the domain of integration. Furthermore, when $Z = \varnothing$, we do not put $\varnothing$ as a subscript and simply write $H^1(\mathcal{G})$, $\mathcal{N}$, $\mathcal{N}^{nod}$, etc.

2.2 Preliminaries

We refer to Appendix A to precise the notion of metric graphs. However, we make precise in the following definition the class of graphs that we consider in this chapter.

[4] To keep notations light, we will denote by $u^{-1}(0)$ the set $\{x \in \mathcal{G} \mid u(x) = 0\}$.

Definition 2.10. We denote by $\mathbf{G_2}$ the class of metric graphs $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ such that:

- $\mathcal{G}$ is connected and has an at most countable number of edges;
- $\deg(v) < \infty$ for every $v \in \mathbb{V}$, where $\deg(v)$ denotes the degree of the vertex v, i.e. the number of edges incident at v;
- $\ell := \inf_{e\in\mathbb{E}} \ell_e > 0$, where ℓ_e denotes the length of the edge e.

Note that a graph $\mathcal{G} \in \mathbf{G_2}$ is noncompact as soon as one of the following two eventualities occurs: i) $\mathcal{G}$ has at least one unbounded edge (i.e. a half-line), ii) the number of edges of $\mathcal{G}$ is infinite.

Remark 2.11. One could add in the definition of $\mathcal{G}$ the assumption that every vertex v satisfies $\deg(v) \neq 2$. Indeed, vertices v of degree 2 can a priori be eliminated from any metric graph, by melting the two edges incident at v into a single edge. In some cases however (see Remark 2.16) the possibility of using vertices of degree 2 turns out to be quite handy. We notice that adding or removing vertices of degree 2 from a graph changes it combinatorially, but not as a metric space, and in this chapter we will identify graphs that differ only by vertices of degree 2.

As anticipated in the presentation of the chapter, we couple equation (2.1) with mixed Kirchhoff and Dirichlet boundary conditions. Given a noncompact graph $\mathcal{G} \in \mathbf{G_2}$, we let $Z \subseteq \mathbb{V}$ denote a set of vertices of degree 1 (possibly empty or infinite) where we impose homogeneous Dirichlet conditions and we set

$$H^1_Z(\mathcal{G}) := \Big\{u \in H^1(\mathcal{G}) \;\Big|\; u(v) = 0 \text{ for every } v \in Z\Big\}.$$

The Nehari manifold associated to J on $H^1_Z(\mathcal{G})$ is

$$\begin{aligned}\mathcal{N}_Z &:= \Big\{u \in H^1_Z(\mathcal{G}) \;\Big|\; u \neq 0,\ J'(u)u = 0\Big\} \\ &= \Big\{u \in H^1_Z(\mathcal{G}) \;\Big|\; u \neq 0,\ \|u'\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p\Big\},\end{aligned}$$

while the nodal Nehari set is

$$\mathcal{N}_Z^{nod} := \Big\{u \in H^1_Z(\mathcal{G}) \;\Big|\; u^\pm \in \mathcal{N}_Z\Big\} = \Big\{u \in H^1_Z(\mathcal{G}) \;\Big|\; u^\pm \neq 0,\ J'(u)u^\pm = 0\Big\}.$$

The nodal Nehari set contains all nodal solutions of $(\text{NLS}_{\mathcal{G},Z})$, but, generally, it is not a manifold. However, the following fundamental property holds for global minimizers on compact graphs.

Proposition 2.12. *Let $\mathcal{G} \in \mathbf{G_2}$ be compact and $\lambda > -\omega_Z(\mathcal{G})$. If $u \in \mathcal{N}_Z^{nod}$ satisfies*

$$J(u) = \inf_{v \in \mathcal{N}_Z^{nod}} J(v),$$

then $J'(u) = 0$.

Proof. The fact that any function realizing the minimum of the action restricted to its nodal Nehari set is in fact a critical point of the action is a very general property, that holds true for a large class of NLS equations including the one we consider in this chapter, and is not specific of graphs. A detailed proof can be found e.g. in [52, Proposition 3.1] in the case of NLS equations (with more general nonlinearities than the one of this chapter) posed on bounded domains of $\mathbb{R}^N$ with homogeneous Dirichlet conditions at the boundary. The proof reported therein uses only abstract tools of Critical Point Theory (in particular, the deformation lemma). For this reason, that argument extends with no modification to compact graphs. □

Obviously $\mathcal{N}_Z^{nod} \subseteq \mathcal{N}_Z$ and, for $u \in \mathcal{N}_Z$, the functional J defined in (2.3) takes the simple form

$$J(u) = \varkappa \|u\|_p^p = \varkappa(\|u'\|_2^2 + \lambda\|u\|_2^2), \qquad \varkappa := \frac{1}{2} - \frac{1}{p}, \tag{2.11}$$

from which we see that J is positive on $\mathcal{N}_Z$. Actually much more can be said, as stated in the next proposition, which rephrases in the present setting an analogous result of [120, Proposition 2.3].

Proposition 2.13. *For every $\lambda > -\omega_Z(\mathcal{G})$ and $p > 2$, there exists a constant $C > 0$ depending only on λ and p such that for all noncompact $\mathcal{G} \in \mathbf{G_2}$,*

$$\inf_{u \in \mathcal{N}_Z} \|u\|_p \geq C > 0.$$

Moreover, if $(u_n)_n \subseteq \mathcal{N}_Z$ satisfies $\sup_n J(u_n) < \infty$, then $(u_n)_n$ is bounded in $H^1(\mathcal{G})$ and

$$\inf_n \|u_n\|_2 > 0, \quad \inf_n \|u_n\|_\infty > 0.$$

As is well known, there is a continuous projection $\pi_\lambda : H_Z^1(\mathcal{G}) \setminus \{0\} \to \mathcal{N}_Z$, defined by

$$\pi_\lambda(u) = n_\lambda(u)u, \qquad n_\lambda(u) = \left(\frac{\|u'\|_2^2 + \lambda\|u\|_2^2}{\|u\|_p^p}\right)^{\frac{1}{p-2}}, \tag{2.12}$$

so that $u \in \mathcal{N}_Z$ if and only if $n_\lambda(u) = 1$. Let us also note that if $u \in H_Z^1(\mathcal{G})$ satisfies $u^\pm \neq 0$, then $\pi_\lambda(u^+) + \pi_\lambda(u^-) \in \mathcal{N}_Z^{nod}$.

Remark 2.14. For every metric graph $\mathcal{G}$ and every set Z of degree 1 vertices, the map

$$t \mapsto \inf_{v \in \mathcal{N}_{t,Z}(\mathcal{G})} J_t(v)$$

is increasing and continuous on $(-\omega_Z(\mathcal{G}), +\infty)$. This property of the action ground state level is general and does not depend on the fact that we are considering the problem on graphs. For this reason, we redirect the interested reader e.g. to [129, Lemma 2.4] for a detailed proof (in the context of open subsets of $\mathbb{R}^N$).

2.3 Proof of the abstract results

In this section we prove the abstract results stated in Theorems 2.3–2.4. The strategy for the proof of the existence results is to construct special minimizing sequences for J on $\mathcal{N}_Z$ or $\mathcal{N}_Z^{nod}$, to avoid problems caused by the noncompactness of the graphs.

Remark 2.15. In this section and the next ones, we will frequently use, without recalling it, the following consequence of the Strong Maximum Principle on metric graphs (see Proposition D.2): if $u \not\equiv 0$ solves problem $(\mathrm{NLS}_{\mathcal{G},Z})$ and if $u \geq 0$ on $\mathcal{G}$, then $u > 0$ on $\mathcal{G} \setminus Z$.

Remark 2.16. The proof of the next results relies on the following approximation procedure: given a noncompact graph $\mathcal{G} \in \mathbf{G_2}$, we construct an increasing sequence $(\mathcal{K}_n)_n \subseteq \mathcal{G}$ of connected compact graphs such that $\bigcup_{n\geq 1} \mathcal{K}_n = \mathcal{G}$, and a sequence $(\chi_n)_n \subseteq H^1(\mathcal{G})$ of cut-off functions such that

$$0 \leq \chi_n \leq 1, \quad \|\chi_n'\|_\infty \leq 1/\ell, \qquad \chi_{n|\mathcal{K}_{n-1}} = 1, \qquad \operatorname{supp} \chi_n \subseteq \mathcal{K}_n,$$

with ℓ as in Definition 2.10.

To describe the graphs $\mathcal{K}_n$ we begin by performing a preliminary operation on $\mathcal{G}$ as follows. On each half-line of $\mathcal{G}$ (if any) we insert vertices of degree 2 at the points of coordinates $k\ell$, $k = 1, 2, \ldots$. Every half-line can now be viewed as a sequence of consecutive edges (each of length ℓ). With some abuse of notation, we still call $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ the new graph obtained in this way (see Remark 2.11).

Let now $v_0 \in \mathcal{G}$ be a fixed vertex. For every $n \geq 1$, let $\mathbb{V}_n$ be the set of vertices of $\mathbb{V}$ that can be reached from v_0 travelling on at most n edges. As each node of $\mathcal{G}$ has finite degree, the sets $\mathbb{V}_n$ are finite and, since $\mathcal{G}$ is connected, for each vertex $v \in \mathcal{G}$ (different from v_0) there exists $n_0(v) \geq 1$ such that v belongs to $\mathbb{V}_n$ for every $n \geq n_0(v)$.

Then we define the graph $\mathcal{K}_n$ as $(\mathbb{V}_n, \mathbb{E}_n)$, where $\mathbb{E}_n$ is the set of edges of $\mathbb{E}$ whose vertices belong to $\mathbb{V}_n$. Clearly, each $\mathcal{K}_n$ is connected and compact, and $\bigcup_{n\geq 1} \mathcal{K}_n = \mathcal{G}$.

Finally, we define χ_n to be equal to 1 on $\mathcal{K}_{n-1}$, to 0 on $\mathcal{G} \setminus \mathcal{K}_n$ and affine on every edge of $\mathcal{K}_n \setminus \mathcal{K}_{n-1}$. All the required properties trivially hold (the bound on χ_n' follows from the fact that all edges of $\mathcal{G}$ have length at least ℓ).

To conclude, note that, given $u \in H^1_Z(\mathcal{G})$, it is straightforward to check that $\chi_n u \to u$ in $H^1(\mathcal{G})$ as $n \to \infty$.

Exploiting Remark 2.16, we now construct suitable minimizing sequences for J on $\mathcal{N}_Z$ and $\mathcal{N}_Z^{nod}$.

Proposition 2.17. *Let $\mathcal{G} \in \mathbf{G_2}$ be noncompact and $\lambda > -\omega_Z(\mathcal{G})$. There exists a minimizing sequence $(u_n)_n \subseteq \mathcal{N}_Z$ for J and $u \in H^1_Z(\mathcal{G})$ such that*

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{G}), \qquad u \geq 0 \qquad \text{and} \qquad J'(u) = 0.$$

Proof. Keeping in mind the notation of Remark 2.16, let $(\mathcal{K}_n)_n$ be the sequence of compact graphs approximating $\mathcal{G}$ and let $\partial\mathcal{K}_n = \mathbb{V}_n \setminus \mathbb{V}_{n-1}$. Define the Hilbert space $H_n := H^1_{Z\cup\partial\mathcal{K}_n}(\mathcal{K}_n)$ and the Nehari manifold associated to J on H_n, namely

$$\mathcal{N}_n := \Big\{u \in H_n \;\Big|\; u \neq 0,\; J'(u)u = 0\Big\}.$$

If $u \in H_n$, it vanishes on $\partial\mathcal{K}_n$ and, after extending it by 0, it can be viewed as a function in $H^1_Z(\mathcal{G})$, that we still denote by u. Therefore, $\mathcal{N}_n \subseteq \mathcal{N}_Z$ for every $n \geq 1$. Let $u_n \in \mathcal{N}_n$ be a ground state for J restricted to H_n, that is,

$$J(u_n) = \inf_{v \in \mathcal{N}_n} J(v).$$

The existence of u_n is standard by the compactness of the embedding of $H^1(\mathcal{K}_n)$ into $L^p(\mathcal{K}_n)$ observing that, by construction, $\omega_{Z\cup\partial\mathcal{K}_n}(\mathcal{K}_n) \geq \omega_Z(\mathcal{G})$. Note also that, as $u \in \mathcal{N}_n$ if and only if $|u| \in \mathcal{N}_n$ and $J(u) = J(|u|)$, we can assume that $u_n \geq 0$ on $\mathcal{G}$.

We claim that $(u_n)_n$ is a minimizing sequence for J on $\mathcal{N}_Z$. First note that, since $\mathcal{K}_{n-1} \subseteq \mathcal{K}_n$ for every n, the sequence $(J(u_n))_n$ is nonincreasing.

Given any $\varepsilon > 0$, let $\overline{u} \in \mathcal{N}_Z$ be such that $J(\overline{u}) \leq \inf\limits_{v \in \mathcal{N}_Z} J(v) + \varepsilon/2$. Let $(\chi_n)_n$ be the sequence of cut-off functions of Remark 2.16. For every n, the function $\widetilde{u}_n := \pi_\lambda(\chi_n \overline{u})$ is in $\mathcal{N}_Z$ and $\operatorname{supp} \widetilde{u}_n \subseteq \mathcal{K}_n$, which means, in particular, that $\widetilde{u}_n$ (restricted to $\mathcal{K}_n$) is in $\mathcal{N}_n$. Moreover, by Remark 2.16 and the continuity of π_λ, as soon as n is large enough we have

$$J(\widetilde{u}_n) \leq \inf_{v \in \mathcal{N}_Z} J(v) + \varepsilon.$$

Therefore, for all n large,

$$J(u_n) = \inf_{v \in \mathcal{N}_n} J(v) \leq J(\widetilde{u}_n) \leq \inf_{v \in \mathcal{N}_Z} J(v) + \varepsilon.$$

Thus $(u_n)_n$ is a minimizing sequence for J on $\mathcal{N}_Z$, and the claim is proved. Since $(u_n)_n$ is bounded in $H^1_Z(\mathcal{G})$ (like all minimizing sequences), up to a subsequence it converges weakly to some $u \in H^1_Z(\mathcal{G})$ that also satisfies $u \geq 0$. Since u_n minimizes J over $\mathcal{N}_n$, it follows that $J'(u_n)\phi = 0$ for every $\phi \in H_n$. As $u \mapsto J'(u)\phi$ is weakly continuous on $H^1_Z(\mathcal{G})$, letting $n \to \infty$ shows that $J'(u)\phi = 0$ for every $\phi \in H_n$ and every n, and thus, by density, that $J'(u) = 0$. □

Proposition 2.18. *Let $\mathcal{G} \in \mathbf{G_2}$ be noncompact and $\lambda > -\omega_Z(\mathcal{G})$. There exists a minimizing sequence $(u_n)_n \subseteq \mathcal{N}^{nod}_Z$ for J and $u \in H^1_Z(\mathcal{G})$ such that*

$$u_n \rightharpoonup u \quad \text{weakly in } H^1(\mathcal{G}) \qquad \text{and} \qquad J'(u) = 0.$$

Proof. The proof is very similar to the one of Proposition 2.17, to which we refer for the notation. Let us define

$$\mathcal{N}^{nod}_n := \left\{ v \in H_n \;\middle|\; v^{\pm} \in \mathcal{N}_n \right\}$$

and, for each n, let $u_n \in \mathcal{N}^{nod}_n$ be a nodal ground state for J restricted to H_n, that is,

$$J(u_n) = \inf_{v \in \mathcal{N}^{nod}_n} J(v).$$

The existence of u_n follows plainly by the compactness of the embedding of $H^1(\mathcal{K}_n)$ into $L^p(\mathcal{K}_n)$ as, for example, in [318, Theorem 18] observing again that

$$\omega_{Z\cup\partial\mathcal{K}_n}(\mathcal{K}_n) \geq \omega_Z(\mathcal{G}).$$

We claim that $(u_n)_n$ is a minimizing sequence for J on $\mathcal{N}^{nod}_Z$. We note that, as above, $(J(u_n))_n$ is nonincreasing. If $u \in \mathcal{N}^{nod}_Z$, we have $(\chi_n u)^{\pm} = \chi_n u^{\pm}$, and both functions are nonzero if n is large enough. By the continuity of π_λ and Remark 2.16, as $n \to \infty$,

$$\pi_\lambda(\chi_n u^+) + \pi_\lambda(\chi_n u^-) \to \pi_\lambda(u^+) + \pi_\lambda(u^-) = u \quad \text{in } H^1_Z(\mathcal{G})\,.$$

Now, given any $\varepsilon > 0$, let $\overline{u} \in \mathcal{N}^{nod}_Z$ satisfy $J(\overline{u}) \leq \inf_{\mathcal{N}^{nod}_Z} J + \varepsilon/2$.

Define $\widetilde{u}_n := \pi_\lambda(\chi_n\overline{u}^+) + \pi_\lambda(\chi_n\overline{u}^-)$, so $\widetilde{u}_n \in \mathcal{N}^{nod}_Z$, $\operatorname{supp}\widetilde{u}_n \subseteq \mathcal{K}_n$, whence $\widetilde{u}_n \in \mathcal{N}^{nod}_n$. Then, for every n large enough,

$$J(u_n) = \inf_{v\in\mathcal{N}^{nod}_n} J(v) \leq J(\widetilde{u}_n) \leq \inf_{v\in\mathcal{N}^{nod}_Z} J(v) + \varepsilon,$$

showing that $(u_n)_n$ is a minimizing sequence for J on $\mathcal{N}^{nod}_Z$.

Since, by Proposition 2.12, $J'(u_n)\phi = 0$ for every $\phi \in H_n$, we conclude exactly as in the proof of Proposition 2.17. □

We are now in position to prove Theorems 2.3–2.4.

Proof of Theorem 2.3. Let us prove the two statements separately.

Proof of (i). Let $(u_n)_n \subseteq \mathcal{N}_Z$ be the minimizing sequence for the functional J on $\mathcal{N}_Z$ constructed in Proposition 2.17 and let $u \geq 0$ be its weak limit. We first show that $u \not\equiv 0$. Indeed, if this were the case, then $u_n \rightharpoonup 0$ in $H^1_Z(\mathcal{G})$, so that

$$\inf_{v \in \mathcal{N}_Z} J(v) = \liminf_{n\to\infty} J(u_n) \geq J^\infty(\mathcal{G}; Z),$$

which is ruled out by assumption (2.4). Now as $u \not\equiv 0$ and $J'(u) = 0$, u is a nontrivial solution of $(\mathrm{NLS}_{\mathcal{G},Z})$. In particular, $u \in \mathcal{N}_Z$ and then, by (2.11) and weak lower semicontinuity,

$$J(u) = \varkappa \|u\|_p^p \leq \liminf_{n\to\infty} \varkappa \|u_n\|_p^p = \liminf_{n\to\infty} J(u_n) = \inf_{v \in \mathcal{N}_Z} J(v)$$

showing that u is a ground state. As such, u is positive on $\mathcal{G} \setminus Z$ by Remark 2.15.

Proof of (ii). Consider the minimizing sequence $(u_n)_n$ given by Proposition 2.18 and its weak limit $u \in H^1_Z(\mathcal{G})$ satisfying $J'(u) = 0$. We first show that $u^\pm \not\equiv 0$. For every n,

$$J(u_n) = J(u_n^+) + J(u_n^-) \geq J(u_n^+) + \inf_{v \in \mathcal{N}_Z} J(v).$$

If, for instance, $u^+ \equiv 0$, then $u_n^+ \rightharpoonup 0$ in $H^1_Z(\mathcal{G})$, so that

$$\inf_{v \in \mathcal{N}_Z^{nod}} J(v) = \liminf_{n\to\infty} J(u_n) \geq \liminf_{n\to\infty} J(u_n^+) + \inf_{v \in \mathcal{N}_Z} J(v) \geq J^\infty(\mathcal{G}; Z) + \inf_{v \in \mathcal{N}_Z} J(v),$$

by definition of $J^\infty(\mathcal{G}; Z)$, which contradicts (2.5). In the same way one proves that $u^- \not\equiv 0$. As $J'(u) = 0$, it follows that u is a non-zero sign changing solution of $(\mathrm{NLS}_{\mathcal{G},Z})$, and hence $u \in \mathcal{N}_Z^{nod}$. Then by weak lower semicontinuity, we conclude that

$$J(u) = \varkappa \|u\|_p^p \leq \varkappa \liminf_{n\to\infty} \|u_n\|_p^p = \liminf_{n\to\infty} J(u_n) = \inf_{v \in \mathcal{N}_Z^{nod}} J(v),$$

namely that u is the required minimizer, i.e. a nodal ground state of $(\mathrm{NLS}_{\mathcal{G},Z})$. □

Proof of Theorem 2.4. Let $u \in \mathcal{N}_Z^{nod}$. Since $u^\pm \in \mathcal{N}_Z$, we have

$$J(u) = J(u^+) + J(u^-) \geq 2 \inf_{v \in \mathcal{N}_Z} J(v),$$

which is (2.6).

Now assume that $u \in \mathcal{N}_Z^{nod}$ satisfies

$$J(u) = \inf_{v \in \mathcal{N}_Z^{nod}} J(v) = 2 \inf_{v \in \mathcal{N}_Z} J(v).$$

Then $J(u^+) = J(u^-) = \inf_{v \in \mathcal{N}_Z} J(v)$, and therefore $u^\pm$ are both ground states of J. As such, by Remark 2.15, they cannot vanish in $\mathcal{G} \setminus Z$, which is a contradiction since $u^\pm \not\equiv 0$. □

2.4 Graphs with at least one half-line

In this section we discuss ground states and nodal ground states for noncompact graphs with at least one half-line. Since for such kind of graphs the bottom of the spectrum of $-\,\mathrm{d}^2/\,\mathrm{d}x^2$ on $H^1_Z(\mathcal{G})$ always satisfies

$$\omega_Z(\mathcal{G}) = 0\,,$$

all the results of this section will hold for every $\lambda \in (0, +\infty)$.

The prototype cases in this context are given by the real line and the half-line, about which everything is known (see e.g. [98, 214]). Since the ground states on $\mathbb{R}$ play a very important role in what follows, we recall briefly their main features. On the real line the only nontrivial L^2 solutions to (2.1) are the *solitons* and are unique up to translations and sign (see Appendix C). Denoting by ϕ_λ the unique positive and even soliton, for every $\lambda > 0$ we have

$$s_\lambda := J_\lambda(\phi_\lambda) = \inf_{v\in\mathcal{N}_\lambda(\mathbb{R})} J_\lambda(v),$$

namely the solitons are the ground states on $\mathbb{R}$ (see e.g. [214, Proposition 3.12]). Similarly, on the half-line (with $Z = \varnothing$) there is a unique nontrivial L^2 solution (up to sign) to (2.1), given by the so-called *half-soliton* ψ_λ, i.e. the restriction of ϕ_λ to $\mathbb{R}^+$. It is the ground state, and

$$J_\lambda(\psi_\lambda) = \inf_{v\in\mathcal{N}_\lambda(\mathbb{R}^+)} J_\lambda(v) = \frac{1}{2}s_\lambda. \tag{2.13}$$

If $Z = \{0\}$ (the vertex of the half-line) there are no nontrivial L^2 solutions to (2.1), as any nontrivial solution of $-u'' + \lambda u = |u|^{p-2}u$ on $\mathbb{R}^+$ that vanishes at the origin corresponds to a periodic orbit in the phase plane associated to the equation and thus is not in $L^2(\mathbb{R}^+)$.

For a general graph with half-lines, a first marker of the importance of the level s_λ is given by the following straightforward property.

Proposition 2.19. *Let $\mathcal{G} \in \mathbf{G_2}$ contain at least one half-line and $\lambda > 0$. Then*

$$\frac{1}{2}s_\lambda \leq \inf_{v\in\mathcal{N}_Z} J(v) \leq s_\lambda. \tag{2.14}$$

Proof. The inequalities can be easily proved using rearrangement techniques (based on the Pólya-Szegő inequality, see Appendix B), arguing exactly as in the proof of [19, Theorem 2.2]. □

In the search for ground states, it is crucial to understand whether one can reverse the second inequality in (2.14) (see e.g. the discussion in [19, 21] in the context of energy ground states of prescribed mass). In [19, Theorems 2.3–2.5], the authors individuated a topological condition on $\mathcal{G}$ under which this can actually be done. To state it, we recall that $\mathbb{V}_\infty$ denotes the set of *vertices at infinity* of $\mathcal{G}$. Note that every vertex at infinity is a vertex of the graph $\mathcal{G}$, but is *not* a point of the metric space $\mathcal{G}$. The assumption introduced in [19] is:

$$\text{for every } e \in \mathbb{E}, \text{ every connected component of the graph } (\mathbb{V}, \mathbb{E} \setminus \{e\}) \text{ contains at least one vertex } v \in \mathbb{V}_\infty. \tag{H}$$

In [19, Theorem 2.3] the authors proved that, if $\mathcal{G}$ satisfies assumption (H), then for every $u \in H^1(\mathcal{G})$ there results $\#u^{-1}(t) \geq 2$ for almost every $t \in (0, \|u\|_\infty)$. The main consequence of this (originally proved in [19, Theorem 2.5] for the problem of prescribed-mass ground states) is described in the following result.

Theorem 2.20 ([120], Theorem 2.6)**.** *If* $\mathcal{G} \in \mathbf{G_2}$ *satisfies assumption* (H) *and* $\lambda > 0$*, then*

$$\inf_{v \in \mathcal{N}} J(v) = s_\lambda$$

and it is never achieved, unless $\mathcal{G}$ *is isometric to* $\mathbb{R}$ *or to a "tower of bubbles" shown in Figure 2.2.*

In this chapter the setting is different from that of [19] and [120] for at least two reasons: first, the boundary conditions are more general and the presence of the set Z must be taken into account; second, we are also interested in nodal ground states. For these reasons it is convenient to reformulate and generalize assumption (H) in a form that is more suited to handle the questions under study. As in the presentation of the chapter, consider the set

$$F(\mathcal{G}) = \Big\{ e \in \mathbb{E} \;\Big|\; \text{at least one connected component of } (\mathbb{V}, \mathbb{E} \setminus \{e\}) \text{ has no vertices in } \mathbb{V}_\infty \cup Z \Big\}$$

and the assumptions

$$\#F(\mathcal{G}) = 0, \tag{H0}$$

$$\#F(\mathcal{G}) \leq 1. \tag{H1}$$

Note that (H0) and (H1) are, respectively, the assumptions in (i) and (ii) in Theorem 2.6. From now on, with a slight abuse of notation, we denote the graph $(\mathbb{V}, \mathbb{E} \setminus \{e\})$ simply by $\mathcal{G} \setminus e$.

To investigate the relations between assumptions (H0) and (H), it is convenient to define a new graph $\widetilde{\mathcal{G}}$ in the following way. If $Z = \varnothing$, we set $\widetilde{\mathcal{G}} = \mathcal{G}$. Otherwise, we replace every (finite) edge e ending at a vertex of Z by a half-line, still called e. We obtain in this way a new graph $\widetilde{\mathcal{G}} = (\widetilde{\mathbb{V}}, \widetilde{\mathbb{E}})$ that has the same number of vertices and edges as $\mathcal{G}$. The only difference is that edges of $\mathcal{G}$ terminating at vertices of Z are replaced, in $\widetilde{\mathcal{G}}$, by half-lines terminating at vertices in $\widetilde{\mathbb{V}}_\infty$.

Then it is easily seen that

$$\mathcal{G} \text{ satisfies (H0)} \quad \Longleftrightarrow \quad \widetilde{\mathcal{G}} \text{ satisfies (H)}. \tag{2.15}$$

Indeed, to say that $\mathcal{G}$ satisfies (H0) means that there are no edges in $\mathbb{E}$ whose removal generates a connected component without vertices in $\mathbb{V}_\infty \cup Z$, namely that for every $e \in \mathbb{E}$, every connected component of $\mathcal{G} \setminus e$ has a vertex in $\mathbb{V}_\infty \cup Z$. But this, read on $\widetilde{\mathcal{G}}$, means that every connected component of $\widetilde{\mathcal{G}} \setminus e$ has a vertex in $\widetilde{\mathbb{V}}_\infty$, which is (H) for $\widetilde{\mathcal{G}}$.

Furthermore, to say that $\#F(\mathcal{G}) = 1$, namely that $F(\mathcal{G}) = \{e\}$ for exactly one edge e, means that the graph $\mathcal{G} \setminus e$ decomposes as

$$\mathcal{G} \setminus e = \mathcal{G}_K \cup \mathcal{G}', \tag{2.16}$$

where $\mathcal{G}_K$ is connected and has no vertices in $\mathbb{V}_\infty \cup Z$, while $\mathcal{G}'$ is connected and contains *all* the vertices of $\mathbb{V}_\infty \cup Z$. Also, there are no edges other than e that permit a decomposition like (2.16). We note that, as $\mathcal{G}$ has at least one half-line, the unique $e \in F(\mathcal{G})$ can never have a vertex in Z. However, e can be a half-line. In this case, though, $\mathcal{G} \setminus e = \mathcal{G}_K \cup \{v_\infty\}$, where v_∞ is the vertex at infinity of the half-line e. Thus in this case the graph $\mathcal{G}$ is made of a set of bounded edges without vertices in Z and a *single* half-line attached to it.

The next result plays a key role in the proof of some of the subsequent results. Roughly, it states that any graph satisfying $\#F(\mathcal{G}) = 1$ can be turned into a graph satisfying (H0) by attaching to it a suitable half-line.

Lemma 2.21. *Let $\mathcal{G} \in \mathbf{G_2}$ be a graph with at least one half-line satisfying $\#F(\mathcal{G}) = 1$. Let e be such that $F(\mathcal{G}) = \{e\}$ and $\mathcal{G}_K$ be the connected component of $\mathcal{G} \setminus e$ as in* (2.16). *Choose a vertex v in $\mathcal{G}_K$ and define a new graph $\widetilde{\mathcal{G}}_v$ by*[a] *$\widetilde{\mathcal{G}}_v = \mathcal{G} \cup h$, where h is a half-line attached at v. Then $\widetilde{\mathcal{G}}_v$ satisfies* (H0).

[a] Shorthand for $(\mathbb{V} \cup \{v_\infty\}, \mathbb{E} \cup \{h\})$, where v_∞ is the vertex at infinity of h.

Proof. Let v_∞ be the vertex at infinity of h and assume by contradiction that $\#F(\widetilde{\mathcal{G}}_v) \geq 1$, namely that there exists $\widetilde{e} \in F(\widetilde{\mathcal{G}}_v)$. We claim that $\widetilde{e} \neq h$. Indeed, removing h from $\widetilde{\mathcal{G}}_v$ would leave v_∞ isolated, splitting $\widetilde{\mathcal{G}}_v \setminus h$ into the two connected components $\mathcal{G}$ and $\{v_\infty\}$. Since both of them contain vertices in $\widetilde{\mathbb{V}}_\infty$, this violates the definition of $\widetilde{e}$.

Similarly, it cannot be $\widetilde{e} = e$: removing e from $\widetilde{\mathcal{G}}_v$, and recalling that h is attached to $\mathcal{G}_K$, would decompose $\widetilde{\mathcal{G}}_v$ into connected components as $(\mathcal{G}_K \cup h) \cup \mathcal{G}'$, violating again the definition of $\widetilde{e}$ as before. We are left with the case where $\widetilde{e}$ is different from both h and e. In this case we have the decomposition

$$\widetilde{\mathcal{G}}_v \setminus \widetilde{e} =: \widetilde{\mathcal{G}}_K \cup \widetilde{\mathcal{G}}',$$

with obvious meaning of the symbols. Notice that, by construction, the half-line h is attached to $\widetilde{\mathcal{G}}'$. Removing h *and* v_∞ from $\widetilde{\mathcal{G}}'$ does not disconnect it and, since $\widetilde{\mathcal{G}}'$ contains at least another vertex in $\mathbb{V}_\infty \cup Z$, we see that $\widetilde{\mathcal{G}}' \setminus (\{v_\infty\}, \{h\})$ is not empty. Therefore $\widetilde{\mathcal{G}}_K$ and $\widetilde{\mathcal{G}}' \setminus (\{v_\infty\}, \{h\})$ are both nonempty, connected, disjoint and their union is $\mathcal{G} \setminus \widetilde{e}$, namely $\widetilde{e} \in F(\mathcal{G})$. Since we also have $e \in F(\mathcal{G})$, this shows that $\#F(\mathcal{G}) \geq 2$, violating the assumption. $\square$

We can now prove that the assumptions (H0) and (H1) are sufficient to rule out the existence of ground states and nodal ground states respectively, as stated in Theorem 2.6.

Proof of Theorem 2.6. We split the proof into two parts.

Part 1: proof of (2.7) *and* (2.9). Up to taking absolute values, it is sufficient to work with nonnegative functions, which we do without further warnings. Since $\mathcal{G}$ contains at least one half-line, Proposition 2.19 guarantees that $\inf_{v \in \mathcal{N}_Z} J(v) \leq s_\lambda$. To prove the reverse inequality under assumption (H0), let $\widetilde{\mathcal{G}}$ be the graph defined after the statement of assumptions (H0)–(H1).

Since, as metric spaces, $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$, every function $u \in H^1_Z(\mathcal{G})$ extended by 0 on $\widetilde{\mathcal{G}} \setminus \mathcal{G}$ can be seen as a function $\widetilde{u} \in H^1(\widetilde{\mathcal{G}})$. Plainly,

$$\|\widetilde{u}\|_{L^q(\widetilde{\mathcal{G}})} = \|u\|_{L^q(\mathcal{G})} \quad \text{for every } q \in [1, +\infty], \qquad \|\widetilde{u}'\|_{L^2(\widetilde{\mathcal{G}})} = \|u'\|_{L^2(\mathcal{G})}.$$

This implies that $\widetilde{u} \in \mathcal{N}(\widetilde{\mathcal{G}})$ and since $\widetilde{\mathcal{G}}$ satisfies (H) (because $\mathcal{G}$ satisfies (H0), see (2.15)),

$$J(u) = \varkappa \|u\|^p_{L^p(\mathcal{G})} = \varkappa \|\widetilde{u}\|^p_{L^p(\widetilde{\mathcal{G}})} = J(\widetilde{u}) \geq \inf_{v \in \mathcal{N}(\widetilde{\mathcal{G}})} J(v) = s_\lambda$$

by Theorem 2.20. As this holds for every (nonnegative) $u \in \mathcal{N}_Z$, (2.9) is proved.

Let us assume now that for some nonnegative $u \in \mathcal{N}_Z$ we have $J(u) = s_\lambda$. Considering, as above, the function $\widetilde{u} \in \mathcal{N}(\widetilde{\mathcal{G}})$, we see that $J(\widetilde{u}) = J(u) = s_\lambda$, namely that $\widetilde{u}$ is a ground state for J on $\mathcal{N}(\widetilde{\mathcal{G}})$. As such, $\widetilde{u}(x) > 0$ for every $x \in \widetilde{\mathcal{G}}$, which shows that $Z = \varnothing$, namely that $\widetilde{\mathcal{G}} = \mathcal{G}$. We then conclude by Theorem 2.20.

Part 2: proof of (2.8) *and* (2.10) We first prove (2.8). By density, for every $\varepsilon > 0$ there exists a nonnegative $u_1 \in \mathcal{N}_Z(\mathcal{G})$ with compact support such that

$$J(u_1) \leq \inf_{v \in \mathcal{N}_Z(\mathcal{G})} J(v) + \varepsilon.$$

Similarly, there exists a nonnegative $u_2 \in \mathcal{N}(\mathbb{R})$ with compact support such that $J(u_2) \le s_\lambda + \varepsilon$. By taking a translation of u_2 (if necessary), we can make sure that its support, identified with an interval on some half-line of $\mathcal{G}$, does not intersect the support of u_1. We then define $w \in H^1(\mathcal{G})$ by

$$w(x) = \begin{cases} u_1(x) & \text{if } x \in \mathcal{G} \setminus \operatorname{supp}(u_2), \\ -u_2(x) & \text{if } x \in \operatorname{supp}(u_2). \end{cases}$$

Obviously, $w \in \mathcal{N}_Z^{nod}$ and we have

$$J(w) = J(u_1) + J(u_2) \le s_\lambda + \inf_{v \in \mathcal{N}_Z} J(v) + 2\varepsilon.$$

Since ε is arbitrary, we conclude.

Let us now prove the reverse inequality in (2.10) under assumption (H1). If $\#F(\mathcal{G}) = 0$, then $\mathcal{G}$ satisfies (H0) and Theorem 2.6 (i) shows that $\inf_{v \in \mathcal{N}_Z} J(v) = s_\lambda$, so that the inequality to be proved reads $\inf_{v \in \mathcal{N}_Z^{nod}} J(v) \ge 2s_\lambda$.

Given any $u \in \mathcal{N}_Z^{nod}$, applying Theorem 2.6 (i) to u^+ and u^- immediately yields

$$J(u) = J(u^+) + J(u^-) > 2s_\lambda\,,$$

the inequality being strict since both u^+ and u^- vanish somewhere on $\mathcal{G}$. This ensures that nodal ground states do not exist in this case.

Suppose now that $\#F(\mathcal{G}) = 1$. Let e be the unique element of $F(\mathcal{G})$ and consider the decomposition (2.16):

$$\mathcal{G} \setminus e = \mathcal{G}_K \cup \mathcal{G}'.$$

Given a vertex v of $\mathcal{G}_K$, for every $u \in \mathcal{N}_Z^{nod}$, at least one among u^+ and u^-, say u^+, vanishes at v. Let $\widetilde{\mathcal{G}}_v$ be the graph constructed in Lemma 2.21, obtained attaching to v a half-line h. Since u^+ vanishes at v, it can be extended to a function $\widetilde{u}^+$ simply by defining it to be 0 on h. Clearly, $\widetilde{u}^+ \in \mathcal{N}_Z(\widetilde{\mathcal{G}}_v)$ and, since $\widetilde{\mathcal{G}}_v$ satisfies (H0) by Lemma 2.21, there results $J(\widetilde{u}^+) \ge s_\lambda$. Then

$$J(u) = J(u^+) + J(u^-) = J(\widetilde{u}^+) + J(u^-) \ge s_\lambda + \inf_{v \in \mathcal{N}_Z} J(v),$$

concluding the proof of (2.10).

It remains to show that the infimum is not achieved when $\#F(\mathcal{G}) = 1$. To this end, it suffices to observe that the inequality used above, $J(\widetilde{u}^+) \ge s_\lambda$, is in fact strict. Indeed if $J(\widetilde{u}^+) = s_\lambda = \inf_{v \in \mathcal{N}(\widetilde{\mathcal{G}})} J(v)$, then $\widetilde{u}^+$ is a ground state on $\widetilde{\mathcal{G}}$, and hence it cannot vanish anywhere, contrary to the fact that $\widetilde{u}^+ \equiv 0$ on h. □

Remark 2.22. The assumptions of Theorem 2.6 are sharp. Indeed, Theorem 2.26 below shows that there exist graphs $\mathcal{G}$ satisfying $\#F(\mathcal{G}) \ge 1$ that admit ground states, while in Theorem 2.34 and Remark 2.36 we exhibit graphs $\mathcal{G}$ satisfying $\#F(\mathcal{G}) \ge 2$ that admit nodal ground states.

Theorem 2.6 shows that nonexistence of ground states or nodal ground states can be determined by purely topological properties of the graph. The situation for existence is, on the contrary, more involved.

In some cases, existence results for ground states based solely on the topology of the graph can be easily obtained when $Z = \varnothing$, as for example if $\mathcal{G}$ has a finite number of edges. In this respect, there is not much to say since the techniques developed in [21, Section 3] for the problem of prescribed-mass minimizers of the energy work in the present setting as well, as we now briefly show.

For the reader's convenience, let us first recall with the next lemma a standard relation between the action of a function and the number of pre-images of each value it attains. These properties are stated in [120, Propositions 2.2 and 2.8] and are consequences of standard rearrangement techniques on graphs. For this reason, we omit the proof of the lemma.

Lemma 2.23. *Let $\mathcal{G}$ be a metric graph whose measure is infinite. Let $u \in \mathcal{N}_{\lambda,Z}$ for a given real $\lambda > 0$ be such that $u \geq 0$ on $\mathcal{G}$ and that $\#u^{-1}(t) \geq K$ for almost every $t \in (0, \max_{\mathcal{G}} u)$ and for some $K \geq 1$. Then $J_\lambda(u) \geq K s_\lambda/2$. Furthermore, if equality holds then,*

- *for almost every $t \in (0, \max_{\mathcal{G}} u)$, $\#u^{-1}(t) = K$;*
- *the support of u has infinite measure;*
- *$u^{-1}(t)$ has zero measure for all $t \in (0, \max_{\mathcal{G}} u)$.*

Since in what follows we are going to use Theorem 2.3, we prove the following characterization of $J^\infty(\mathcal{G}; Z)$.

Proposition 2.24. *Let $\mathcal{G} \in \mathbf{G_2}$ be a noncompact graph with a finite number of edges and $\lambda > 0$. Then*

$$J_\lambda^\infty(\mathcal{G}; Z) = s_\lambda. \tag{2.17}$$

Proof. By density, for every $\varepsilon > 0$ there exists $u = u_\varepsilon \in \mathcal{N}(\mathbb{R})$ with compact support such that $J(u) \leq s_\lambda + \varepsilon$.

For every n large enough, the function $u_n(x) := u(x - n)$ is supported in $\mathbb{R}^+$ and, as such, it can be seen as an element of $\mathcal{N}_Z$ by placing its support on a half-line of $\mathcal{G}$ and then extending it by 0 outside its support. Clearly, $u_n \rightharpoonup 0$ in $H^1(\mathcal{G})$ and $\liminf_n J(u_n) \leq s_\lambda + \varepsilon$. Since ε is arbitrary, the " $\leq$ " part is proved.

For the reverse inequality, let $(u_n)_n \subseteq \mathcal{N}_Z$ be a sequence converging weakly to 0 in $H^1(\mathcal{G})$ with $J_\lambda(u_n) \to J_\lambda^\infty(\mathcal{G}; Z)$. We can assume $u_n \geq 0$ for every n since otherwise we replace it by $|u_n|$ that is still in $\mathcal{N}_Z$. By weak convergence, we also see that $u_n \to 0$ in $L^\infty_{\text{loc}}(\mathcal{G})$. Hence, if ε_n denotes the maximum of u_n on the set of all bounded edges of $\mathcal{G}$, clearly $\varepsilon_n \to 0$.

Therefore, letting $v_n := (u_n - \varepsilon_n)^+$, we see from Proposition 2.13 that $v_n \not\equiv 0$ for every n large enough. Since $\mathcal{G}$ contains at least one half-line, by construction $\# v_n^{-1}(t) \geq 2$ for every $t \in (0, \max v_n)$ and every n, since v_n vanishes on the set of all bounded edges, and the same holds for $\pi_\lambda(v_n)$. So $J(\pi_\lambda(v_n)) \geq s_\lambda$ by Lemma 2.23. Furthermore, as $n \to \infty$,

$$n_\lambda(v_n)^{p-2} = \frac{\|v_n'\|_2^2 + \lambda \|v_n\|_2^2}{\|v_n\|_p^p} \leq \frac{\|u_n'\|_2^2 + \lambda \|u_n\|_2^2}{\|u_n\|_p^p + o(1)} = 1 + o(1), \tag{2.18}$$

entailing

$$s_\lambda \leq J(\pi_\lambda(v_n)) = J\Big(n_\lambda(v_n) v_n\Big) = \varkappa n_\lambda(v_n)^p \|v_n\|_p^p \leq \varkappa(1+o(1))\|u_n\|_p^p = J(u_n) + o(1),$$

from which we obtain $\liminf_n J(u_n) \geq s_\lambda$. Since this holds for every sequence converging weakly to 0, the proof is complete. □

Remark 2.25. The assumption that the graph has a finite number of edges in Proposition 2.24 cannot be removed. This can be seen considering for instance the following example. On a real line we insert, for each integer $k \geq 1$, a node v_k at the point of coordinate k and a terminal edge L_k of length k, by identifying v_k with an endpoint of L_k (Figure 2.5). By density, for every $\varepsilon > 0$ there exist $k \geq 1$ and $u_k \in \mathcal{N}(\mathbb{R}^+)$ with compact support in $[0, k]$ such that $J(u_k) \leq \frac{s_\lambda}{2} + \varepsilon$. Since u_k can be considered as a function on the edge L_k, we obtain a sequence $(u_k)_k \in \mathcal{N}(\mathcal{G})$ that converges weakly to 0 and such that $\liminf_k J_\lambda(u_k) \leq \dfrac{s_\lambda}{2} + \varepsilon$. This proves that $J_\lambda^\infty(\mathcal{G}) \leq \frac{s_\lambda}{2}$. By Proposition 2.19, we obtain in fact $J_\lambda^\infty(\mathcal{G}) = \frac{s_\lambda}{2}$.

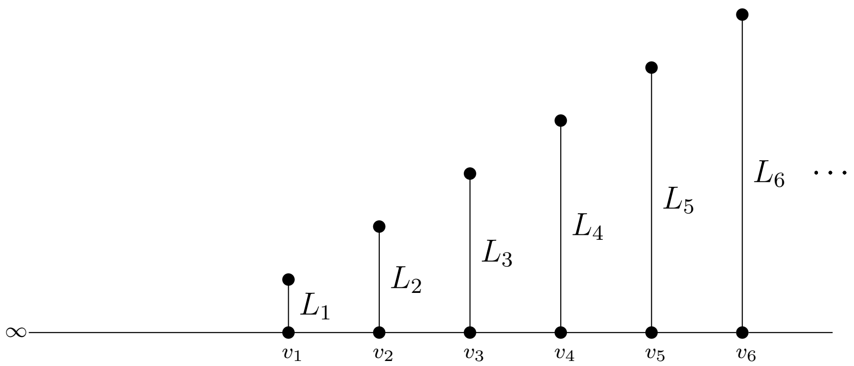


Figure 2.5: The graph $\mathcal{G}$ described in Remark 2.25

Having established (2.17), Theorem 2.3 yields existence of a ground state on a noncompact graph with a finite number of edges as soon as one can prove that $\inf_{v \in \mathcal{N}_Z} J(v) < s_\lambda$. Observe that this condition is analogous to the one appearing in the fixed mass case. As we anticipated above, such inequality can be shown to hold for a number of graphs with $Z = \varnothing$ exploiting only topological properties, by the use of the "graph surgery" techniques developed in [21, Section 3].

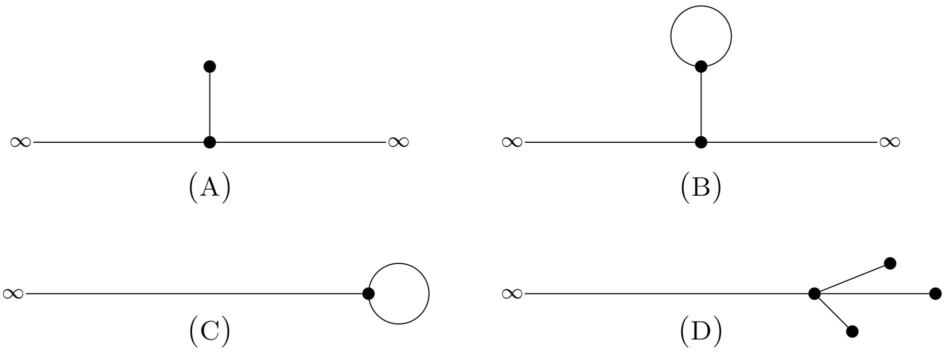


Figure 2.6: Some graphs with $Z = \varnothing$ admitting ground states. (A): line with a pendant; (B): signpost; (C): tadpole; (D): 3–fork.

Theorem 2.26. *For every $\lambda > 0$, every graph $\mathcal{G}$ depicted in Figure 2.6, for every length of its edges, satisfies*

$$\inf_{v \in \mathcal{N}} J(v) < s_\lambda$$

and therefore admits a ground state.

Proof. The inequality can be proved starting with a soliton on $\mathbb{R}$, using suitable rearrangement techniques, exactly as in the final part of [Section 3][21]. The basic idea is that, on each of these graphs, one can construct explicit functions built from pieces of the positive even soliton ϕ_λ on $\mathbb{R}$ and pieces of its monotone rearrangement on $\mathbb{R}^+$, then projected on the Nehari manifold.

For instance, on the tadpole, such a function coincides on the loop of the graph (of total length L) with the restriction of the soliton to the interval $[-L/2, L/2]$, and on the half-line with the monotone rearrangement on $\mathbb{R}^+$ of the restriction of the soliton to $\mathbb{R} \setminus [-L/2, L/2]$. The construction on the other graphs in Figure 2.6 is analogous. Since rearrangements always lower the normalizing factor n_λ defined in (2.12), it is then easy to check that these functions realize action levels strictly smaller than that of the soliton s_λ. Existence of a ground state follows then from Theorem 2.3. □

When Z is not empty, the existence of a ground state is harder to obtain and further conditions of metrical nature have to be imposed. Indeed, the next theorem shows that, if a graph hosts a ground state, the diameter of the set $\mathcal{B}$ of all bounded edges cannot be arbitrarily small. Recall that $\mathrm{diam}(\mathcal{B})$ is given by the supremum of lengths of the shortest paths between any two points of $\mathcal{B}$.

Theorem 2.27. *There exists a constant $C > 0$ depending only on $\lambda > 0$ and p such that, for every $\mathcal{G} \in \mathbf{G_2}$ with at least one half-line and every $Z \neq \varnothing$ such that $\inf_{v \in \mathcal{N}_Z} J(v)$ is achieved, we have*

$$\operatorname{diam}(\mathcal{B}) \geq C,$$

where, as above, $\mathcal{B}$ is the set of all bounded edges of $\mathcal{G}$.

Proof. Let $u \in \mathcal{N}_Z$ satisfy $J(u) = \inf_{v \in \mathcal{N}_Z} J(v)$. As usual, we can assume that $u \geq 0$. Let us show that u attains its L^∞-norm in $\mathcal{B}$. Suppose by contradiction that u instead attains its maximum at y on a half-line h. Then $\# u^{-1}(t) \geq 2$ for every $0 < t < \max u$. Indeed, t is attained at least once on h and once on a path γ joining y to a point in Z. Thus Lemma 2.23 and Proposition 2.19 imply that $J(u) \geq s_\lambda = \inf_{v \in \mathcal{N}_Z} J(v)$.

Let us show that the inequality must be strict, which gives the desired contradiction. If it was not, Lemma 2.23 would imply that $\# u^{-1}(t) = 2$ for almost every $t \in (0, \max u)$. Since u already has two preimages on $h \cup \gamma$, that means that u must be constant on $\mathcal{G} \setminus (h \cup \gamma)$. This contradicts the last assertion of Lemma 2.23 unless $\mathcal{G} = h \cup \gamma$. But this last equality is neither possible because for the half-line with a zero Dirichlet boundary condition, there are no nonzero solutions as it was mentioned at the beginning of this section.

Let then $\bar{x} \in \mathcal{B}$ be such that $\|u\|_\infty = u(\bar{x})$. By the Cauchy–Schwarz inequality, (2.11) and Proposition 2.19, letting z be any vertex in Z, we have

$$\|u\|_\infty = |u(\bar{x})| = |u(\bar{x}) - u(z)| \leq \sqrt{\operatorname{diam}(\mathcal{B})}\|u'\|_{L^2(\mathcal{G})} \leq \sqrt{\operatorname{diam}(\mathcal{B})}\sqrt{\frac{s_\lambda}{\varkappa}}$$

which, coupled with Proposition 2.13, yields

$$C \leq \|u\|_p^p \leq \|u\|_\infty^{p-2}\|u\|_2^2 \leq \frac{1}{\lambda}\operatorname{diam}(\mathcal{B})^{\frac{p}{2}-1}\left(\frac{s_\lambda}{\varkappa}\right)^{\frac{p}{2}},$$

for a suitable constant $C > 0$ depending on λ and p only, and we conclude. □

Remark 2.28. Comparing Theorems 2.26 and 2.27 highlights the effect of the set Z on the existence of ground states. One may wonder whether Theorem 2.27 can be improved to obtain a universal lower bound involving only the total length of the edges with a vertex in Z, rather than the whole set $\mathcal{B}$. However, this cannot be done in general: it is easy to exhibit graphs where ground states do exist and the length of the edges with vertices in Z is arbitrarily small. To see this, let $\mathcal{G}$ be any given graph with $Z = \varnothing$ and such that $\inf_{v \in \mathcal{N}(\mathcal{G})} J(v) < s_\lambda$ (e.g. any of the graphs in Figure 2.6). Exploiting for instance the approximation procedure described in Remark 2.16, one can construct a function $u \in \mathcal{N}(\mathcal{G})$ so that $J(u) < s_\lambda$ and the support of u is contained in a suitable neighborhood of $\mathcal{B}$.

In particular, there exists $M > 0$ such that the restriction of u to each half-line of $\mathcal{G}$ satisfies $u \equiv 0$ on $[M, +\infty)$. For every $\ell > 0$, let then $\mathcal{G}_\ell$ be the graph obtained by attaching a single edge of length ℓ at the point $x = M$ of one of the half-lines of $\mathcal{G}$, and assume that the vertex of degree 1 of this edge is the only vertex in Z. Clearly, one can think of u as a function in $\mathcal{N}_Z(\mathcal{G}_\ell)$ for every ℓ, so that $\inf_{v\in\mathcal{N}_Z(\mathcal{G}_\ell)} J(v) \leq J(u) < s_\lambda$, thus implying existence of ground states in $\mathcal{N}_Z(\mathcal{G}_\ell)$ by Theorem 2.3 and Proposition 2.24.

In the case of nodal ground states, it is not even needed to have $Z \neq \varnothing$ to recover the analogue of Theorem 2.27.

Theorem 2.29. *There exists a constant $C > 0$ depending only on $\lambda > 0$ and p such that, for every $\mathcal{G} \in \mathbf{G_2}$ with at least one half-line and every Z such that $\inf_{v\in\mathcal{N}_Z^{nod}} J(v)$ is achieved, we have*

$$\operatorname{diam}(\mathcal{B}) \geq C,$$

where, as above, $\mathcal{B}$ is the set of all bounded edges of $\mathcal{G}$.

Proof. Let u be a nodal ground state. Observe that if u^+ attains its L^∞-norm on a half-line, as in Theorem 2.27, we prove that $J_\lambda(u^+) > s_\lambda$. Hence, we have $J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-) > s_\lambda + \inf_{v\in\mathcal{N}_Z(\mathcal{G})} J_\lambda(v)$, which contradicts (2.8). The same is valid for u^-. Hence both u^+ and u^- attain their L^∞-norm on $\mathcal{B}$ only. Thus u changes sign in $\mathcal{B}$ and as such, has a zero in $\mathcal{B}$. We then conclude as in Theorem 2.27 working on u^+. □

In view of Theorems 2.6–2.27–2.29, it is clear that a suitable combination of topological and metrical features is needed to guarantee existence of ground states with $Z \neq \varnothing$ and nodal ground states. Towards this direction, we conclude the discussion of this section with two general procedures to construct graphs where ground states and nodal ground states do exist. The first one is genuinely of metrical nature, in that it is completely independent of the topology of the graph. The second one mixes topological and metrical properties.

In the next statement, by *pendant* we mean a finite length terminal edge whose vertex of degree 1 is not in Z.

Theorem 2.30. *There exists a constant $C > 0$ depending only on $\lambda > 0$ and p such that, for every noncompact graph $\mathcal{G} \in \mathbf{G_2}$ with a finite number of edges,*

1) if $\mathcal{G}$ has a pendant of length $a \geq C$, then $\inf_{v\in\mathcal{N}_Z} J(v)$ is achieved;

2) if $\mathcal{G}$ has two pendants of lengths $a_1, a_2 \geq C$, then $\inf_{v\in\mathcal{N}_Z^{nod}} J(v)$ is achieved.

Remark 2.31. Observe again that the assumption that $\mathcal{G}$ has a finite number of edges cannot be removed. This can be easily seen as follows. For point 1) it is enough to consider the graph $\mathcal{G}$ in Remark 2.25, for which $\inf_{v\in\mathcal{N}_\lambda(\mathcal{G})} J(v) = \frac{s_\lambda}{2}$. Therefore, if u were a (positive) ground state, by Lemma 2.23, almost every $t \in (0, \max u)$ would be attained only once on $\mathcal{G}$, which is incompatible with the presence of vertices of degree 3. For point 2) one can simply consider any periodic graph, since such graphs admit no nodal ground state by Theorem 2.7.

Remark 2.32. Observe that in Theorem 2.30 the constant C is the same for ground states and for nodal ground states and depends on the presence of the pendants but not on the rest of the graph. This will be of great help in section 2.6.

Proof of Theorem 2.30. Let $\psi_\lambda \in H^1(\mathbb{R}^+)$ be the positive half-soliton satisfying, by (2.13), $J(\psi_\lambda) = s_\lambda/2$. By density, there exists a function $u_1 \in \mathcal{N}(\mathbb{R}^+)$ supported in some interval $[0, C]$ such that

$$J(u_1) < \frac{3}{4}s_\lambda$$

(for example, one may take $(\psi_\lambda - \delta)^+$ with δ small and then project it on $\mathcal{N}_\lambda(\mathbb{R}^+)$).

Part 1) Let $\mathcal{G}_a$ be a graph with at least one pendant of length a. If a is larger than C, we may consider $[0, C]$ as contained in the pendant, identifying $x = 0$ with its vertex of degree 1. Extending u_1 by 0 on the remaining part of $\mathcal{G}_a$, we obtain a function $\widetilde{u}_1 \in \mathcal{N}_Z(\mathcal{G}_a)$ such that

$$J(\widetilde{u}_1) = J(u_1) < \frac{3}{4}s_\lambda.$$

The existence of a ground state follows from Proposition 2.24 and Theorem 2.3.

Part 2) We denote by $\mathcal{G}_{a_1,a_2}$ a graph with two pendants e_1, e_2 of lengths a_1, a_2 and we show that if $a_1 \geq C$ and $a_2 \geq C$, then

$$\inf_{v\in\mathcal{N}_Z^{nod}(\mathcal{G}_{a_1,a_2})} J(v) < J^\infty(\mathcal{G}_{a_1,a_2}; Z) + \inf_{v\in\mathcal{N}_Z(\mathcal{G}_{a_1,a_2})} J(v) \tag{2.19}$$

which, via Theorem 2.3, establishes the existence of a nodal ground state. Notice that, by Propositions 2.19–2.24, for every a_1, a_2,

$$J^\infty(\mathcal{G}_{a_1,a_2}; Z) = s_\lambda, \qquad \inf_{v\in\mathcal{N}_Z(\mathcal{G}_{a_1,a_2})} J(v) \geq \frac{1}{2}s_\lambda,$$

so that

$$J^\infty(\mathcal{G}_{a_1,a_2}; Z) + \inf_{v\in\mathcal{N}_Z(\mathcal{G}_{a_1,a_2})} J(v) \geq \frac{3}{2}s_\lambda.$$

Now, using the same u_1 as in 1), we define

$$\widetilde{u}_1(x) = \begin{cases} u_1(x) & \text{if } x \in [0, a_1] \subseteq e_1, \\ -u_1(x) & \text{if } x \in [0, a_2] \subseteq e_2, \\ 0 & \text{elsewhere on } \mathcal{G}_{a_1,a_2}. \end{cases}$$

Clearly $\widetilde{u}_1 \in \mathcal{N}_Z^{nod}(\mathcal{G}_{a_1,a_2})$ and

$$J(\widetilde{u}_1) = 2J(u_1) < \frac{3}{2}s_\lambda.$$

This proves (2.19) and concludes the proof. □

We now discuss the second procedure to find nodal ground states. The idea is to take two graphs admitting ground states and connect them by the addition of a faraway edge. So, let $\mathcal{G}^1$, $\mathcal{G}^2$ be any two noncompact graphs with a finite number of edges for which $\inf_{v\in\mathcal{N}_{Z^i}(\mathcal{G}^i)} J(v) < s_\lambda$. Given $L > 0$, we glue together $\mathcal{G}^1$ and $\mathcal{G}^2$ by taking a new edge of length 1, attaching its first endpoint to the point $x = L$ on a half-line h^1 of $\mathcal{G}^1$ and its second endpoint to the point $x = L$ on a half-line h^2 of $\mathcal{G}^2$. We call $\mathcal{G}_L$ the resulting graph (see Figure 2.7) and we let the set Z_L of vertices of degree 1 in $\mathcal{G}_L$ with homogeneous Dirichlet conditions be given by the union of the corresponding sets of vertices in $\mathcal{G}^1$ and $\mathcal{G}^2$.

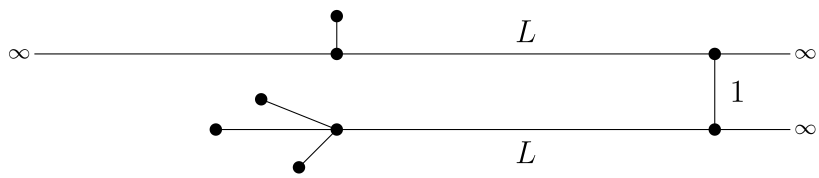


Figure 2.7: Example of a graph $\mathcal{G}_L$ as in Theorem 2.34, constructed starting with two graphs in Figure 2.6. If the vertical edge on the right is sufficiently far from the pendants of the graph, nodal ground states exist.

Lemma 2.33. *Let $\lambda > 0$ and $\mathcal{G}^1$, $\mathcal{G}^2$, $\mathcal{G}_L$ be the graphs described above. Then*

$$\lim_{L\to\infty} \inf_{v\in\mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) = \min\left(\inf_{v\in\mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v), \inf_{v\in\mathcal{N}_{Z^2}(\mathcal{G}^2)} J(v)\right).$$

Proof. Without loss of generality, assume that

$$c_1 := \inf_{v\in\mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v) \le \inf_{v\in\mathcal{N}_{Z^2}(\mathcal{G}^2)} J(v) =: c_2. \tag{2.20}$$

For every $\varepsilon > 0$ there exists a function $u_\varepsilon \in \mathcal{N}_{Z^1}(\mathcal{G}^1)$ with compact support such that $J(u_\varepsilon) \le c_1 + \varepsilon$. For every L large enough, we can view u_ε as a function in $\mathcal{N}_{Z_L}(\mathcal{G}_L)$, after extending it to zero on $\mathcal{G}_L$ outside its support. Therefore

$$\limsup_{L\to\infty} \inf_{v\in\mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) \le \limsup_{L\to\infty} J(u_\varepsilon) = J(u_\varepsilon) \le c_1 + \varepsilon$$

and since ε is arbitrary we deduce that

$$\limsup_{L\to\infty} \inf_{v\in\mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) \le c_1. \tag{2.21}$$

We now prove a complementary inequality. For every L, let $u_L \in \mathcal{N}_{Z_L}(\mathcal{G}_L)$ satisfy

$$J(u_L) \le \inf_{v\in\mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) + \frac{1}{L} \tag{2.22}$$

and notice that by (2.11), (2.21) and (2.22), u_L is bounded independently of L. Let

$$u_L(x_L) = \min_{h^1\cap[0,L]} u_L(x), \quad u_L(y_L) = \min_{h^2\cap[0,L]} u_L(x)$$

and set

$$\delta_L = \max\left(u_L(x_L), u_L(y_L)\right).$$

Since u_L is uniformly bounded in $L^2(\mathcal{G}_L)$, $\delta_L \to 0$ as $L \to \infty$.

Consider the function $(u_L - \delta_L)^+ \in H^1_{Z_L}(\mathcal{G}_L)$, which does not vanish identically, for all L large, by Proposition 2.13. Exactly as in (2.18), $n_\lambda\left((u_L-\delta_L)^+\right) \le 1 + o(1)$ as $L \to \infty$.

Set $v_L = \pi_\lambda\left((u_L-\delta_L)^+\right)$. Now $v_L \in \mathcal{N}_{Z_L}(\mathcal{G}_L)$, it vanishes at x_L, y_L and

$$J(v_L) = \varkappa n_\lambda\left((u_L-\delta_L)^+\right)^p \|(u_L-\delta_L)^+\|_p^p \le \varkappa(1+o(1))\|u_L\|_p^p = J(u_L) + o(1) \tag{2.23}$$

as $L \to \infty$.

We now cut $\mathcal{G}_L$ at x_L and y_L, splitting it into three parts $\overline{\mathcal{G}}^1 \subseteq \mathcal{G}^1$, $\overline{\mathcal{G}}^2 \subseteq \mathcal{G}^2$ and $\mathcal{G}^3 = \mathcal{G}_L \setminus (\overline{\mathcal{G}}^1 \cup \overline{\mathcal{G}}^2)$. We call v_i $(i = 1, 2)$ the two vertices of $\mathcal{G}^3$ created on h^i (see Figure 2.8).

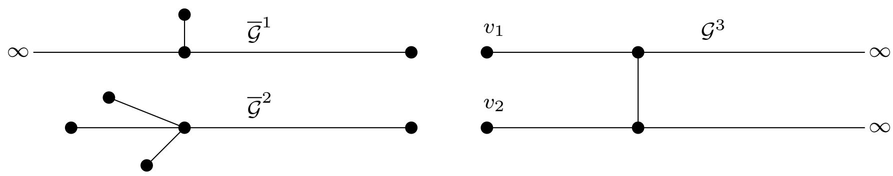


Figure 2.8: The graph $\mathcal{G}_L$ of Figure 2.7 splits into the three graphs $\overline{\mathcal{G}}^1$, $\overline{\mathcal{G}}^2$ and $\mathcal{G}^3$.

We can use v_L to construct a function $v_L^1 \in H^1_{Z^1}(\mathcal{G}^1)$ by setting

$$v_L^1(x) = \begin{cases} v_L(x) & \text{if } x \in \overline{\mathcal{G}}^1, \\ 0 & \text{elsewhere on } \mathcal{G}^1 \end{cases}$$

and in the same way we contruct a function $v_L^2 \in H^1_{Z^2}(\mathcal{G}^2)$. Finally, we call v_L^3 the restriction of v_L to $\mathcal{G}^3$. Setting $Z^3 = \{v_1, v_2\}$, by construction there results $v_L^3 \in H^1_{Z^3}(\mathcal{G}^3)$.

If $v_L^i \neq 0$, then there exists $\theta_i \in \mathbb{R}$ so that $v_L^i \in \mathcal{N}_{\theta_i, Z^i}(\mathcal{G}^i)$. Taking $\theta_i = 0$ if $v_L^i = 0$ and recalling that $v_L \in \mathcal{N}_{Z_L}(\mathcal{G}_L)$, we obtain

$$\lambda = \frac{\|v_L^1\|_2^2}{\|v_L\|_2^2}\theta_1 + \frac{\|v_L^2\|_2^2}{\|v_L\|_2^2}\theta_2 + \frac{\|v_L^3\|_2^2}{\|v_L\|_2^2}\theta_3. \tag{2.24}$$

Furthermore,

$$J(v_L) = \varkappa(\|v_L^1\|_p^p + \|v_L^2\|_p^p + \|v_L^3\|_p^p) \geq \varkappa \max\left\{\|v_L^1\|_p^p, \|v_L^2\|_p^p, \|v_L^3\|_p^p\right\}.$$

Since, by (2.24), λ is a convex combination of the θ_i's, at least one of the three will satisfy $\theta_i \geq \lambda$.

If $\theta_1 \geq \lambda$, by Remark 2.14, $\varkappa\|v_L^1\|_p^p \geq c_1$. If $\theta_2 \geq \lambda$, by (2.20), $\varkappa\|v_L^2\|_p^p \geq c_2 \geq c_1$. If $\theta_3 \geq \lambda$, $\varkappa\|v_L^3\|_p^p \geq \inf_{\mathcal{N}_{\lambda,Z^3}(\mathcal{G}_3)} J \geq s_\lambda \geq c_1$ since $\mathcal{G}^3$ satisfies (H0) and by the assumptions on $\mathcal{G}^1$. In each case we deduce, via (2.23), that

$$J(u_L) \geq J(v_L) + o(1) \geq c_1 + o(1)$$

as $L \to \infty$, so that by (2.22),

$$\liminf_L \inf_{v \in \mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) \geq \liminf_L \Big(J(u_L) - 1/L\Big) \geq c_1.$$

In view of (2.21), this ends the proof. □

Theorem 2.34. *Let $\lambda > 0$ and $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}_L$ be the graphs considered above. If L is large enough, then there exist nodal ground states on $\mathcal{G}_L$.*

Proof. Without loss of generality, we assume that

$$\min\left(\inf_{v \in \mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v), \inf_{v \in \mathcal{N}_{Z^2}(\mathcal{G}^2)} J(v)\right) = \inf_{v \in \mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v).$$

Let $\varepsilon := \frac{1}{3}\Big(s_\lambda - \inf_{v \in \mathcal{N}_{Z^2}(\mathcal{G}^2)} J(v)\Big)$. By Lemma 2.33, we choose L so large that

$$\inf_{v \in \mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) \geq \inf_{v \in \mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v) - \varepsilon \tag{2.25}$$

and that there exist nonnegative $u^i \in \mathcal{N}_{Z^i}(\mathcal{G}^i)$, with compact support satisfying

$$J(u^i) \leq \inf_{v \in \mathcal{N}_{Z^i}(\mathcal{G}^i)} J(v) + \varepsilon.$$

In particular, there is $M > 0$ such that the restriction of u^i to each half-line of $\mathcal{G}^i$ vanishes on $[M, +\infty)$. Hence, for every $L \geq M$, we define $w : \mathcal{G}_L \to \mathbb{R}$ as

$$w(x) := \begin{cases} u^1(x) & \text{if } x \in \mathcal{G}^1, \\ -u^2(x) & \text{if } x \in \mathcal{G}^2, \\ 0 & \text{elsewhere on } \mathcal{G}_L, \end{cases}$$

where with a slight abuse of notation we still denote by $\mathcal{G}^1$, $\mathcal{G}^2$ the corresponding subgraphs of $\mathcal{G}_L$.

Clearly, $w \in \mathcal{N}_{Z_L}^{nod}(\mathcal{G}_L)$ and, by (2.25) and the choice of ε, we have

$$\begin{aligned}\inf_{v\in\mathcal{N}_{Z_L}^{nod}(\mathcal{G}_L)} J(v) \le J(w) = J(u^1) + J(u^2) \\ &\le \inf_{v\in\mathcal{N}_{Z^1}(\mathcal{G}^1)} J(v) + \inf_{v\in\mathcal{N}_{Z^2}(\mathcal{G}^2)} J(v) + 2\varepsilon \\ &< \inf_{v\in\mathcal{N}_{Z_L}(\mathcal{G}_L)} J(v) + s_\lambda,\end{aligned}$$

implying that nodal ground states exist by Theorem 2.3 and Proposition 2.24. □

Remark 2.35. Concrete examples of graphs fulfilling the hypotheses of Theorem 2.34 can be produced starting for instance from any of the graphs in Figure 2.6 (see e.g. Figure 2.7). Note that, since $\inf_{v\in\mathcal{N}_Z(\mathcal{G})} J(v) < s_\lambda$ implies $\#F(\mathcal{G}) \ge 1$, by construction we have $\#F(\mathcal{G}_L) \ge 2$.

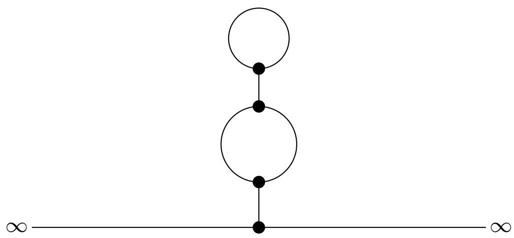


Figure 2.9: A graph with $\#F(\mathcal{G}) = 2$ where nodal ground states never exist, independently of the length of the edges.

Remark 2.36. Theorem 2.6 (ii) and Theorem 2.29 show that, if (H1) holds, or if the set of all bounded edges of $\mathcal{G}$ is too small, nodal ground states never exist, whereas Theorem 2.34 proves that there exist graphs with $\#F(\mathcal{G}) \ge 2$ and a sufficiently large compact core where nodal ground states do exist. However, even though the former provides sufficient conditions for nonexistence, the latter are not sufficient conditions for existence. It is in fact not difficult to produce examples of graphs with $\#F(\mathcal{G}) = 2$, and compact core of arbitrary size, where nodal ground states do not exist.

For instance, consider the graph in Figure 2.9. If u is a nodal ground state on this graph, we know that either u is identically equal to zero on the two half-lines or it has constant sign on them. Assume thus that $u \ge 0$ on the half–lines. Then, since u^+ is not identically zero, u^+ vanishes on the set $\mathcal{B}$ of the bounded edges of $\mathcal{G}$ at least at a point different from the vertex of the half-lines. We then have that u^+ has always at least two preimages for every $t \in (0, \max u^+)$ by Proposition 2.38 and hence $J(u^+) \ge s_\lambda$ by Lemma 2.23. In view of Remark 2.15, as $u^- \in \mathcal{N}_Z(\mathcal{G})$ vanishes somewhere on the graph, we have also that $J(u^-) > \inf_{v\in\mathcal{N}_Z(\mathcal{G})} J(v)$. This implies that

$$J(u) = J(u^-) + J(u^+) > \inf_{v\in\mathcal{N}_Z(\mathcal{G})} J(v) + s_\lambda,$$

which contradicts (2.8).

2.5 Graphs with infinitely many bounded edges

In this section we extend our discussion about ground states and nodal ground states to graphs with infinitely many edges whose length is uniformly bounded. In particular, we focus on two subclasses of such graphs that have already been considered in the literature: periodic graphs and regular trees.

2.5.1 Periodic graphs

Throughout this section, when we speak of a periodic metric graph we mean a graph fulfilling [68, Definition 4.1.1]. We avoid reporting the full details of the definition here. For our purposes, it is enough to recall that, if $\mathcal{G}$ is a periodic graph, then there exists a number $n \in \mathbb{Z}^{\geq 1}$ and a compact subset $W \subset \mathcal{G}$, called a *fundamental domain* of $\mathcal{G}$, such that

$$\mathcal{G} = \bigcup_{k \in \mathbb{Z}^n} W_k \,,$$

where W_k is a copy of W for every $k \in \mathbb{Z}^n$, and $W_i \cap W_j$ contains at most finitely many points for every $i \neq j$. In this case, we say that $\mathcal{G}$ is a $\mathbb{Z}^n$-periodic graph.

Since we are concerned with problems involving the homogeneous Dirichlet conditions on a subset Z of the vertices of $\mathcal{G}$, we specify that when $\mathcal{G}$ is a $\mathbb{Z}^n$-periodic graph, we only consider here $\mathbb{Z}^n$-periodic subsets Z (that is, $Z \cap W_k$ is a copy of $Z \cap W$ for every $k \in \mathbb{Z}^n$).

Proof of Theorem 2.7. We address independently the case of ground states and nodal ground states.

Part 1: existence of ground states. Let $(u_n)_n \subseteq \mathcal{N}_Z$ be a minimizing sequence such that $\lim_n J(u_n) = \inf_{v \in \mathcal{N}_Z} J(v)$. Exploiting the periodicity of $\mathcal{G}$ and Z, we can assume with no loss of generality that u_n attains its L^∞ norm on W_0, for every n. Since $(u_n)_n$ is bounded in $H^1(\mathcal{G})$, up to subsequences $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$ and $u_n \to u$ in $L^\infty_{\text{loc}}(\mathcal{G})$ as $n \to \infty$. Furthermore, $u \not\equiv 0$ on $\mathcal{G}$ because, if this were not the case, by the strong convergence of (u_n) to u in $L^\infty(W_0)$ we would have $\|u_n\|_{L^\infty(\mathcal{G})} = \|u_n\|_{L^\infty(W_0)} \to 0$ as $n \to \infty$, contradicting Proposition 2.13.

Let us remark that, if $u_n \to u$ in $L^2(\mathcal{G})$ then, using standard Gagliardo–Nirenberg inequalities (see Corollary A.8), we deduce that $u_n \to u$ in $L^p(\mathcal{G})$ and, by weak lower semicontinuity, $n_\lambda(u) \leq 1$, so that

$$\inf_{v \in \mathcal{N}_Z} J(v) \leq J(\pi_\lambda(u)) = \varkappa n_\lambda(u)^p \|u\|_p^p \leq \lim_n \varkappa \|u_n\|_p^p = \inf_{v \in \mathcal{N}_Z} J(v) \,,$$

i.e. $\pi_\lambda(u)$ is a ground state.

Let us thus show that u_n converges to u strongly in $L^2(\mathcal{G})$. Let us assume by contradiction that

$$\liminf_n \|u_n - u\|_2^2 > 0 \,.$$

Let $\theta \in \mathbb{R}$ and $(\lambda_n)_n \subseteq \mathbb{R}$ be such that $u \in \mathcal{N}_{\theta,Z}$, $u_n - u \in \mathcal{N}_{\lambda_n,Z}$ for every n.

By the weak convergence of $(u_n)_n$ to u in $H^1(\mathcal{G})$, the Brezis–Lieb lemma [86] and the fact that $u_n \in \mathcal{N}_{\lambda,Z}$, we have

$$\begin{aligned}
\lambda_n &= \frac{\|u_n - u\|_p^p - \|u_n' - u'\|_2^2}{\|u_n - u\|_2^2} \\
&= \frac{\|u_n\|_p^p - \|u_n'\|_2^2 - \|u\|_p^p + \|u'\|_2^2 + o(1)}{\|u_n - u\|_2^2} \\
&= \frac{\lambda\|u_n\|_2^2 - \theta\|u\|_2^2 + o(1)}{\|u_n - u\|_2^2} \\
&= \lambda + \frac{(\lambda - \theta)\|u\|_2^2 + o(1)}{\|u_n - u\|_2^2} \\
&= \lambda + \frac{\|u\|_2^2}{\|u_n - u\|_2^2}(\lambda - \theta) + o(1) \qquad (2.26)
\end{aligned}$$

as $n \to \infty$. Applying again Brezis–Lieb lemma, we obtain

$$\inf_{v \in \mathcal{N}_{\lambda,Z}} J_\lambda(v) = \lim_n \varkappa\|u_n\|_p^p = \lim_n \varkappa\|u_n - u\|_p^p + \varkappa\|u\|_p^p \,. \qquad (2.27)$$

Keeping in mind that $\lambda > 0$, we distinguish three cases.

- If $\theta > \lambda$, then, by Remark 2.14,

$$\lim_n \varkappa\|u_n - u\|_p^p + \varkappa\|u\|_p^p \geq \varkappa\|u\|_p^p \geq \inf_{v \in \mathcal{N}_{\theta,Z}} J_\theta(v) > \inf_{v \in \mathcal{N}_{\lambda,Z}} J_\lambda(v).$$

- If $\theta = \lambda$, we see from (2.26) that $\lambda_n \to \lambda$ as $n \to \infty$. Therefore, using again Remark 2.14,

$$\lim_n \varkappa\|u_n - u\|_p^p + \varkappa\|u\|_p^p \geq \liminf_n \inf_{v \in \mathcal{N}_{\lambda_n,Z}} J_{\lambda_n}(v) + \inf_{v \in \mathcal{N}_{\theta,Z}} J_\theta(v) = 2 \inf_{v \in \mathcal{N}_{\lambda,Z}} J_\lambda(v) \,.$$

- If $\theta < \lambda$, we have $\liminf_n \lambda_n > \lambda$ and, similarly,

$$\lim_n \varkappa\|u_n - u\|_p^p + \varkappa\|u\|_p^p \geq \liminf_n \inf_{v \in \mathcal{N}_{\lambda_n,Z}} J_{\lambda_n}(v) > \inf_{v \in \mathcal{N}_{\lambda,Z}} J_\lambda(v).$$

In all three cases, (2.27) yields a contradiction.

Part 2: nonexistence of nodal ground states. By Theorem 2.4, it is enough to show that $\inf_{v\in\mathcal{N}_Z^{nod}} J(v) \le 2\inf_{v\in\mathcal{N}_Z} J(v)$. To this end, given $\varepsilon > 0$, let $u \in \mathcal{N}_Z$ be such that $J(u) \le \inf_{v\in\mathcal{N}_Z} J(v) + \varepsilon$. With no loss of generality, we can take such u to be nonnegative, compactly supported on $\mathcal{G}$ and attaining its L^∞ norm on W_0 (it is for instance enough to apply Remark 2.16 to a suitable ground state of J in $\mathcal{N}_Z$, that exists by the first part of the proof). Hence, there exists $M > 0$ such that $\text{supp}(u) \subseteq \bigcup_{|k|\le M} W_k$. Let then $\bar{u} \in \mathcal{N}_Z$ be a translation of u on $\mathcal{G}$ such that $\text{supp}(\bar{u}) \subseteq \bigcup_{|k|>M} W_k$ and define $w : \mathcal{G} \to \mathbb{R}$ as

$$w(x) := \begin{cases} u(x) & \text{if } x \in \text{supp}(u), \\ -\bar{u}(x) & \text{if } x \in \text{supp}(\bar{u}), \\ 0 & \text{elsewhere on } \mathcal{G}\,. \end{cases}$$

By construction, w belongs to $\mathcal{N}_Z^{nod}$ and we have

$$J(w) = J(u) + J(\bar{u}) \le 2 \inf_{v\in\mathcal{N}_Z} J(v) + 2\varepsilon.$$

Given the arbitrariness of $\varepsilon > 0$, we conclude. □

Remark 2.37. Observe that $J^\infty(\mathcal{G}; Z) = \inf_{v\in\mathcal{N}_Z} J(v)$ for every periodic graph $\mathcal{G}$ and every set Z with the same periodicity. This is the reason why it is not possible to rely directly on the abstract result of Theorem 2.3 to prove Theorem 2.7.

2.5.2 Regular trees

Recall that a regular tree is an acyclic, noncompact metric graph with edges all of the same length and vertices all of the same degree $d \ge 3$ (unrooted tree), with the possible exception of a single vertex of degree 1 (rooted tree). Note that, if $\mathcal{G}$ is an unrooted tree, then necessarily $Z = \varnothing$ since every vertex has degree at least 3, whereas if $\mathcal{G}$ is a rooted tree either $Z = \varnothing$ or it coincides with the root of $\mathcal{G}$ (i.e. the unique vertex of degree 1).

We divide the proof of Theorem 2.8 in two parts, proving first statements (i)–(ii) on ground states and then statement (iii) on nodal ground states.

Proof of Theorem 2.8. We split the proof in several steps.

Step 1: ground states when $\mathcal{G}$ is an unrooted tree. Let $(u_n)_n \subseteq \mathcal{N}$ be such that $\lim_n J(u_n) = \inf_{v\in\mathcal{N}} J(v)$. Exploiting the symmetry of $\mathcal{G}$, it is not restrictive to assume that u_n attains its L^∞ norm in the same fixed edge of $\mathcal{G}$, for every n. Indeed, the problem is invariant under any isometry of $\mathcal{G}$ and the isometry group of the tree $\mathcal{G}$ acts transitively on the edges of $\mathcal{G}$. Hence, arguing as in the proof of Theorem 2.7 shows that the weak limit in $H^1(\mathcal{G})$ of $(u_n)_n$ provides a desired ground state for J in $\mathcal{N}$.

Step 2: ground states when $\mathcal{G}$ is a rooted tree and $Z = \varnothing$. Let r be the root of $\mathcal{G}$, $d \ge 3$ be the degree of each vertex of $\mathcal{G}$ different from the root, and $\overline{\mathcal{G}}$ be the unrooted tree obtained by gluing together d copies of $\mathcal{G}$ at their roots.

Let us first prove that

$$J^\infty(\mathcal{G}) = \inf_{v\in\mathcal{N}_{\{r\}}(\mathcal{G})} J(v) = \inf_{v\in\mathcal{N}(\overline{\mathcal{G}})} J(v). \tag{2.28}$$

In order to prove the equalities in (2.28), given any function $u \in \mathcal{N}_{\{r\}}(\mathcal{G})$, we construct a sequence $(u_n)_n \subseteq \mathcal{N}_{\{r\}}(\mathcal{G})$ converging weakly to 0 in $H^1(\mathcal{G})$ by translating u along $\mathcal{G}$ and extending it by 0 on the remaining part of the graph. This proves that $J^\infty(\mathcal{G}) \le \inf_{v\in\mathcal{N}_{\{r\}}(\mathcal{G})} J(v)$.

Next, let $(u_n)_n \subseteq \mathcal{N}(\mathcal{G})$ be a sequence converging weakly to 0 in $H^1(\mathcal{G})$ and such that $J(u_n) \to J^\infty(\mathcal{G})$. Since $u_n(r) \to 0$ by $L^\infty_{\mathrm{loc}}(\mathcal{G})$ convergence, we can assume without loss of generality that each u_n satisfies $u_n(r) = 0$, namely that $u_n \in \mathcal{N}_{\{r\}}(\mathcal{G})$. This shows that $J^\infty(\mathcal{G}) \ge \inf_{v\in\mathcal{N}_{\{r\}}(\mathcal{G})} J(v)$, and the first equality is proved.

To prove the second equality, notice that any $u \in \mathcal{N}_{\{r\}}(\mathcal{G})$ can be seen as an element of $\mathcal{N}(\overline{\mathcal{G}})$, after extending it by 0 on $\overline{\mathcal{G}} \setminus \mathcal{G}$. On the other hand, any $u \in \mathcal{N}(\overline{\mathcal{G}})$ with compact support can be considered (when translated in such a way that $r \notin \mathrm{supp}(u)$) as an element of $\mathcal{N}_{\{r\}}(\mathcal{G})$. By density this is enough to conclude the proof of (2.28).

By (2.28) and Theorem 2.3, in order to prove the existence of a ground state, it is sufficient to show that

$$\inf_{v\in\mathcal{N}(\mathcal{G})} J(v) < \inf_{v\in\mathcal{N}(\overline{\mathcal{G}})} J(v)\,. \tag{2.29}$$

To this end, let $u \in \mathcal{N}(\overline{\mathcal{G}})$ be a positive ground state of J in $\mathcal{N}(\overline{\mathcal{G}})$, whose existence is guaranteed by the previous step. Take a vertex v of $\overline{\mathcal{G}}$ and split the graph at v into d disjoint rooted trees $\mathcal{G}_i$, $i = 1, \dots, d$. For every i, let $u_i > 0$ be the restriction of u to $\mathcal{G}_i$ and $\lambda_i \in \mathbb{R}$ be such that $u_i \in \mathcal{N}_{\lambda_i}(\mathcal{G}_i)$. Since $u \in \mathcal{N}(\overline{\mathcal{G}})$ and $u > 0$ on $\overline{\mathcal{G}}$, we have

$$\lambda = \sum_{i=1}^{d} \frac{\|u_i\|^2_{L^2(\mathcal{G}_i)}}{\|u\|^2_{L^2(\overline{\mathcal{G}})}}\lambda_i\,,$$

so that

$$\lambda \le (\max_{1\le i\le d} \lambda_i) \sum_{i=1}^{d} \frac{\|u_i\|^2_{L^2(\mathcal{G}_i)}}{\|u\|^2_{L^2(\overline{\mathcal{G}})}} = \max_{1\le i\le d} \lambda_i\,.$$

Hence, there exists $j \in \{1, \dots, d\}$ such that $n_\lambda(u_j) \le 1$.

Since each $\mathcal{G}_i$ is a copy of $\mathcal{G}$, we then have

$$\begin{aligned}
\inf_{v\in\mathcal{N}(\mathcal{G})} J(v) &\leq J(\pi_\lambda(u_j)) \\
&= \varkappa\, n_\lambda(u_j)^p \|u_j\|^p_{L^p(\mathcal{G}_j)} \\
&< \varkappa \sum_{i=1}^{d} \|u_i\|^p_{L^p(\mathcal{G}_i)} \\
&= \varkappa\, \|u\|^p_{L^p(\overline{\mathcal{G}})} \\
&= \inf_{v\in\mathcal{N}(\overline{\mathcal{G}})} J(v)\,,
\end{aligned}$$

that is (2.29).

Step 3: ground states when $\mathcal{G}$ is a rooted tree and $Z \neq \varnothing$. Since Z coincides with the root of $\mathcal{G}$, as in the first part of Step 2 we have

$$\inf_{v\in\mathcal{N}_Z(\mathcal{G})} J(v) = \inf_{v\in\mathcal{N}(\overline{\mathcal{G}})} J(v),$$

where $\overline{\mathcal{G}}$ is the unrooted tree corresponding to $\mathcal{G}$ as above. That the problem on $\mathcal{G}$ has no ground state follows by the fact that, if u were a ground state in $\mathcal{N}_Z(\mathcal{G})$, it would be also a ground state in $\mathcal{N}(\overline{\mathcal{G}})$, as any function on $\mathcal{G}$ vanishing at the root can be regarded as a function on $\overline{\mathcal{G}}$ as well after extending it by 0. Since this is impossible because ground states never vanish on $\overline{\mathcal{G}}$, we conclude.

Step 4: nodal ground states when $\mathcal{G}$ is an unrooted tree or $\mathcal{G}$ is a rooted tree and $Z \neq \varnothing$. If $\mathcal{G}$ is an unrooted tree, exploiting again its symmetry, it is easy to adapt the argument developed in the proof of Theorem 2.7 to show again that

$$\inf_{v\in\mathcal{N}^{nod}(\mathcal{G})} J(v) = 2 \inf_{v\in\mathcal{N}(\mathcal{G})} J(v),$$

and likewise, if $\mathcal{G}$ is a rooted tree with $Z \neq \varnothing$, that

$$\inf_{v\in\mathcal{N}_Z^{nod}(\mathcal{G})} J(v) = 2 \inf_{v\in\mathcal{N}_Z(\mathcal{G})} J(v).$$

This implies that nodal ground states do not exist by Theorem 2.4.

Step 5: nodal ground states when $\mathcal{G}$ is a rooted tree and $Z = \varnothing$. Since, given any $u \in \mathcal{N}^{nod}$, at least one between u^+ and u^- vanishes at the root r, it follows that

$$\inf_{v\in\mathcal{N}^{nod}(\mathcal{G})} J(v) = \inf_{v\in\mathcal{N}(\mathcal{G})} J(v) + \inf_{v\in\mathcal{N}_{\{r\}}(\mathcal{G})} J(v).$$

Arguing as in the proof of Theorem 2.4, this immediately implies that nodal ground states do not exist. □

2.6 Qualitative properties of nodal ground states

The first result of this section concerns the *nodal domains* (i.e. the connected components of $\mathcal{G} \setminus u^{-1}(0)$) of any minimizer u in $\mathcal{N}^{nod}_{\lambda,Z}(\mathcal{G})$.

Theorem 2.38. *Let $\mathcal{G} \in \mathbf{G_2}$ and $\lambda > -\omega_Z(\mathcal{G})$. Let $u \in \mathcal{N}_Z^{nod}$ be a nodal ground state. Then u has exactly two nodal domains.*

Proof. Assume for contradiction that there are at least three nodal domains. Up to a change of sign, we can make sure that on at least two of them u is positive, and we call $\mathcal{G}_1$ one of the two. Since u solves (2.1), multiplying by u and integrating on $\mathcal{G}_1$ we have

$$\int_{\mathcal{G}_1} (|u'|^2 + \lambda|u|^2 - |u|^p)\,\mathrm{d}x = uu'_{|\partial\mathcal{G}_1} + \int_{\mathcal{G}_1} (-u'' + \lambda u - |u|^{p-2}u)u\,\mathrm{d}x = 0, \quad (2.30)$$

because on $\partial\mathcal{G}_1$ either $u = 0$ or $u' = 0$ (this happens at vertices of degree 1 not in Z). Now, we define $v \in H^1_Z(\mathcal{G})$ by

$$v(x) := \begin{cases} u(x) & \text{if } x \in \mathcal{G} \setminus \mathcal{G}_1, \\ 0 & \text{if } x \in \mathcal{G}_1, \end{cases}$$

and we observe that $v^- = u^- \in \mathcal{N}_Z$ and that v^+ (which is not identically zero by construction) satisfies

$$\begin{aligned} &\int_{\mathcal{G}} (|(v^+)'|^2 + \lambda|v^+|^2 - |v^+|^p)\,\mathrm{d}x \\ &= \int_{\mathcal{G}} (|(u^+)'|^2 + \lambda|u^+|^2 - |u^+|^p)\,\mathrm{d}x - \int_{\mathcal{G}_1} (|u'|^2 + \lambda|u|^2 - |u|^p)\,\mathrm{d}x \\ &= 0 \end{aligned}$$

by (2.30) and because $u^+ \in \mathcal{N}_Z$. Therefore $v \in \mathcal{N}_Z^{nod}$ and

$$J(v) = \varkappa\|v\|^p_{L^p(\mathcal{G})} = \varkappa\|u\|^p_{L^p(\mathcal{G})} - \varkappa\|u\|^p_{L^p(\mathcal{G}_1)} < \varkappa\|u\|^p_{L^p(\mathcal{G})} = J(u),$$

violating the minimality of u. □

To conclude, we are left to prove Theorem 2.9. To this end, we will actually prove three independent statements, the full proof of Theorem 2.9 then following by their combination. Each of these statements exhibits a graph supporting a nodal ground state with nodal set respectively given by

1) k isolated points;

2) $m \geq 2$ half-lines all attached to the same vertex;

3) n line segments all attached to the same vertex.

These three constructions, though mutually independent, can all be carried out on the same kind of graph, that we now describe. Given a positive integer N and $L > 0$, let v_1, v_2 be two vertices joined by N edges $e_1, \dots, e_N$, each of length L. Attach to v_1 a pendant and a half-line and do the same to v_2. In this way we obtain the graph $\mathcal{G}_{N,L}$ depicted in Figure 2.10. Throughout, we fix $\lambda > 0$ and the length of the two pendants so that nodal ground states in $\mathcal{N}_\lambda^{nod}(\mathcal{G}_{N,L})$ exist (independently of any other feature of the graph), which is possible by Theorem 2.30.

Proof of 1). Here we show that, for a suitable choice of L, the graph $\mathcal{G}_{k,L}$ admits a nodal ground state u such that $u^{-1}(0)$ consists of k isolated points.

Proposition 2.39. *For every positive integer k, there exists $\overline{L} > 0$, depending on λ and k, such that, for every $L \geq \overline{L}$, every nodal ground state u on $\mathcal{G}_{k,L}$ has a nodal set of the form $u^{-1}(0) = \{x_1, \dots, x_k\}$, where x_i belongs to the interior of the edge e_i.*

To prove this proposition we consider also the graph $\overline{\mathcal{G}}_{k+1}$ made of $k+1$ half-lines and a pendant all attached at the same vertex. The length of the pendant of $\overline{\mathcal{G}}_{k+1}$ coincides with that of the two pendants of $\mathcal{G}_{k,L}$. Hence, by Theorem 2.30, ground states exist in $\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})$.

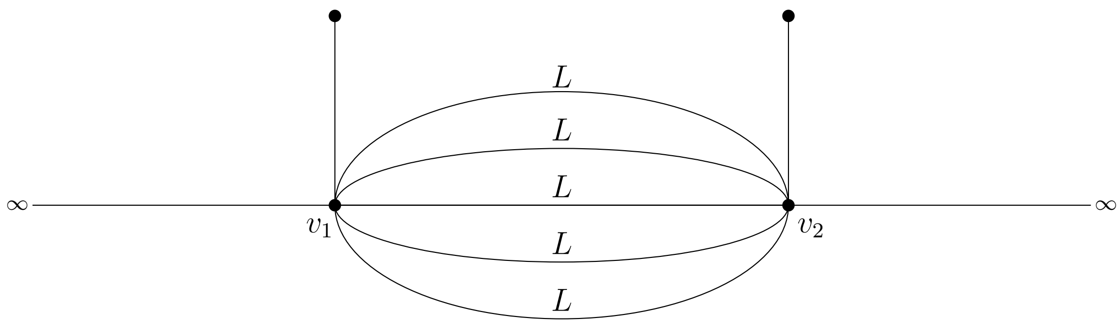


Figure 2.10: The graph $\mathcal{G}_{N,L}$ with $N = 5$

Lemma 2.40. *For every positive integer k, we have*

$$\lim_{L\to\infty} \inf_{v \in \mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v) = \inf_{v \in \mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v).$$

Proof. The argument is similar to that in the proof of Lemma 2.33. Using suitable compactly supported functions in $\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})$, one immediately checks that

$$\limsup_{L\to\infty} \inf_{v \in \mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v) \leq \inf_{v \in \mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v).$$

To show that

$$\liminf_{L\to\infty} \inf_{v\in\mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v) \geq \inf_{v\in\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v), \tag{2.31}$$

it is enough to note that, if $u_L \in \mathcal{N}_\lambda(\mathcal{G}_{k,L})$ satisfies

$$J(u_L) \leq \inf_{v\in\mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v) + \frac{1}{L},$$

then

$$\max_{1\leq i\leq k} \min_{x\in e_i} |u_L(x)| \to 0 \qquad \text{as } L\to\infty.$$

This allows one to obtain (2.31) working exactly as in the proof of Lemma 2.33. □

Lemma 2.41. *If $L\to\infty$, then*

$$\limsup_{L\to\infty} \inf_{v\in\mathcal{N}_\lambda^{nod}(\mathcal{G}_{k,L})} J(v) \leq 2 \inf_{v\in\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v).$$

Proof. The proof follows the same lines as the one of Theorem 2.34. □

Proof of Proposition 2.39. Let u be a nodal ground state in $\mathcal{N}_\lambda^{nod}(\mathcal{G}_{k,L})$.

Step 1: for L long enough, either $u\equiv 0$ on the pendants or it has no zero on their closure. Assume by contradiction that $u \not\equiv 0$ on a pendant p, but $u(x_0)=0$ for some x_0 on p. With no loss of generality, let $u>0$ at the vertex of degree one of p. Outside p, $u<0$ thanks to Theorem 2.38. Denoting as usual by $|p|$ the length of p, since u is a solution to (2.1), we have $u^+\in\mathcal{N}_\lambda(0,|p|)$ with $u^+(|p|)=0$, so that

$$J(u^+) \geq \inf_{\substack{v\in\mathcal{N}_\lambda(0,|p|)\\ v(|p|)=0}} J(v) > \inf_{v\in\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v).$$

This is because the pendant of $\overline{\mathcal{G}}_{k+1}$ can be identified with the interval $[0,|p|]$, but u^+ is not a ground state in $\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})$ as ground states never vanish.

Letting then

$$\delta := \inf_{\substack{v\in\mathcal{N}_\lambda(0,|p|)\\ v(|p|)=0}} J(v) - \inf_{v\in\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v) > 0$$

and recalling that $u^-\in\mathcal{N}_\lambda(\mathcal{G}_{k,L})$, it follows

$$\inf_{v\in\mathcal{N}_\lambda^{nod}(\mathcal{G}_{k,L})} J(v) = J(u) = J(u^-) + J(u^+) \geq \inf_{v\in\mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v) + \inf_{v\in\mathcal{N}_\lambda(\overline{\mathcal{G}}_{k+1})} J(v) + \delta,$$

contradicting Lemmas 2.40–2.41 for L large enough.

Step 2: there holds $u(v_1)u(v_2) < 0$. Assume that this is not the case. Since u solves (2.1), on any of the two half-lines either $u \equiv 0$ or it never vanishes.

Combining with Step 1, this implies that there exist $i \in \{1, \ldots, k\}$ and $\bar{x}_1$, $\bar{x}_2 \in e_i \cup \{v_1, v_2\}$ with $u(\bar{x}_1) = u(\bar{x}_2) = 0$ and, for all $x \in (\bar{x}_1, \bar{x}_2)$, $u \neq 0$. Without loss of generality, let $u > 0$ on $(\bar{x}_1, \bar{x}_2)$. By Theorem 2.38, we know that $u < 0$ on the remaining part of the graph and $u^- \in \mathcal{N}_\lambda(\mathcal{G}_{k,L})$, while we can think of u^+ as a function in $\mathcal{N}_\lambda(\mathbb{R})$ with compact support. Hence we have

$$\inf_{v \in \mathcal{N}_\lambda^{nod}(\mathcal{G}_{k,L})} J(v) = J(u^+) + J(u^-) > s_\lambda + \inf_{v \in \mathcal{N}_\lambda(\mathcal{G}_{k,L})} J(v),$$

which contradicts Theorem 2.6.

Step 3: conclusion. The previous steps ensure that $u^{-1}(0) \subseteq \bigcup_{i=1}^N e_i$ and that it is a finite union of points by uniqueness of the solution of the Cauchy problem for (2.1). The uniqueness of the zero of u on each e_i follows then by Theorem 2.38. □

Proof of 2): here, we prove the following result.

Proposition 2.42. *Let $m \geq 2$. There exists a graph $\overline{\mathcal{G}}$ that admits a nodal ground state u such that $u^{-1}(0)$ is the union of $m \geq 2$ half-lines attached at the same vertex.*

The graph $\overline{\mathcal{G}}$ is obtained from the graph $\mathcal{G}_{1,L}$ attaching m half-lines at a suitable point. Before proving Proposition 2.42 we establish the following lemma.

Lemma 2.43. *Let $\mathcal{G}$ be a noncompact graph with a finite number of edges. Let $\widetilde{\mathcal{G}}$ be a graph obtained from $\mathcal{G}$ attaching $m \geq 2$ half-lines $h_1, \ldots, h_m$ at one of its points $\mathtt{p}$. If there exists a nodal ground state in $\mathcal{N}_\lambda^{nod}(\widetilde{\mathcal{G}})$, then*

$$\inf_{v \in \mathcal{N}_\lambda^{nod}(\widetilde{\mathcal{G}})} J(v) \geq \inf_{v \in \mathcal{N}_\lambda^{nod}(\mathcal{G})} J(v). \tag{2.32}$$

Proof. Let $\tilde{u}$ be a nodal ground state on $\widetilde{\mathcal{G}}$ and assume without loss of generality that $\tilde{u}(\mathtt{p}) \geq 0$. Denote by u the restriction of $\tilde{u}$ on $\mathcal{G}$ and by ϕ_i the restriction of $\tilde{u}$ to the half-line h_i, for $i = 1, \ldots, m$.

If $\tilde{u}(\mathtt{p}) = 0$, since $\tilde{u}$ solves (2.1), each ϕ_i vanishes identically. Hence, u belongs to $\mathcal{N}_\lambda^{nod}(\mathcal{G})$, $J(\tilde{u}) = J(u)$ and (2.32) follows.

If $\tilde{u}(\mathtt{p}) > 0$, each ϕ_i coincides with a portion of the same soliton ϕ_λ. With a slight abuse of notation we denote by $\phi_i'(\mathtt{p})$ the outgoing derivative of ϕ_i at $\mathtt{p}$ along h_i. Note that $\phi_i'(\mathtt{p}) < 0$ for every i. Indeed, if on the contrary we had for instance $\phi_1'(\mathtt{p}) \geq 0$, then the restriction of $\tilde{u}$ to the union of the h_i's would contain at least one full soliton ϕ_λ, so that $\|\tilde{u}\|_{L^p(\cup_i h_i)}^p \geq \|\phi_\lambda\|_p^p$.

This would lead to

$$\begin{aligned}\inf_{v\in\mathcal{N}_\lambda^{nod}(\widetilde{\mathcal{G}})} J(v) &= J(\widetilde{u})\\ &= \varkappa(\|\widetilde{u}^+\|_p^p + \|\widetilde{u}^-\|_p^p)\\ &> \varkappa\|\phi_\lambda\|_p^p + \inf_{v\in\mathcal{N}_\lambda(\widetilde{\mathcal{G}})} J(v)\\ &= s_\lambda + \inf_{v\in\mathcal{N}_\lambda(\widetilde{\mathcal{G}})} J(v)\end{aligned}$$

which contradicts (2.8).

As, for all $i \in \{1,\dots,m\}$, ϕ_i is a solution to (2.1), we have in particular $\phi_i \in \mathcal{N}_{\theta_i}(h_i)$ with

$$\begin{aligned}\theta_i &= \frac{\int_{h_i}\phi_i^p\,\mathrm{d}x - \int_{h_i}(\phi_i')^2\,\mathrm{d}x}{\int_{h_i}\phi_i^2\,\mathrm{d}x}\\ &= \frac{\int_{h_i}\phi_i^p\,\mathrm{d}x + \phi_i(\mathtt{p})\phi_i'(\mathtt{p}) + \int_{h_i}\phi_i''\phi_i\,\mathrm{d}x}{\int_{h_i}\phi_i^2\,\mathrm{d}x}\\ &= \frac{\lambda\int_{h_i}\phi_i^2\,\mathrm{d}x + \phi_i(\mathtt{p})\phi_i'(\mathtt{p})}{\int_{h_i}\phi_i^2\,\mathrm{d}x}\\ &< \lambda\end{aligned}$$

since ϕ_i is a portion of ϕ_λ and $\phi_i'(\mathtt{p}) < 0$. Letting then μ be the number such that $u^+ \in \mathcal{N}_\mu(\mathcal{G})$, there holds

$$\lambda = \sum_{i=1}^m \frac{\|\phi_i\|_{L^2(h_i)}^2}{\|\widetilde{u}^+\|_{L^2(\widetilde{\mathcal{G}})}^2}\theta_i + \frac{\|u^+\|_{L^2(\mathcal{G})}^2}{\|\widetilde{u}^+\|_{L^2(\widetilde{\mathcal{G}})}^2}\mu$$

which, combined with the preceding inequality, yields $\mu > \lambda$.

Since, analogously to Remark 2.14, for a given λ the map

$$\mu \mapsto \inf_{v\in\mathcal{N}_{\mu,\lambda}^{nod}(\mathcal{G})} \frac{1}{2}\|v'\|_{L^2(\mathcal{G})}^2 + \frac{\mu}{2}\|v^+\|_{L^2(\mathcal{G})}^2 + \frac{\lambda}{2}\|v^-\|_{L^2(\mathcal{G})}^2 - \frac{1}{p}\|v\|_{L^p(\mathcal{G})}^p,$$

where

$$\mathcal{N}_{\mu,\lambda}^{nod}(\mathcal{G}) := \left\{v \in H^1(\mathcal{G}) \mid v^+ \in \mathcal{N}_\mu(\mathcal{G}) \text{ and } v^- \in \mathcal{N}_\lambda(\mathcal{G})\right\}$$

is increasing, we have

$$\begin{aligned}\inf_{v\in\mathcal{N}_\lambda^{nod}(\widetilde{\mathcal{G}})} J(v) &= J(\widetilde{u})\\ &= \varkappa(\|\widetilde{u}^+\|_p^p + \|\widetilde{u}^-\|_p^p)\\ &\geq \varkappa(\|u^+\|_p^p + \|u^-\|_p^p)\\ &\geq \inf_{v\in\mathcal{N}_{\mu,\lambda}^{nod}(\mathcal{G})} J(v)\\ &> \inf_{v\in\mathcal{N}_\lambda^{nod}(\mathcal{G})} J(v),\end{aligned}$$

which concludes the proof. □

Proof of Proposition 2.42. Consider the graph $\mathcal{G}_{1,L}$ with $L \geq \overline{L}$ which is given by Proposition 2.39. On this graph, by Theorem 2.30 and Proposition 2.39, we have a nodal ground state u with $u^{-1}(0) = \{x_0\}$. Let now $\overline{\mathcal{G}}$ be the graph obtained from $\mathcal{G}_{1,L}$ attaching m half-lines at the point x_0 and let $\overline{u} \in \mathcal{N}_\lambda^{nod}(\overline{\mathcal{G}})$ be the function obtained extending u by 0 on each of the additional half-lines.

By Theorem 2.30, nodal ground states exist on $\overline{\mathcal{G}}$ and, by Lemma 2.43,

$$\inf_{v\in\mathcal{N}_\lambda^{nod}(\mathcal{G}_{1,L})} J(v) = J(u) = J(\overline{u}) \geq \inf_{v\in\mathcal{N}_\lambda^{nod}(\overline{\mathcal{G}})} J(v) \geq \inf_{v\in\mathcal{N}_\lambda^{nod}(\mathcal{G}_{1,L})} J(v).$$

This proves that $\overline{u}$ is a nodal ground state on $\bar{\mathcal{G}}$ and hence the existence of a nodal ground state whose nodal set is given by m half-lines attached at the same point. □

Proof of 3): here we prove the following statement.

Proposition 2.44. *Let n be a nonnegative integer. There exist a graph $\overline{\mathcal{G}}$, a subset $Z \subseteq \mathbb{V} \setminus \mathbb{V}_\infty$ of its vertices of degree 1 and a nodal ground state $u \in \mathcal{N}_{\lambda,Z}^{nod}(\overline{\mathcal{G}})$ such that $u^{-1}(0)$ consists of n line segments attached at the same point, each of length smaller than or equal to $\frac{\varkappa}{2s_\lambda}\left(\frac{p\lambda}{2}\right)^{2/(p-2)}$.*

Similarly to construction 2), the graph $\overline{\mathcal{G}}$ will be obtained from $\mathcal{G}_{1,L}$ attaching n line segments at one of its points. To do this we need the next lemma.

Lemma 2.45. *Let $\mathcal{G}$ be a noncompact graph with a finite number of edges. Let $\widetilde{\mathcal{G}}$ be a graph obtained from $\mathcal{G}$ attaching n line segments $s_1, \ldots, s_n$ at one of its points $\mathtt{p}$. Assume that each line segment has a length smaller than or equal to a number $S > 0$ and ends at a vertex with the Dirichlet boundary condition. Suppose also that $\widetilde{u}$ and S are such that $\widetilde{u}$ is a nodal ground state on $\widetilde{\mathcal{G}}$ and that $S \leq \frac{\varkappa}{J(\widetilde{u})}\left(\frac{p\lambda}{2}\right)^{2/(p-2)}$. Then*

$$\inf_{v\in\mathcal{N}_{\lambda,Z}^{nod}(\widetilde{\mathcal{G}})} J(v) \geq \inf_{v\in\mathcal{N}_{\lambda,Z}^{nod}(\mathcal{G})} J(v).$$

Proof. We proceed in the same way as in the proof of Lemma 2.43. With no loss of generality, let $\widetilde{u}(\mathtt{p}) \geq 0$. Denote by u the restriction of $\widetilde{u}$ to $\mathcal{G}$ and by u_i the restriction of u to s_i for every i. Moreover, let $u_i'(\mathtt{p})$ be the outward derivative of u_i at $\mathtt{p}$ along s_i. Note again that, as $\widetilde{u}$ is a nodal ground state, $u_i(\mathtt{p})u_i'(\mathtt{p}) \leq 0$. Indeed, if this were not the case, we would have $u_i'(\mathtt{p}) > 0$ and, since u_i satisfies the Dirichlet condition at the end of s_i, by a phase plane analysis we would have $u_i(x_0) := \max u_i \geq \max \phi_\lambda = \left(\frac{p\lambda}{2}\right)^{1/(p-2)}$.

Considering the first zero $x_1 \in s_i$ of u_i it would then follow

$$\left(\frac{p\lambda}{2}\right)^{1/(p-2)} \le u_i(x_0) - u_i(x_1) = \int_{x_1}^{x_0} u_i'(s)\,ds \le \sqrt{x_0 - x_1}\|\tilde{u}'\|_2 < \sqrt{SJ(\tilde{u})/\varkappa},$$

which contradicts the choice of S. The rest of the proof follows as in that of Lemma 2.43. □

Proof of Proposition 2.44. The proof is the same as the one of Proposition 2.42, using Lemma 2.45 instead of Lemma 2.43 and observing that, by Theorem 2.30, $J(\tilde{u}) \le 2s_\lambda$. □

Remark 2.46. Graphs fulfilling Theorem 2.9 can be obtained combining *ad libitum* the constructions 1), 2), 3). The general result is a graph as the one depicted in Figure 2.11.

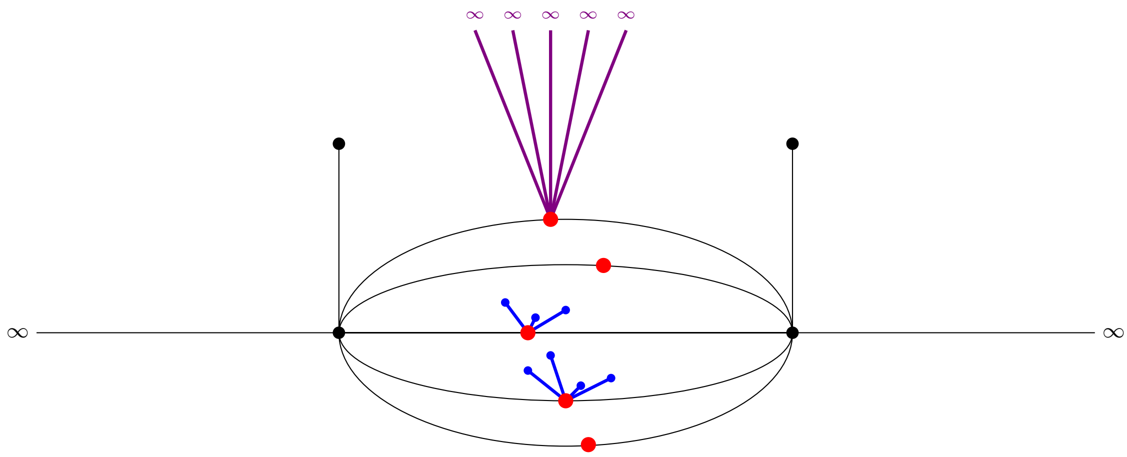


Figure 2.11: Example of a graph as in Theorem 2.9 hosting a nodal ground state whose nodal set (thick on the picture) is made of two isolated points, two groups of three and four line segments respectively, and a group of five half-lines.

Chapter 3
A new approach to prescribed mass (nodal) solutions of (NLS)

3.1 Presentation of the chapter

The present chapter focuses on *normalized* solutions of nonlinear Schrödinger (NLS) equations with homogeneous Dirichlet boundary conditions on bounded domains, namely solutions of the problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \|u\|^2_{L^2(\Omega)} = \mu. \end{cases} \tag{3.1}$$

Here, $\Omega \subset \mathbb{R}^N$ is a connected bounded open set, λ and μ are real parameters, and the nonlinearity exponent satisfies

$$p \in (2, 2^*), \quad 2^* = \frac{2N}{N-2} \quad (2^* = \infty \text{ if } N = 1, 2).$$

The attribute normalized for a function u solving (3.1) comes from the fact that its L^2-norm (usually called the *mass*) is prescribed a priori. In our setting, this means that the parameter $\mu > 0$ is given, whereas λ (sometimes called the chemical potential or the frequency) is an unknown of the problem. As is well known, the problem has a variational structure, since weak solutions of (3.1) are critical points of the *energy* functional $E : H^1_0(\Omega) \to \mathbb{R}$

$$E(u, \Omega) := \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)}$$

on the L^2-sphere

$$\mathcal{M}_\mu(\Omega) := \left\{ u \in H^1_0(\Omega) \;\middle|\; \|u\|^2_{L^2(\Omega)} = \mu \right\},$$

λ arising then as a Lagrange multiplier.

The study of normalized solutions for NLS equations gathered a constantly growing interest in the last decades. In particular, among all solutions with fixed mass, a specific attention can be naturally devoted to *least energy normalized solutions*, i.e. functions u solving (3.1) and satisfying

$$E(u, \Omega) = \inf \{E(v, \Omega) \mid v \text{ solves (3.1) for some } \lambda \in \mathbb{R}\}.$$

In the seminal papers [99, 183] least energy positive solutions are identified for the problem on the whole $\mathbb{R}^N$ in the L^2-subcritical $p < 2 + \frac{4}{N}$ and L^2-supercritical $p > 2 + \frac{4}{N}$ regimes, respectively. When $p < 2 + \frac{4}{N}$, these least energy solutions can be found by solving the minimization problem

$$\inf_{u \in \mathcal{M}_\mu(\mathbb{R}^N)} E(u, \mathbb{R}^N),$$

which is attained for every $\mu > 0$. Conversely, when p is L^2-supercritical this is no longer possible, as the energy E is unbounded from below on $\mathcal{M}_\mu(\mathbb{R}^N)$ for every μ, and different approaches (e.g. of mountain pass type) are needed. Since [183], normalized solutions on $\mathbb{R}^N$ have been largely investigated in various settings (see e.g. [46, 48, 51, 55, 190, 194, 195, 245, 303, 304, 329] and references therein) and a comprehensive theory is by now available.

On the contrary, on bounded domains the literature is more limited, to date, and the general portrait is less understood. Given the boundedness of the domain Ω, in the L^2-subcritical regime $p < 2 + \frac{4}{N}$ least energy solutions always exist and they are again the global minimizers of E on the whole set $\mathcal{M}_\mu(\Omega)$. The same is true for masses smaller than a threshold (independent of Ω) in the L^2-critical case $p = 2 + \frac{4}{N}$.

When $p > 2 + \frac{4}{N}$, instead, even existence of positive solutions (not necessarily least energy) is more involved, because many crucial properties of the problem on $\mathbb{R}^N$ are no longer available on bounded domains (as e.g. the invariance under dilations of the ambient space). To the best of our knowledge, a complete description of the set of normalized positive solutions is available only when Ω is a ball and it is given in [259] (similar results have then been obtained also for NLS systems in [260]). As for general domains, existence results for solutions (not necessarily positive) with fixed Morse index have been derived when p is L^2-supercritical in [273, 274], whereas in [269] specific positive solutions are constructed for large masses when $p < 2 + \frac{4}{N}$, small masses when $p > 2 + \frac{4}{N}$, and masses close to an explicit value when $p = 2 + \frac{4}{N}$.

The aim of the present chapter is to fit in this research line focusing on the following questions:

1) how to find least energy normalized solutions in the L^2-supercritical regime?

2) how to find normalized nodal solutions?

As far as we know, to date both questions are essentially open. As for 1), the only available result we are aware of is the already mentioned one on the ball reported in [259], that identifies the normalized solution with minimal energy among *positive* ones. Even in this special setting, it is not known whether this is also the least energy solution among *all* solutions with the same mass. For domains other than the ball, nothing seems to be known.

The situation concerning 2) is even worse, due to the lack of general existence results for normalized nodal solutions. Actually, all papers in the literature either restrict their attention to positive solutions, or do not allow to recover any specific information on the sign of the solutions under exam.

Though at a first sight questions 1) and 2) may appear somehow far from each other, a common feature that may perhaps explain the lack of results in both directions is the absence of suitable variational frameworks to tackle them.

This is readily understood when looking for L^2-supercritical least energy solutions, for which we already observed that it is not possible to simply minimize E on the whole manifold $\mathcal{M}_\mu(\Omega)$ (as one does in the L^2-subcritical regime).

Such a difficulty is all the more severe for normalized nodal solutions, for which a proper variational framework involving the energy is not available even when p is L^2-subcritical. For instance, one may be first tempted to consider

$$\inf_{\substack{u\in\mathcal{M}_\mu(\Omega)\\ u^\pm\not\equiv 0}} E(u,\Omega),$$

where u^+ and $u^- = \min(u,0)$ are the positive and negative parts of u, but it is evident that this number is never attained, as it coincides with the infimum of E on the whole $\mathcal{M}_\mu(\Omega)$ (with no sign constraint). Even a slightly more sophisticated approach considering the two-parameter minimization problem

$$\inf_{\substack{u^+\in\mathcal{M}_{\mu_1}(\Omega),u^-\in\mathcal{M}_{\mu_2}(\Omega)\\ \mu=\mu_1+\mu_2}} E(u,\Omega),$$

leads to seemingly insuperable difficulties.

In this chapter, we tackle both 1) and 2) using a unified approach: we take advantage of already available existence results for solutions of the problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

for a *given* $\lambda \in \mathbb{R}$, that is (3.1) without the mass constraint, and we characterize the dependence on λ of the mass of such solutions. Indeed, for a fixed $\lambda \in \mathbb{R}$, it is well known that positive (up to a change of sign) solutions of (3.2) can be found variationally e.g. by considering (for any $p \in (2, 2^*)$) the minimization problem

$$\mathcal{J}_\Omega(\lambda) := \inf_{u\in\mathcal{N}_\lambda(\Omega)} J_\lambda(u,\Omega)$$

for the *action* functional $J_\lambda(\,\cdot\,,\Omega) : H_0^1(\Omega) \to \mathbb{R}$

$$J_\lambda(u,\Omega) := \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)}$$

constrained to the associated *Nehari manifold*

$$\begin{aligned}\mathcal{N}_\lambda(\Omega) &:= \Big\{u \in H^1_0(\Omega) \setminus \{0\} \;\Big|\; J'_\lambda(u,\Omega)u = 0\Big\} \\ &= \Big\{u \in H^1_0(\Omega) \setminus \{0\} \;\Big|\; \|\nabla u\|^2_{L^2(\Omega)} + \lambda \|u\|^2_{L^2(\Omega)} = \|u\|^p_{L^p(\Omega)}\Big\}.\end{aligned}$$

Similarly, when looking for nodal solutions one can consider the problem

$$\mathcal{J}^{nod}_\Omega(\lambda) := \inf_{\mathcal{N}^{nod}_\lambda(\Omega)} J_\lambda(u,\Omega),$$

where

$$\mathcal{N}^{nod}_\lambda(\Omega) := \Big\{u \in H^1_0(\Omega) \;\Big|\; u^\pm \in \mathcal{N}_\lambda(\Omega)\Big\}$$

is the *nodal Nehari set*. Depending on the value of λ, existence of solutions of these two problems, usually called *action ground states* and *nodal action ground states* respectively, is essentially well known (see Section 3.2 below for further details).

The main contribution of the present chapter is the following complete characterization of the masses of all action and nodal action ground states.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $p \in (2, 2^*)$, let*

$$\begin{aligned}M_p(\Omega) &:= \Big\{\|u\|^2_{L^2(\Omega)} \;\Big|\; u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}_\Omega(\lambda) \text{ for some } \lambda \in \mathbb{R}\Big\} \\ M^{nod}_p(\Omega) &:= \Big\{\|u\|^2_{L^2(\Omega)} \;\Big|\; u \in \mathcal{N}^{nod}_\lambda(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}^{nod}_\Omega(\lambda) \text{ for some } \lambda \in \mathbb{R}\Big\}\end{aligned} \tag{3.3}$$

be the set of masses of all action ground states and nodal action ground states, respectively. Then

(i) if $p < 2 + \frac{4}{N}$, then $M_p(\Omega) = M^{nod}_p(\Omega) = (0, \infty)$;

(ii) if $p = 2 + \frac{4}{N}$, then there exist $0 < \mu_p, \mu^{nod}_p < \infty$ such that either $M_p = (0, \mu_p)$ or $M_p = (0, \mu_p]$, and either $M^{nod}_p = (0, \mu^{nod}_p)$ or $M^{nod}_p = (0, \mu^{nod}_p]$;

(iii) if $p > 2 + \frac{4}{N}$, then there exist $0 < \mu_p, \mu^{nod}_p < \infty$ such that $M_p = (0, \mu_p]$ and $M^{nod}_p = (0, \mu^{nod}_p]$.

Notice that Theorem 3.1 holds for any bounded open subset Ω in $\mathbb{R}^N$, and this high generality makes its proof far from trivial. Indeed, taking for granted the existence of action ground states (resp. nodal action ground states) u_λ at fixed λ, one may be tempted to try to characterize the set M_p (resp. M^{nod}_p) by studying the map $\lambda \mapsto \|u_\lambda\|_{L^2(\Omega)}$.

However, in principle such a map is not even well defined, as ground states need not be unique, and in any case its regularity is by no means guaranteed. Actually, in other contexts where the dependence on λ of the mass of a curve of solutions u_λ is relevant, it is common to assume from the very beginning to work with a C^1 curve of solutions (as one does e.g. in the standard stability theory for Hamiltonian systems [172, 301, 330]). For ground states on a general bounded domain, this level of regularity is too strong.

Instead, the proof of Theorem 3.1 does not require any regularity assumption of this sort and exploits a different perspective. Roughly, we will show that M_p (resp. M_p^{nod}) is the range of the derivative of the action ground state level $\mathcal{J}_\Omega$ (resp. of $\mathcal{J}_\Omega^{nod}$) so that Theorem 3.1 can also be seen as a Darboux-type theorem for $\mathcal{J}_\Omega$ and $\mathcal{J}_\Omega^{nod}$ (see Remark 3.19).

Let us now discuss the impact of Theorem 3.1 on questions 1)-2) above. With respect to 2), since even the existence of one nodal solution of (3.1) at prescribed mass μ is in general an open problem, we have the following result, that is an immediate corollary of Theorem 3.1.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and $p \in (2, 2^*)$. Then*

(i) if $p < 2 + \frac{4}{N}$, there exists a nodal solution of (3.1) for every $\mu > 0$;

(ii) if $p = 2 + \frac{4}{N}$, there exists a nodal solution of (3.1) for every $\mu \in (0, \mu_p^{nod})$, where μ_p^{nod} is as in Theorem 3.1(ii);

(iii) if $p > 2 + \frac{4}{N}$, there exists a nodal solution of (3.1) for every $\mu \in (0, \mu_p^{nod}]$, where μ_p^{nod} is as in Theorem 3.1(iii).

Remark 3.3. Clearly, a statement analogous to Theorem 3.2 can be given for normalized positive solutions of (3.1) too, with μ_p in place of μ_p^{nod}. When $p \leq 2 + \frac{4}{N}$, this does not extend the existence results for positive solutions already available in the literature. On the contrary, it is more relevant in the L^2-supercritical case. Indeed, in this regime, our approach provides a simple technique to exhibit normalized solutions for a whole interval of masses $(0, \mu_p]$ and, as far as we know, similar results have been previously obtained only through more technically demanding constructions (see e.g. [273]).

Since Theorem 3.2 provides regimes of nonlinearities and masses for which the set of normalized nodal solutions is not empty, it is then natural to wonder whether one can identify the *least energy nodal solutions*, i.e. u solving (3.1) such that $u^\pm \neq 0$ and

$$E(u, \Omega) = \inf\{E(v, \Omega) \mid v \text{ is a nodal solution of (3.1) for some } \lambda \in \mathbb{R}\}.$$

Actually, our method allows to answer in the affirmative to this question in the L^2-subcritical and L^2-critical regimes. To state this result, let

$$\mu_N := 2 \inf_{u \in \mathcal{N}_1(\mathbb{R}^N)} \left(\frac{1}{2} \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2} \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2+4/N} \|u\|^{2+4/N}_{L^{2+4/N}(\mathbb{R}^N)} \right). \quad (3.4)$$

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and either*

(i) $p < 2 + \frac{4}{N}$ *and* $\mu > 0$*; or*

(ii) $p = 2 + \frac{4}{N}$ *and* $\mu < 2\mu_N$*, where* μ_N *is the number in* (3.4).

Then there exists a least energy normalized nodal solution with mass μ. Moreover, every least energy normalized nodal solution u is a nodal action ground state in $\mathcal{N}^{nod}_\lambda(\Omega)$, where λ is the number associated to u in (3.1).

Theorem 3.4 says that, in the above regimes, least energy normalized nodal solutions are nodal action ground states. Observe that, at the critical power $p = 2 + \frac{4}{N}$, we are able to prove this fact only for masses strictly smaller than the threshold $2\mu_N \leq \mu_p^{nod}$, although nodal solutions exist for every $\mu \leq \mu_p^{nod}$ (Theorem 3.1).

Remark 3.5. Again, the analogue of Theorem 3.4 can be stated and proved for least energy normalized positive solutions. In this case, solutions with minimal energy exist for every mass when $p < 2 + \frac{4}{N}$, and for every mass strictly smaller than μ_N when $p = 2 + \frac{4}{N}$, and they are also action ground states in $\mathcal{N}_\lambda(\Omega)$ for suitable λ. However, this is already well known, since in this range of p and μ it is easily seen that E admits global minimizers on $H^1_\mu(\Omega)$, and that such minimizers are also action ground states was recently proved in [129, Theorem 1.3].

Remark 3.6. Combining our results with those of [259], we obtain a perhaps unexpected consequence. When Ω is a ball, $p = 2 + 4/N$, and for $\mu \in [\mu_N, 2\mu_N)$ there exist least energy normalized nodal solutions with mass μ, by Theorem 3.4. By [259, Theorem 1.5], there are no positive solutions of mass μ. This means that *least energy solutions of mass μ are nodal.*

Theorem 3.4 (and its counterpart for positive solutions) gives no insight in the L^2-supercritical regime. However, it makes perfect sense to wonder whether normalized solutions with minimal energy are action ground states also when $p > 2 + \frac{4}{N}$. At present, we are not able to answer this question for any general bounded and open set Ω in $\mathbb{R}^N$, but we can partially solve the problem at least for star-shaped domains of $\mathbb{R}^N$. The next theorem summarizes our results in this direction, that provide our main contribution with respect to question 1) above.

Theorem 3.7. *Let Ω be bounded, open, smooth and star-shaped, $p \in \left(2 + \frac{4}{N}, 2^*\right)$, and μ_p, μ_p^{nod} be as in Theorem 3.1. Then there exists a least energy normalized solution for every $\mu \leq \mu_p$, and there exists a least energy normalized nodal solution for every $\mu \leq \mu_p^{nod}$. Moreover, there exist $\overline{\mu}_p \leq \mu_p$ and $C_p > 0$ such that every least energy normalized solution u with mass $\mu \leq \overline{\mu}_p$ is an action ground state in $\mathcal{N}_\lambda(\Omega)$, where λ is the number associated to u in* (3.1) *and satisfies $\lambda \leq C_p$. Analogously, there exist $\overline{\mu}_p^{nod} \leq \mu_p^{nod}$ and $C_p^{nod} > 0$ such that every least energy normalized nodal solution v with mass $\mu \leq \overline{\mu}_p^{nod}$ is a nodal action ground state in $\mathcal{N}_{\lambda'}^{nod}(\Omega)$, where λ' is the number associated to v in* (3.1) *and satisfies $\lambda' \leq C_p^{nod}$.*

Note that Theorem 3.7 not only shows that, in certain regimes of masses, least energy solutions are again action ground states, but it also proves that the corresponding frequency λ of such ground states is bounded from above uniformly in λ. In fact, in the L^2-supercritical regime it is not difficult to show that action ground states have small masses both when λ is close to $-\gamma_1$ and when λ is large. Theorem 3.7 suggests that, even though they have the same masses, small frequency ground states are energetically convenient. This kind of property had been observed before when Ω is a ball (see [259, Theorem 1.7 and Remark 6.4]), and to some extent one can interpret Theorem 3.7 as a first step towards a proof of a result of this sort for general domains.

It is an open problem to understand whether the content of Theorem 3.7 remains true when Ω is not star-shaped. Note that, in the proof of Theorem 3.7 reported in Section 3.5 below, the star-shapedness assumption plays a role not only to show that action ground states are least energy normalized solutions, but also to prove that a least energy normalized solution actually exists.

To conclude, we wish to point out that the argument developed in the present chapter is not limited to NLS equations (3.1) with a pure power nonlinearity and homogeneous Dirichlet conditions at the boundary. On the contrary, since it can be generalized to other boundary conditions or nonlinearities, the chapter actually provides a new approach to the study of normalized solutions of NLS equations, that one can try to exploit whenever a suitable Nehari manifold associated to the problem under exam is available. Moreover, this work can be seen as a further step in the investigation of the relation between the action and the energy approaches to the search of solutions of NLS equations, thus extending the first analyses in this direction recently started in [129, 190].

The remainder of the chapter is organized as follows. Section 3.2 recalls some known facts and proves preliminary existence results for nodal action ground states. Section 3.3 provides a detailed analysis of the nodal action ground state level $\mathcal{J}_\Omega^{nod}$, whereas Section 3.4 gives the proof of Theorem 3.1 on the masses of action ground states. Finally, Section 3.5 completes the proof of the main results of the chapter, namely Theorems 3.2–3.4–3.7.

Notation. Throughout, we will use shorter notations for norms as $\|u\|_q$, avoiding to write the domain of integration whenever it is clear by the context.

3.2 Existence results for nodal action ground states

This section discusses existence and non-existence of nodal action ground states on open bounded subsets Ω of $\mathbb{R}^N$. We recall that, if Ω is smooth, existence of such states has been proved in [53, Theorem 1.1]. Hence, here we limit ourselves to prove some basic estimates that allow us to extend this already known result to general open and bounded sets (without regularity assumptions).

We start by recalling the picture for action ground states. The following result concerning the action was proved in [129, Theorem 1.5, Lemma 2.4 and Remark 2.5]. Here, γ_1 denotes the first eigenvalue of $-\Delta$ with homogeneous Dirichlet conditions at the boundary of Ω.

Proposition 3.8. *For every $p \in (2, 2^*)$, the following properties hold.*

(i) For every $\lambda \leq -\gamma_1$, $\mathcal{J}_\Omega(\lambda) = 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ do not exist.

(ii) For every $\lambda > -\gamma_1$, $\mathcal{J}_\Omega(\lambda) > 0$ and action ground states in $\mathcal{N}_\lambda(\Omega)$ exist.

(iii) The function $\mathcal{J}_\Omega : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\gamma_1, +\infty)$.

(iv) Letting $Q_p(\lambda) = \{\|u\|_2^2 : u \in \mathcal{N}_\lambda(\Omega)$ and $J_\lambda(u, \Omega) = \mathcal{J}_\Omega(\lambda)\}$, there results

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{J}_\Omega(\lambda + \varepsilon) - \mathcal{J}_\Omega(\lambda)}{\varepsilon} = \frac{1}{2} \inf Q_p(\lambda) \leq \frac{1}{2} \sup Q_p(\lambda) = \lim_{\varepsilon \to 0^-} \frac{\mathcal{J}_\Omega(\lambda + \varepsilon) - \mathcal{J}_\Omega(\lambda)}{\varepsilon},$$

Moreover, for every λ outside an at most countable set, all action ground states have the same mass (i.e., $Q_p(\lambda)$ is a singleton).

Remark 3.9. It is well known that the threshold $-\gamma_1$ appearing in the preceding result is also a threshold for the existence of constant sign solutions. Precisely, if $\lambda \le -\gamma_1$ then (3.2) has no nonzero solutions u with $u \ge 0$. It is also well known that if $\lambda > -\gamma_1$ and u is a nonzero solution with $u \ge 0$, then $u > 0$ in Ω.

We now establish a similar picture for nodal action ground states. In this setting, a major role is played by the second eigenvalue γ_2 of $-\Delta$ with homogeneous Dirichlet conditions at the boundary. Requiring only $\lambda > -\gamma_2$ poses some problems that one does not encounter in the study of signed ground states. For instance, the inequality $\|\nabla u\|_2^2 + \lambda\|u\|_2^2 > 0$ does not hold for every u and checking it for a given u requires some care. Also, we will have to estimate the norms of the positive and negative parts of functions separately, something that is not directly readable from the functional J_λ. For these reasons we proceed with single statements instead of collecting them all together as in Proposition 3.8.

We recall that, given $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ such that $\|\nabla u\|_2^2 + \lambda\|u\|_2^2 > 0$, defining

$$n_\lambda(u) := \left(\frac{\|\nabla u\|_2^2 + \lambda\|u\|_2^2}{\|u\|_p^p}\right)^{\frac{1}{p-2}},$$

we have $n_\lambda(u)u \in \mathcal{N}_\lambda(\Omega)$. We also recall that if $u \in \mathcal{N}_\lambda(\Omega)$, then

$$J_\lambda(u, \Omega) = \kappa\|u\|_p^p = \kappa\Big(\|\nabla u\|_2^2 + \lambda\|u\|_2^2\Big), \qquad \kappa = \frac{1}{2} - \frac{1}{p}, \tag{3.5}$$

a fact we will tacitly use in the proofs.

Remark 3.10. The fact that nodal action ground states in $\mathcal{N}_\lambda^{nod}(\Omega)$, when they exist, are solutions of problem (3.2) is well known (see e.g. [52, Proposition 3.1]).

The next proposition is the nodal analogue of Proposition 3.8(i).

Proposition 3.11. *For every $p \in (2, 2^*)$ and every $\lambda \le -\gamma_2$, $\mathcal{J}_\Omega^{nod}(\lambda) = 0$ and nodal action ground states in $\mathcal{N}_\lambda^{nod}(\Omega)$ do not exist.*

Proof. Fix any $\lambda \le -\gamma_2$ and let $\varphi_2 \in H_0^1(\Omega)$ be an eigenfunction corresponding to $\gamma_2 = \gamma_2(\Omega)$. Denoting by $\Omega^+ := \{x \in \Omega \mid \varphi_2(x) > 0\}$ and $\Omega^- := \{x \in \Omega \mid \varphi_2(x) < 0\}$, there results, as is well known, $\gamma_2(\Omega) = \gamma_1(\Omega^+) = \gamma_1(\Omega^-)$. Then, by assumption, $\lambda \le -\gamma_1(\Omega^+) = -\gamma_1(\Omega^-)$. By Proposition 3.8($i$) we deduce that

$$\mathcal{J}_\Omega^{nod}(\lambda) \le \inf_{v \in \mathcal{N}_\lambda(\Omega^+)} J_\lambda(v, \Omega^+) + \inf_{v \in \mathcal{N}_\lambda(\Omega^-)} J_\lambda(v, \Omega^-) = 0.$$

Since $\mathcal{J}_\Omega^{nod}(\lambda)$ is always nonnegative by (3.5), we see that $\mathcal{J}_\Omega^{nod}(\lambda) = 0$. Furthermore, again by (3.5), if u were a nodal ground state in $\mathcal{N}_\lambda^{nod}(\Omega)$, we would have $\|u\|_p = 0$, which is impossible as $0 \notin \mathcal{N}_\lambda^{nod}(\Omega)$. □

In the following, let

$$C(p) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2}{\|u\|_p}.$$

Proposition 3.12. *For every $p \in (2, 2^*)$, there exist positive constants C_1, C_2 such that for every $\lambda \geq -\gamma_2$,*

$$\mathcal{J}_\Omega^{nod}(\lambda) \leq C_1(\lambda+\gamma_2)^{\frac{p}{p-2}} \tag{3.6}$$

$$\mathcal{J}_\Omega^{nod}(\lambda) \geq C_2 \min\left(1, \frac{\lambda+\gamma_2}{\gamma_2}\right)^{\frac{p}{p-2}} \tag{3.7}$$

Proof. To prove (3.6), notice that when $\lambda > -\gamma_2$, if $\varphi_2 \in H_0^1(\Omega)$ is an eigenfunction associated to γ_2, then $\|\nabla \varphi_2^\pm\|_2^2 + \lambda \|\varphi_2^\pm\|_2^2 = (\lambda+\gamma_2)\|\varphi_2^\pm\|_2^2 > 0$, so that $n_\lambda(\varphi_2^\pm)$ is well defined and $n_\lambda(\varphi_2^+)\varphi_2^+ + n_\lambda(\varphi_2^-)\varphi_2^- \in \mathcal{N}_\lambda^{nod}(\Omega)$. Hence,

$$\begin{aligned}
\mathcal{J}_\Omega^{nod}(\lambda) &\leq J_\lambda\Big(n_\lambda(\varphi_2^+)\varphi_2^+ + n_\lambda(\varphi_2^-)\varphi_2^-, \Omega\Big) \\
&= \kappa n_\lambda(\varphi_2^+)^p \|\varphi_2^+\|_p^p + \kappa n_\lambda(\varphi_2^-)^p \|\varphi_2^-\|_p^p \\
&= \kappa(\lambda+\gamma_2)^{\frac{p}{p-2}} \left(\left(\frac{\|\varphi_2^+\|_2}{\|\varphi_2^+\|_p}\right)^{\frac{2p}{p-2}} + \left(\frac{\|\varphi_2^-\|_2}{\|\varphi_2^-\|_p}\right)^{\frac{2p}{p-2}} \right) \\
&=: C_1(\lambda+\gamma_2)^{\frac{p}{p-2}},
\end{aligned}$$

which is (3.6). To prove (3.7) we use an argument taken from [74]. Given $u \in \mathcal{N}_\lambda^{nod}(\Omega)$, we notice that, plainly, there exists $s \in (0,1)$ such that

$$u_s := su^+ + (1-s)u^-$$

is L^2–orthogonal to the eigenspace E_1 associated with γ_1. Since $n_\lambda(u_s)$ is well defined, we see that the function $v := n_\lambda(u_s)u_s$ belongs to $\mathcal{N}_\lambda(\Omega) \cap E_1^\perp$, and we write it as $v = \alpha u^+ + \beta u^-$ for some $\alpha, \beta > 0$.

Then, as u^+ and u^- belong to $\mathcal{N}_\lambda(\Omega)$,

$$J_\lambda(v,\Omega) = J_\lambda(\alpha u^+,\Omega) + J_\lambda(\beta u^-,\Omega) \leq J_\lambda(u^+,\Omega) + J_\lambda(u^-,\Omega) = J_\lambda(u,\Omega), \tag{3.8}$$

since, by definition of Nehari manifold, $J_\lambda(tu^\pm,\Omega) \leq J_\lambda(u^\pm,\Omega)$ for every $t > 0$. Now, if $\lambda \geq 0$, obviously

$$\|\nabla v\|_2^2 \leq \|\nabla v\|_2^2 + \lambda\|v\|_2^2,$$

while, if $\lambda \in (-\gamma_2, 0)$,

$$\frac{\lambda+\gamma_2}{\gamma_2}\|\nabla v\|_2^2 = \|\nabla v\|_2^2 + \frac{\lambda}{\gamma_2}\|\nabla v\|_2^2 \leq \|\nabla v\|_2^2 + \lambda\|v\|_2^2$$

because $v \in E_1^\perp$.

From the two preceding inequalities we obtain

$$\min\left(1, \frac{\lambda+\gamma_2}{\gamma_2}\right)\|\nabla v\|_2^2 \le \|\nabla v\|_2^2 + \lambda\|v\|_2^2 = \|v\|_p^p \le C(p)^{-p}\|\nabla v\|_2^p,$$

so that

$$\|\nabla v\|_2 \ge C(p)^{\frac{p}{p-2}} \min\Big(1, \frac{\lambda+\gamma_2}{\gamma_2}\Big)^{\frac{1}{p-2}}.$$

Thus,

$$\begin{aligned} J_\lambda(v,\Omega) = \kappa\Big(\|\nabla v\|_2^2 + \lambda\|v\|_2^2\Big) &\ge \kappa \min\Big(1, \frac{\lambda+\gamma_2}{\gamma_2}\Big)\|\nabla v\|_2^2 \\ &\ge \kappa C(p)^{\frac{2p}{p-2}} \min\Big(1, \frac{\lambda+\gamma_2}{\gamma_2}\Big)^{\frac{p}{p-2}}, \end{aligned}$$

which proves (3.7) (with $C_2 = \kappa C(p)^{\frac{2p}{p-2}}$) using (3.8) and taking the infimum over $u \in \mathcal{N}_\lambda^{nod}(\Omega)$. □

Remark 3.13. For future reference, we notice that an estimate similar to (3.7) holds for $\mathcal{J}_\Omega$: there exists $C > 0$ such that for every $\lambda \ge -\gamma_1$,

$$\mathcal{J}_\Omega(\lambda) \ge C \min\left(1, \frac{\lambda+\gamma_1}{\gamma_1}\right)^{\frac{p}{p-2}}.$$

To check this, it is enough to notice that, for every $u \in \mathcal{N}_\lambda(\Omega)$ with $\lambda \in (-\gamma_1, 0)$,

$$\frac{\lambda+\gamma_1}{\gamma_1}\|\nabla u\|_2^2 = \|\nabla u\|_2^2 + \frac{\lambda}{\gamma_1}\|\nabla u\|_2^2 \le \|\nabla u\|_2^2 + \lambda\|u\|_2^2$$

and proceed exactly as in the proof of Proposition 3.12.

The following "a priori" type result will be used in the proof of existence of nodal action ground states on arbitrary domains and will also provide a useful tool for the next sections.

Proposition 3.14. *Let $(\Omega_n)_{n\ge1}$ be a sequence of connected open sets such that $\Omega_n \subseteq \Omega_{n+1}$ for all n and $\Omega = \bigcup_{n\ge1}\Omega_n$. Let $(\alpha_n)_{n\ge1} \subseteq (-\gamma_2, +\infty)$ be a sequence converging to $\lambda \in (-\gamma_2, +\infty)$, where $\gamma_2 = \gamma_2(\Omega)$ is the second Dirichlet eigenvalue of $-\Delta$ on Ω.*

Suppose that, for every n, $u_n \in H_0^1(\Omega)$ is a nodal action ground state in $\mathcal{N}_{\alpha_n}^{nod}(\Omega_n)$, extended by 0 on $\Omega\setminus\Omega_n$. Then, up to subsequences, $(u_n)_n$ converges in $H_0^1(\Omega)$ to a function u, which is a nodal action ground state in $\mathcal{N}_\lambda^{nod}(\Omega)$ and

$$\lim_{n\to\infty} \mathcal{J}_{\Omega_n}^{nod}(\alpha_n) = \mathcal{J}_\Omega^{nod}(\lambda) = J_\lambda(u,\Omega). \tag{3.9}$$

Proof. Let $\varphi_2 \in H_0^1(\Omega_1)$ be an eigenfunction corresponding to $\gamma_2(\Omega_1) \geq \gamma_2(\Omega)$, extended by 0 to Ω. Since for every $n \geq 1$

$$\|\nabla\varphi_2^\pm\|_2^2 + \alpha_n\|\varphi_2^\pm\|_2^2 = (\gamma_2(\Omega_1) + \alpha_n)\|\varphi_2^\pm\|_2^2 > (\gamma_2(\Omega_1) - \gamma_2(\Omega))\|\varphi_2^\pm\|_2^2 \geq 0,$$

we see that the numbers $n_{\alpha_n}(\varphi_2^\pm)$ are well defined. Therefore

$$\begin{aligned} J_{\alpha_n}(u_n, \Omega) &= \inf_{v \in \mathcal{N}_{\alpha_n}^{nod}(\Omega_n)} J_{\alpha_n}(v, \Omega_n) \\ &\leq J_{\alpha_n}\Big(n_{\alpha_n}(\varphi_2^+)\varphi_2^+ + n_{\alpha_n}(\varphi_2^-)\varphi_2^-, \Omega_n\Big) \\ &= \kappa\Big(\gamma_2(\Omega_1) + \alpha_n\Big)^{\frac{p}{p-2}} \left(\left(\frac{\|\varphi_2^+\|_2}{\|\varphi_2^+\|_p}\right)^{\frac{2p}{p-2}} + \left(\frac{\|\varphi_2^-\|_2}{\|\varphi_2^-\|_p}\right)^{\frac{2p}{p-2}}\right), \end{aligned}$$

from which we deduce (the sequence α_n being convergent) that $J_{\alpha_n}(u_n, \Omega)$ is bounded. By (3.5), this implies that u_n is bounded in $L^p(\Omega)$ and hence in $L^2(\Omega)$, because Ω is bounded. Again by (3.5), we obtain that u_n is bounded in $H_0^1(\Omega)$. Therefore, up to subsequences, $(u_n)_n$ converges weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ to $u \in H_0^1(\Omega)$. Noticing that, if $u_n \in \mathcal{N}_{\alpha_n}^{nod}(\Omega_n)$, then we also have $u_n \in \mathcal{N}_{\alpha_n}^{nod}(\Omega)$, (3.7) shows that

$$\begin{aligned} \kappa\|u\|_p^p = \kappa \lim_n \|u_n\|_p^p = \lim_n J_{\alpha_n}(u_n, \Omega_n) &\geq \liminf_n \mathcal{J}_\Omega^{nod}(\alpha_n) \\ \geq C_2 \liminf_n \min\left(1, \frac{\alpha_n + \gamma_2}{\gamma_2}\right)^{\frac{p}{p-2}} &= C_2 \min\left(1, \frac{\lambda + \gamma_2}{\gamma_2}\right)^{\frac{p}{p-2}} > 0 \end{aligned}$$

because $\alpha_n \to \lambda > -\gamma_2$. Therefore $u \not\equiv 0$.

Given any $\varphi \in \mathcal{C}_c^\infty(\Omega)$, we have $\operatorname{supp}(\varphi) \subseteq \Omega_n$ for all n large enough, and since by Remark 3.10 u_n is a solution of (3.2) in Ω_n with multiplier α_n,

$$\int_\Omega \nabla u_n \cdot \nabla\varphi + \alpha_n \int_\Omega u_n\varphi = \int_\Omega |u_n|^{p-2}u_n\varphi$$

for all n large enough. Letting $n \to \infty$, we see that u solves problem (3.2) with $\lambda = \lim_n \alpha_n$.

Next we show that u is nodal. Assume by contradiction that, say, $u \geq 0$. Then, by Remark 3.9, $\lambda > -\gamma_1$. Since, as above, $u_n^\pm \in \mathcal{N}_{\alpha_n}(\Omega)$, by Proposition 3.8 we see that

$$\kappa\|u^-\|_p^p = \kappa \lim_n \|u_n^-\|_p^p = \lim_n J_{\alpha_n}(u_n^-, \Omega_n) \geq \liminf_n \mathcal{J}_\Omega(\alpha_n) = \mathcal{J}_\Omega(\lambda) > 0,$$

i.e. a contradiction. Hence, $u \in \mathcal{N}_\lambda^{nod}(\Omega)$.

It remains to prove that u is a nodal action ground state and that (3.9) holds. First notice that, by the already proved convergences of u_n and $\alpha_n \to \lambda$,

$$\lim_n \mathcal{J}^{nod}_{\Omega_n}(\alpha_n) = \lim_n J_{\alpha_n}(u_n, \Omega_n) = \lim_n \kappa \|u_n\|_p^p = \kappa \|u\|_p^p = J_\lambda(u, \Omega) \geq \mathcal{J}^{nod}_{\Omega}(\lambda), \tag{3.10}$$

because $u_n \in \mathcal{N}^{nod}_{\alpha_n}(\Omega_n)$ and $u \in \mathcal{N}^{nod}_\lambda(\Omega)$.

We now establish the reversed inequality. To this aim, given any $\varepsilon > 0$, it is easily seen, by density, that there exists a function $v \in \mathcal{N}^{nod}_\lambda(\Omega) \cap C^\infty_c(\Omega)$ such that $J_\lambda(v, \Omega) \leq \mathcal{J}^{nod}_\Omega(\lambda) + \varepsilon$. Since v has compact support in Ω, actually $v \in \mathcal{N}^{nod}_\lambda(\Omega_n)$ for every n large enough. Now, as $n \to \infty$,

$$n_{\alpha_n}(v^+)^{p-2} = \frac{\|\nabla v^+\|_2^2 + \alpha_n \|v^+\|_2^2}{\|v^+\|_p^p} = 1 + (\alpha_n - \lambda) \frac{\|v^+\|_2^2}{\|v^+\|_p^p} = 1 + o(1),$$

and the same for $n_{\alpha_n}(v^-)$. In particular, $n_{\alpha_n}(v^+)v^+ + n_{\alpha_n}(v^-)v^- \in \mathcal{N}_{\alpha_n}(\Omega_n)$ and

$$\begin{aligned} \mathcal{J}^{nod}_{\Omega_n}(\alpha_n) &\leq J_{\alpha_n}\left(n_{\alpha_n}(v^+)v^+ + n_{\alpha_n}(v^-)v^-, \Omega_n\right) = \kappa n_{\alpha_n}(v^+)^p \|v^+\|_p^p + \kappa n_{\alpha_n}(v^-)^p \|v^-\|_p^p \\ &\leq (1 + o(1))\kappa \|v\|_p^p = (1 + o(1)) J_\lambda(v, \Omega) \leq (1 + o(1)) \left(\mathcal{J}^{nod}_\Omega(\lambda) + \varepsilon\right), \end{aligned}$$

from which we obtain

$$\lim_n \mathcal{J}^{nod}_{\Omega_n}(\alpha_n) \leq \mathcal{J}^{nod}_\Omega(\lambda) + \varepsilon.$$

Recalling that ε is arbitrary and coupling with (3.10), we see that u is a nodal action ground state and (3.9) holds. This also shows that the convergence of u_n to u is strong in $H^1_0(\Omega)$ and completes the proof. □

We can now prove that nodal action ground states exist on arbitrary bounded open sets Ω when $\lambda > -\gamma_2(\Omega)$.

Proposition 3.15. *For every $p \in (2, 2^*)$ and every $\lambda > -\gamma_2$, nodal action ground states in $\mathcal{N}^{nod}_\lambda(\Omega)$ exist.*

Proof. A proof that nodal action ground states exist when $\lambda > -\gamma_1$ can be found in [93, 318]. This result was extended in [53, Theorem 1.1] to cover all $\lambda > -\gamma_2$, assuming Ω is smooth (regularity is used to turn $\mathcal{N}^{nod}_\lambda(\Omega)$ into a manifold, see [53, Lemma 3.2]). Here we slightly extend [53, Theorem 1.1] to arbitrary domains.

Using [119, Proposition 8.2.1], there exists a sequence of connected and bounded open sets Ω_n with smooth boundary such that $\Omega_n \subseteq \Omega_{n+1}$ for all n and $\Omega = \bigcup_{n\geq 1} \Omega_n$. Let $\lambda > -\gamma_2(\Omega)$. By inclusion, $\gamma_2(\Omega_n) \geq \gamma_2(\Omega)$ for all $n \geq 1$, so that $\lambda > -\gamma_2(\Omega_n)$. By [53, Theorem 1.1], there exists a nodal action ground state $u_n \in \mathcal{N}^{nod}_\lambda(\Omega_n)$. Applying Proposition 3.14, the sequence $(u_n)_n$ converges, up to subsequences, to a nodal action ground state in $\mathcal{N}^{nod}_\lambda(\Omega)$. □

3.3 The level of nodal action ground states

In this section we collect some properties of the nodal action ground state level that will be used later on. We begin by a lower bound on the L^p norm of positive and negative parts of nodal action ground states.

Lemma 3.16. *For every $\alpha > -\gamma_2$ there exists $c_\alpha > 0$ such that, for all $\lambda \geq \alpha$ and all $u \in \mathcal{N}_\lambda^{nod}(\Omega)$ such that $J_\lambda(u,\Omega) = \mathcal{J}_\Omega^{nod}(\lambda)$, there results*

$$\|u^\pm\|_p \geq c_\alpha. \tag{3.11}$$

Proof. We argue by contradiction. Assume then that there exist sequences $(\lambda_n)_n$ and $(u_n)_n$ such that, for every n, $\lambda_n \geq \alpha$ and $u_n \in \mathcal{N}_{\lambda_n}^{nod}(\Omega)$ is a nodal action ground state such that, for instance,

$$\|u_n^+\|_p \to 0 \tag{3.12}$$

as $n \to \infty$. Observe that this implies that $\mathcal{J}_\Omega(\lambda_n) \to 0$ as $n \to \infty$, so that $\limsup_n \lambda_n \leq -\gamma_1$ by Proposition 3.8. Hence, $(\lambda_n)_n$ is bounded and, up to subsequences, we may assume that $\lambda_n \to a$ for some $a \geq \alpha$. In particular, $a > -\gamma_2$. Then, Proposition 3.14 implies that, up to subsequences again, $(u_n)_n$ converges in $H^1(\Omega)$ to some nodal ground state u in $\mathcal{N}_a^{nod}(\Omega)$. In particular, $u^+ \neq 0$. Since $(u_n^+)_n$ converges strongly in $L^p(\Omega)$ to u^+, this contradicts (3.12). □

We can now provide an analogue of Proposition 3.8 in the nodal setting.

Proposition 3.17. *For every $p \in (2, 2^*)$,*

i) $\mathcal{J}_\Omega^{nod} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\gamma_2, +\infty)$;

ii) for every $\lambda > -\gamma_2$, let

$$Q_p^{nod}(\lambda) := \Big\{ \|u\|_2^2 \mid u \in \mathcal{N}_\lambda^{nod}(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}_\Omega^{nod}(\lambda) \Big\}.$$

Then,

$$\begin{aligned}
\limsup_{\varepsilon \to 0^+} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} &\leq \frac{1}{2} \inf Q_p^{nod}(\lambda) \\
&\leq \frac{1}{2} \sup Q_p^{nod}(\lambda) \\
&\leq \liminf_{\varepsilon \to 0^-} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon}.
\end{aligned} \tag{3.13}$$

Furthermore, for almost every λ, all nodal action ground states have the same mass (i.e. $Q_p^{nod}(\lambda)$ is a singleton).

Proof. To establish i), note first that the continuity of $\mathcal{J}_\Omega^{nod}$ follows directly by Proposition 3.14 taking $\Omega_n = \Omega$ for every n.

Since $\mathcal{J}_\Omega^{nod} \equiv 0$ on $(-\infty, -\gamma_2]$ by Proposition 3.11, to obtain the desired monotonicity of $\mathcal{J}_\Omega^{nod}$ we only have to prove that $\mathcal{J}_\Omega^{nod}$ is increasing for $\lambda \geq -\gamma_2$. To check this, it is enough to show that every $\lambda > -\gamma_2$ has a neighborhood where $\mathcal{J}_\Omega^{nod}$ is increasing. To this aim, let $\lambda > -\gamma_2$ be arbitrarily fixed. We claim that there exists $\delta > 0$ such that, for every $\alpha, \beta \in (\lambda - \delta, \lambda + \delta)$ with $\alpha < \beta$, if $u_\beta \in \mathcal{N}_\beta^{nod}(\Omega)$ satisfies $J_\beta(u_\beta, \Omega) = \mathcal{J}_\Omega^{nod}(\beta)$, then $\|\nabla u_\beta^\pm\|_2^2 + \alpha\|u_\beta^\pm\|_2^2 > 0$. Indeed, since Ω is bounded and $p > 2$, $\|u\|_2 \leq C\|u\|_p$ for every $u \in H_0^1(\Omega)$, with $C = |\Omega|^{\frac{p-2}{2p}}$. Hence,

$$\begin{aligned}\|\nabla u_\beta^\pm\|_2^2 + \alpha\|u_\beta^\pm\|_2^2 &= \|\nabla u_\beta^\pm\|_2^2 + \beta\|u_\beta^\pm\|_p^p + (\alpha-\beta)\|u_\beta^\pm\|_2^2 = \|u_\beta^\pm\|_p^p + (\alpha-\beta)\|u_\beta^\pm\|_2^2 \\ &\geq \|u_\beta^\pm\|_p^p + C^2(\alpha-\beta)\|u_\beta^\pm\|_p^2 > 0\end{aligned}$$

if $\beta - \alpha$ (namely δ) is small enough (note that, by Lemma 3.16, $\|u_\beta^\pm\|_p$ are uniformly bounded away from zero if β is bounded away from $-\gamma_2$). This shows that $n_\alpha(u_\beta^\pm)$ is well defined. So, letting α, β and u_β be as above and noticing that $n_\alpha(u_\beta^\pm) < 1$, we have

$$\begin{aligned}\mathcal{J}_\Omega^{nod}(\alpha) &\leq J_\alpha\left(n_\alpha(u_\beta^+)u_\beta^+ + n_\alpha(u_\beta^-)u_\beta^-, \Omega\right) = \kappa\Big(n_\alpha(u_\beta^+)^p\|u_\beta^+\|_p^p + n_\alpha(u_\beta^-)^p\|u_\beta^-\|_p^p\Big) \\ &< \kappa\|u_\beta\|_p^p = J_\beta(u_\beta, \Omega) = \mathcal{J}_\Omega^{nod}(\beta), \end{aligned} \tag{3.14}$$

which shows that $\mathcal{J}_\Omega^{nod}$ is increasing around every $\lambda > -\gamma_2$.

To complete the proof of i), it remains to show that $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz continuous. To this end, we first prove (3.13). For the first inequality, let $\lambda > -\gamma_2$ and let $u \in \mathcal{N}_\lambda^{nod}(\Omega)$ be any function such that $J_\lambda(u, \Omega) = \mathcal{J}_\Omega^{nod}(\lambda)$ (at least one such nodal action ground state exists by Proposition 3.15). For every $\varepsilon > 0$, we have, as in (3.14),

$$\begin{aligned}\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda) &\leq J_{\lambda+\varepsilon}(n_{\lambda+\varepsilon}(u^+)u^+ + n_{\lambda+\varepsilon}(u^-)u^-, \Omega) - J_\lambda(u, \Omega) \\ &= \kappa\Big[\Big(n_{\lambda+\varepsilon}(u^+)^p - 1\Big)\|u^+\|_p^p + \Big(n_{\lambda+\varepsilon}(u^-)^p - 1\Big)\|u^-\|_p^p\Big].\end{aligned}$$

Since, as $\varepsilon \to 0^+$,

$$n_{\lambda+\varepsilon}(u^\pm)^p = 1 + \frac{\varepsilon p}{p-2}\frac{\|u^\pm\|_2^2}{\|u^\pm\|_p^p} + o(\varepsilon),$$

we have

$$\begin{aligned}\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda) &\leq \kappa\left(\frac{\varepsilon p}{p-2}\frac{\|u^+\|_2^2}{\|u^+\|_p^p} + o(\varepsilon)\right)\|u^+\|_p^p \\ &\quad + \kappa\left(\frac{\varepsilon p}{p-2}\frac{\|u^-\|_2^2}{\|u^-\|_p^p} + o(\varepsilon)\right)\|u^-\|_p^p \\ &= \frac{\varepsilon}{2}\|u\|_2^2 + o(\varepsilon).\end{aligned}$$

Therefore, for all $u \in \mathcal{N}_\lambda^{nod}(\Omega)$,

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} \le \frac{1}{2}\|u\|_2^2,$$

proving the first part of (3.13).

The argument for the last inequality in (3.13) is the same, after noticing that $n_{\lambda+\varepsilon}(u) > 0$ for every negative ε small enough.

We can now prove that $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz continuous on $[-\gamma_2, +\infty)$ (the claim is trivial on $(-\infty, -\gamma_2]$). Note first that, by the first inequality in (3.13), the fact that $\|u\|_2^p \le C\|u\|_p^p = 2pC\mathcal{J}_\Omega^{nod}(\lambda)/(p-2)$ if u is a nodal action ground state in $\mathcal{N}_\lambda^{nod}(\Omega)$ and the continuity of $\mathcal{J}_\Omega^{nod}$, for every compact interval $K \subset [-\gamma_2, +\infty)$ there exists a constant $L > 0$ such that

$$\sup_{\lambda\in K} \limsup_{\varepsilon\to 0^+} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} \le \frac{L}{2}. \tag{3.15}$$

Then, given $\alpha \in K$, consider the function $f(\beta) := \mathcal{J}_\Omega^{nod}(\beta) - \mathcal{J}_\Omega^{nod}(\alpha) - L(\beta-\alpha)$, defined for every $\beta \in K$ such that $\beta \ge \alpha$. Assume by contradiction that there exists β such that $f(\beta) > 0$. Since, by definition of f and L, $f(s)$ is smaller than $f(\alpha) = 0$ for every s in a right neighborhood of α, then f would have a minimum point $\bar\beta$ in the interior of K. But this is impossible, since at $\bar\beta$ it should be $\limsup_{\varepsilon\to 0^+} \frac{\mathcal{J}_\Omega^{nod}(\bar\beta+\varepsilon) - \mathcal{J}_\Omega^{nod}(\bar\beta)}{\varepsilon} \ge L$, contradicting (3.15). Hence, f is non-positive on K, that is $\mathcal{J}_\Omega^{nod}$ is L-Lipschitz continuous on K. This completes the proof of *i*).

Finally, since $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz continuous, it is in particular differentiable for almost every $\lambda > -\gamma_2$. If $\mathcal{J}_\Omega^{nod}$ is differentiable at some λ, all inequalities in (3.13) are equalities, and $Q_p^{nod}(\lambda)$ is a singleton, concluding the proof of *ii*). □

We now turn to the asymptotic behavior of $\mathcal{J}_\Omega^{nod}(\lambda)$.

Proposition 3.18. *Let μ_N be the number defined in* (3.4). *Then*

$$\lim_{\lambda\to+\infty} \frac{\mathcal{J}_\Omega^{nod}(\lambda)}{\lambda} = \begin{cases} +\infty & \text{if } p \in \left(2, 2+\frac{4}{N}\right), \\ \mu_N & \text{if } p = 2+\frac{4}{N}, \\ 0 & \text{if } p \in \left(2+\frac{4}{N}, 2^*\right). \end{cases} \tag{3.16}$$

Proof. It is just a slight variant of [129, Lemma 2.4], where the analogous properties are proved for $\mathcal{J}_\Omega$. We will therefore be rather sketchy here. For every $\mu > 0$, set

$$\mathcal{E}_\Omega(\mu) := \inf_{u\in\mathcal{M}_\mu(\Omega)} E(u, \Omega).$$

In [129, Proposition 2.3], it has been shown that for every $p \in (2, 2^*)$, every $\lambda \in \mathbb{R}$ and every $\mu > 0$,

$$\mathcal{J}_\Omega(\lambda) \geq \mathcal{E}_\Omega(2\mu) + \lambda\mu$$

(be careful that in [129] the manifold $\mathcal{M}_\mu(\Omega)$ is defined as the set of functions in $H^1_0(\Omega)$ whose L^2 norm is equal to 2μ, whereas here functions in $\mathcal{M}_\mu(\Omega)$ have L^2 norm equal to μ). If u is any element in $\mathcal{N}^{nod}_\lambda(\Omega)$, then $u^\pm \in \mathcal{N}_\lambda(\Omega)$ and, by the previous inequality,

$$J_\lambda(u, \Omega) = J_\lambda(u^+, \Omega) + J_\lambda(u^-, \Omega) \geq 2\mathcal{J}_\Omega(\lambda) \geq 2(\mathcal{E}_\Omega(2\mu) + \lambda\mu).$$

Taking the infimum over $u \in \mathcal{N}^{nod}_\lambda(\Omega)$ we conclude that for every $p \in (2, 2^*)$, every $\lambda \in \mathbb{R}$ and every $\mu > 0$,

$$\mathcal{J}^{nod}_\Omega(\lambda) \geq 2\left(\mathcal{E}_\Omega(2\mu) + \lambda\mu\right). \tag{3.17}$$

We now distinguish cases according to the value of p.

Case 1: $p \in (2, 2 + 4/N)$. It is well known that in this case $\mathcal{E}_\Omega(2\mu)$ is finite for every $\mu > 0$. Therefore, by (3.17),

$$\liminf_{\lambda\to+\infty} \frac{\mathcal{J}^{nod}_\Omega(\lambda)}{\lambda} \geq \liminf_{\lambda\to+\infty} \frac{2(\mathcal{E}_\Omega(2\mu) + \lambda\mu)}{\lambda} = 2\mu.$$

Since μ is arbitrary, the conclusion follows.

Case 2: $p = 2 + 4/N$. By [129, Lemma 2.1], $\mathcal{E}_\Omega(\mu_N) = 0$, so that again by (3.17), for every $\lambda > 0$, we have

$$\mathcal{J}^{nod}_\Omega(\lambda) \geq 2\left(\mathcal{E}(\mu_N) + \lambda\frac{\mu_N}{2}\right) = \lambda\mu_N,$$

yielding $\mathcal{J}^{nod}_\Omega(\lambda)/\lambda \geq \mu_N$. Conversely, in [129, Lemma 2.4] it is shown that, for sufficiently large λ, there exists a function $v_\lambda \in \mathcal{N}_\lambda(\Omega)$ compactly supported in a small ball B contained in Ω and such that $\|v_\lambda\|_2^2 = \mu_N$ and $E(v_\lambda, B)/\lambda = o(1)$ as $\lambda \to +\infty$. Let w_λ be a translation of v_λ supported in another ball contained in Ω but disjoint from B. Clearly, $v_\lambda - w_\lambda \in \mathcal{N}^{nod}_\lambda(\Omega)$, $\|v_\lambda - w_\lambda\|_2^2 = 2\mu_N$ and $E(v_\lambda - w_\lambda, \Omega)/\lambda = o(1)$ as $\lambda \to +\infty$. Hence,

$$\limsup_{\lambda\to+\infty} \frac{\mathcal{J}^{nod}_\Omega(\lambda)}{\lambda} \leq \limsup_{\lambda\to+\infty} \frac{J_\lambda(v_\lambda - w_\lambda, \Omega)}{\lambda} = \lim_{\lambda\to+\infty} \frac{E(v_\lambda - w_\lambda, \Omega) + \lambda\mu_N}{\lambda} = \mu_N,$$

and Case 2 is proved.

Case 3: $p \in (2 + 4/N, 2^*)$. Again as in [129, Lemma 2.4], for large λ we can construct a function $v_\lambda \in \mathcal{N}_\lambda(\Omega)$ supported in a small ball $B \subset \Omega$ and such that $J(v_\lambda, \Omega)/\lambda \to 0$ as $\lambda \to +\infty$ (thanks to $p > 2 + 4/N$). It suffices now to define w_λ as in Case 2 to see that $v_\lambda - w_\lambda \in \mathcal{N}^{nod}_\lambda(\Omega)$ and

$$\limsup_{\lambda\to+\infty} \frac{\mathcal{J}^{nod}_\Omega(\lambda)}{\lambda} \leq \lim_{\lambda\to+\infty} \frac{J_\lambda(v_\lambda - w_\lambda, \Omega)}{\lambda} = 2\lim_{\lambda\to+\infty} \frac{J_\lambda(v_\lambda, \Omega)}{\lambda} = 0.$$

Since (for $\lambda > 0$), $\mathcal{J}^{nod}_\Omega(\lambda)/\lambda \geq 0$ by (3.7), the proof is complete. □

3.4 Proof of Theorem 3.1

The purpose of this section is to prove the characterization of the masses of all action ground states and nodal action ground states as stated in Theorem 3.1.

To this end, recall the definition of the sets $M_p(\Omega), M_p^{nod}(\Omega)$ given in (3.3), and note that

$$M_p(\Omega) = \bigcup_{\lambda\in\mathbb{R}} Q_p(\lambda), \qquad M_p^{nod}(\Omega) = \bigcup_{\lambda\in\mathbb{R}} Q_p^{nod}(\lambda),$$

where $Q_p(\lambda), Q_p^{nod}(\lambda)$ are the sets in Proposition 3.8 and in Proposition 3.17, respectively. Furthermore, for every $p \in (2, 2^*)$, we denote

$$\mu_p := 2\sup\left\{\mathcal{J}_\Omega'(\lambda) \mid \mathcal{J}_\Omega \text{ is differentiable at } \lambda\right\}, \tag{3.18}$$

$$\mu_p^{nod} := 2\sup\left\{(\mathcal{J}_\Omega^{nod})'(\lambda) \mid \mathcal{J}_\Omega^{nod} \text{ is differentiable at } \lambda\right\}, \tag{3.19}$$

and we recall that μ_N is the number defined in (3.4).

Remark 3.19. We will see that although $\mathcal{J}_\Omega$ (respectively $\mathcal{J}_\Omega^{nod}$) may fail to be differentiable on a set of measure zero, *every* value $\mu \in (0, \frac{1}{2}\mu_p)$ (resp. $(0, \frac{1}{2}\mu_p^{nod})$) is achieved by $\mathcal{J}_\Omega'$ (resp. $(\mathcal{J}_\Omega^{nod})'$). This is the Darboux-type property we mentioned in the Introduction.

The rest of this section is devoted to show that μ_p, μ_p^{nod} above provide the desired thresholds of Theorem 3.1, and that they are equal to $+\infty$ when $p < 2+\frac{4}{N}$ and finite when $p \geq 2+\frac{4}{N}$. Since the argument is exactly the same both for action ground states and for nodal action ground states, here we report it in details only for the nodal case. The argument is divided in the following series of lemmas.

Lemma 3.20. *For every $p \in (2, 2^*)$, let*

$$g(\lambda) := \liminf_{\varepsilon\to 0^-} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} \qquad \text{and} \qquad \overline{g} := 2\sup_{\lambda\in\mathbb{R}} g(\lambda). \tag{3.20}$$

Then the following holds.

1) if $\mu \in (0, \overline{g})$, then $\mu \in M_p^{nod}(\Omega)$;

2) if $\mu > \overline{g}$, then $\mu \notin M_p^{nod}(\Omega)$.

Proof. Note first that, by Proposition 3.15 and Proposition 3.17, $\overline{g} > 0$. We fix any $\mu \in (0, \overline{g})$ and we take $\overline{\lambda}$ such that $\mu < 2g(\overline{\lambda}) \leq \overline{g}$. Note that if $\overline{g}$ is not attained, then $\overline{\lambda}$ exists by definition; if $\overline{g}$ is attained, we take $\overline{\lambda}$ such that $2g(\overline{\lambda}) = \overline{g}$. Since $\mu < 2g(\overline{\lambda})$, there exists $\delta > 0$ such that for every $\varepsilon \in (-\delta, 0)$,

$$2\frac{\mathcal{J}_\Omega^{nod}(\overline{\lambda}+\varepsilon) - \mathcal{J}_\Omega^{nod}(\overline{\lambda})}{\varepsilon} > \frac{1}{2}(\mu + 2g(\overline{\lambda})),$$

or

$$\mathcal{J}_\Omega^{nod}(\overline{\lambda}+\varepsilon) < \mathcal{J}_\Omega^{nod}(\overline{\lambda}) + \frac{\varepsilon}{4}(\mu + 2g(\overline{\lambda})). \tag{3.21}$$

Define now the function $f_\mu : [-\gamma_2, \overline{\lambda}] \to \mathbb{R}$ as

$$f_\mu(\lambda) = \mathcal{J}_\Omega^{nod}(\lambda) - \frac{\mu}{2}\lambda.$$

Since f_μ is continuous, it has a global minimum point $\widetilde{\lambda} \in [-\gamma_2, \overline{\lambda}]$. Notice that $\widetilde{\lambda} \neq -\gamma_2$ because, by (3.6), $f_\mu(\lambda) < f_\mu(-\gamma_2)$ in a right neighborhood of $-\gamma_2$. Similarly, $\widetilde{\lambda} \neq \overline{\lambda}$ because for $\varepsilon \in (-\delta, 0)$, by (3.21),

$$\begin{aligned} f_\mu(\widetilde{\lambda}) \le f_\mu(\overline{\lambda}+\varepsilon) &= \mathcal{J}_\Omega^{nod}(\overline{\lambda}+\varepsilon) - \frac{\mu}{2}\overline{\lambda} - \frac{\varepsilon}{2}\mu < \mathcal{J}_\Omega^{nod}(\overline{\lambda}) + \frac{\varepsilon}{4}(\mu + 2g(\overline{\lambda})) - \frac{\mu}{2}\overline{\lambda} - \frac{\varepsilon}{2}\mu \\ &= f_\mu(\overline{\lambda}) + \varepsilon\,\frac{2g(\overline{\lambda}) - \mu}{4} < f_\mu(\overline{\lambda}). \end{aligned}$$

Therefore $\widetilde{\lambda} \in (-\gamma_2, \overline{\lambda})$ and $f_\mu(\widetilde{\lambda}+\varepsilon) - f_\mu(\widetilde{\lambda}) \ge 0$ for every $|\varepsilon|$ small enough. Now, if $\varepsilon < 0$,

$$\begin{aligned} \frac{\mathcal{J}_\Omega^{nod}(\widetilde{\lambda}+\varepsilon) - \mathcal{J}_\Omega^{nod}(\widetilde{\lambda})}{\varepsilon} - \frac{\mu}{2} &= \frac{\mathcal{J}_\Omega^{nod}(\widetilde{\lambda}+\varepsilon) - \mu\widetilde{\lambda}/2 - \varepsilon\mu/2 - \mathcal{J}_\Omega^{nod}(\widetilde{\lambda}) + \mu\widetilde{\lambda}/2}{\varepsilon} \\ &= \frac{f_\mu(\widetilde{\lambda}+\varepsilon) - f_\mu(\widetilde{\lambda})}{\varepsilon} \le 0, \end{aligned}$$

whence

$$2\limsup_{\varepsilon\to 0^-} \frac{\mathcal{J}_\Omega^{nod}(\widetilde{\lambda}+\varepsilon) - \mathcal{J}_\Omega^{nod}(\widetilde{\lambda})}{\varepsilon} \le \mu.$$

The same computation, for $\varepsilon > 0$ leads to

$$2\liminf_{\varepsilon\to 0^+} \frac{\mathcal{J}_\Omega^{nod}(\widetilde{\lambda}+\varepsilon) - \mathcal{J}_\Omega^{nod}(\widetilde{\lambda})}{\varepsilon} \ge \mu.$$

These inequalities, coupled with (3.13) show that $\mathcal{J}_\Omega^{nod}$ is differentiable at $\widetilde{\lambda}$ and that $(\mathcal{J}_\Omega^{nod})'(\widetilde{\lambda}) = \mu$. Thus, every nodal action ground state in $\mathcal{N}_{\widetilde{\lambda}}^{nod}(\Omega)$ has mass μ, and $\mu \in M_p^{nod}(\Omega)$. This proves 1).

To prove the second statement, just notice that if $\mu > \overline{g}$ is the mass of a nodal action ground state u in some $\mathcal{N}_\lambda^{nod}(\Omega)$, then by (3.13),

$$\overline{g} < \mu \le \sup Q_p^{nod}(\lambda) \le 2\liminf_{\varepsilon\to 0^-} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} = 2g(\lambda) \le \overline{g},$$

which is impossible. □

Remark 3.21. The main argument in the proof of the preceding lemma consists in minimizing, given μ, the function $f_\mu(\lambda) = \mathcal{J}_\Omega^{nod}(\lambda) - \frac{\mu}{2}\lambda$ over the compact set $[-\gamma_2, \overline{\lambda}]$. This approach is used to unify the three cases of p L^2-subcritical, L^2-critical or L^2-supercritical. We note, for future reference, that for L^2-subcritical p and all μ, or for L^2-critical p and $\mu < 2\mu_N$, the function f_μ can be minimized on $[-\gamma_2, +\infty)$, obtaining then a global minimum. Indeed, in these cases f_μ is coercive, as one immediately sees by writing

$$f_\mu(\lambda) = \lambda\left(\frac{\mathcal{J}_\Omega^{nod}(\lambda)}{\lambda} - \frac{\mu}{2}\right)$$

and taking into account the asymptotic behavior of $\mathcal{J}_\Omega^{nod}(\lambda)/\lambda$ described in (3.16). Actually one can easily see that f_μ can be minimized on $\mathbb{R}$ (as $\mathcal{J}_\Omega^{nod}(\lambda) = 0$ for $\lambda \leq -\gamma_2$).

Lemma 3.22. *For every $p \in (2, 2^*)$, let $g(\lambda)$ and $\overline{g}$ be as in* (3.20)*. Then, recalling* (3.19)*,*

$$\mu_p^{nod} = \overline{g}.$$

Proof. We provide a proof for completeness. Let $A \subseteq \mathbb{R}$ be the set of points where $\mathcal{J}_\Omega^{nod}$ is differentiable, so that $\mu_p^{nod} = 2\sup_A(\mathcal{J}_\Omega^{nod})'$. Obviously, for every $\lambda \in A$, $(\mathcal{J}_\Omega^{nod})'(\lambda) = g(\lambda)$, and therefore

$$\mu_p^{nod} = 2\sup_{\lambda\in A}(\mathcal{J}_\Omega^{nod})'(\lambda) = 2\sup_{\lambda\in A} g(\lambda) \leq \overline{g}.$$

Conversely, let $\lambda \in \mathbb{R}$ and observe that, for every $\varepsilon < 0$, since $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz continuous,

$$\begin{aligned}\mathcal{J}_\Omega^{nod}(\lambda) - \mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) &= \int_{\lambda+\varepsilon}^{\lambda} (\mathcal{J}_\Omega^{nod})'(t)\,dt \\ &\leq (-\varepsilon) \sup_{A\cap[\lambda+\varepsilon,\lambda]} (\mathcal{J}_\Omega^{nod})' \\ &\leq (-\varepsilon)\sup_A(\mathcal{J}_\Omega^{nod})' \\ &= -\varepsilon\frac{\mu_p^{nod}}{2}.\end{aligned}$$

Dividing by $-\varepsilon > 0$ we obtain

$$\frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} \leq \frac{\mu_p^{nod}}{2},$$

whence

$$2g(\lambda) = 2\liminf_{\varepsilon\to 0^-} \frac{\mathcal{J}_\Omega^{nod}(\lambda+\varepsilon) - \mathcal{J}_\Omega^{nod}(\lambda)}{\varepsilon} \leq \mu_p^{nod}.$$

Since this holds for every $\lambda \in \mathbb{R}$, we obtain $\overline{g} \leq \mu_p^{nod}$. □

Remark 3.23. In view of the previous lemma, the conclusion of Lemma 3.20 can be written as

1) if $\mu \in (0, \mu_p^{nod})$, then $\mu \in M_p^{nod}(\Omega)$;

2) if $\mu > \mu_p^{nod}$, then $\mu \notin M_p^{nod}(\Omega)$.

Lemma 3.24. *There results*

$$\mu_p^{nod} \begin{cases} = +\infty & \text{if } p < 2 + 4/N \\ < +\infty & \text{if } p \geq 2 + 4/N. \end{cases}$$

Proof. First let $p < 2+4/N$. By Remark 3.23, it is enough to show that there exist nodal action ground states of arbitrarily large mass. We will do it by estimating from below the mass of elements of $\mathcal{N}_\lambda^{nod}(\Omega)$ in terms of λ. To this aim, take any $u \in \mathcal{N}_\lambda^{nod}(\Omega)$. By the Gagliardo-Nirenberg inequality

$$\|u\|_p^p \leq K_p \|u\|_2^{p-\alpha} \|\nabla u\|_2^\alpha, \qquad \alpha = N\left(\frac{p}{2} - 1\right),$$

noticing that $\alpha < 2$ since p is L^2-subcritical, using the Young inequality and writing $\mu = \|u\|_2^2$, we deduce that

$$\|\nabla u\|_2^2 + \lambda\mu = \|u\|_p^p \leq K_p \frac{2-\alpha}{2} \mu^{\frac{p-\alpha}{2}\frac{2}{2-\alpha}} + \frac{\alpha}{2}\|\nabla u\|_2^2 = K_p \frac{2-\alpha}{2}\mu^{\frac{p-\alpha}{2-\alpha}} + \frac{\alpha}{2}\|\nabla u\|_2^2.$$

Therefore

$$\lambda\mu \leq \left(1 - \frac{\alpha}{2}\right)\|\nabla u\|_2^2 + \lambda\mu \leq K_p \frac{2-\alpha}{2}\mu^{\frac{p-\alpha}{2-\alpha}},$$

whence $\mu \geq C\lambda^{\frac{2-\alpha}{p-2}} \to +\infty$ as $\lambda \to +\infty$.

Assume now that $p \geq 2 + 4/N$ and take again $u \in \mathcal{N}_\lambda^{nod}(\Omega)$. Denoting by C (possibly different but) universal constants, by (3.6) we notice that

$$\mu = \|u\|_2^2 \leq C\|u\|_p^2 = C\left(\mathcal{J}_\Omega^{nod}(\lambda)\right)^{\frac{2}{p}} \leq C(\lambda + \gamma_2)^{\frac{2}{p-2}}.$$

This shows that μ is bounded when λ ranges in a bounded subset of $\mathbb{R}$. By (3.16), $\mathcal{J}_\Omega^{nod}(\lambda)/\lambda$ is bounded on $[1, +\infty)$ since $p \geq 2 + 4/N$. Therefore, for every $\lambda \geq 1$,

$$\lambda\mu = \lambda\|u\|_2^2 \leq \|\nabla u\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p = \frac{1}{\kappa}\mathcal{J}_\Omega^{nod}(\lambda) \leq C\lambda$$

and the proof is complete. □

Remark 3.25. We notice that, when $p = 2 + 4/N$, recalling (3.4),

$$\mu_p^{nod} \geq 2\mu_N.$$

Indeed, for every $\lambda > -\gamma_2$, since $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz continuous,

$$\mathcal{J}_\Omega^{nod}(\lambda) = \int_{-\gamma_2}^{\lambda} (\mathcal{J}_\Omega^{nod})'(t)\, dt \leq (\lambda + \gamma_2)\frac{\mu_p^{nod}}{2}$$

so that, by (3.16),

$$\mu_N = \lim_{\lambda\to\infty} \frac{\mathcal{J}_\Omega^{nod}(\lambda)}{\lambda} \leq \lim_{\lambda\to\infty} \frac{\lambda + \gamma_2}{\lambda}\frac{\mu_p^{nod}}{2} = \frac{\mu_p^{nod}}{2}.$$

Proof of Theorem 3.1, nodal case. Point (i) follows by Lemmas 3.20–3.22–3.24.

When $p \geq 2 + 4/N$, the same results show that μ_p^{nod} is finite, that $(0, \mu_p^{nod}) \subset M_p^{nod}(\Omega)$, and that $M_p^{nod}(\Omega)\cap(\mu_p^{nod}, +\infty) = \varnothing$. This proves point (ii) for $p = 2+\frac{4}{N}$.

To complete the proof of point (iii) too, we are left to show that $\mu_p^{nod} \in M_p^{nod}(\Omega)$ when p is L^2-supercritical. To this aim, let $\mu_n \nearrow \mu_p^{nod}$ and let u_n be a sequence of nodal action ground states of mass μ_n in some $\mathcal{N}_{\lambda_n}^{nod}(\Omega)$, with $\lambda_n \in (-\gamma_2, +\infty)$ for every n. We notice that no subsequence of λ_n can converge to $-\gamma_2$, since in this case, for such subsequence (not relabeled), we would have

$$\mu_n = \|u_n\|_2^2 \leq C\|u_n\|_p^2 = C\left(\frac{1}{\kappa}\mathcal{J}_\Omega^{nod}(\lambda_n)\right)^{2/p} \to 0$$

by the continuity of $\mathcal{J}_\Omega^{nod}$. Similarly, no subsequence of λ_n can tend to $+\infty$, because we would have

$$\lambda_n\mu_n \leq \|\nabla u_n\|_2^2 + \lambda_n\mu_n = \|u_n\|_p^p = \frac{1}{\kappa}\mathcal{J}_\Omega^{nod}(\lambda_n),$$

entailing that $\mu_n \leq \frac{1}{\kappa}\mathcal{J}_\Omega^{nod}(\lambda_n)/\lambda_n \to 0$ as $n \to \infty$ by Proposition 3.18 (since $p > 2 + \frac{4}{N}$).

Therefore, we can assume that, up to subsequences, $\lambda_n \to \lambda \in (-\gamma_2, +\infty)$ as $n \to \infty$. Then, by Proposition 3.14, up to subsequences we have that u_n converges in $H_0^1(\Omega)$ to a nodal action ground state $u \in \mathcal{N}_\lambda^{nod}(\Omega)$. Since $\|u\|_2^2 = \mu_p^{nod}$, we see that $\mu_p^{nod} \in M_p^{nod}(\Omega)$ and the proof is complete. □

3.5 Proof of Theorems 3.2–3.4–3.7

In this section we prove the main results of the chapter regarding normalized nodal solutions and least energy normalized solutions, i.e. Theorems 3.2–3.4–3.7.

The proof of Theorems 3.2–3.4 is a consequence of the discussion developed in Section 3.4. In particular, Theorem 3.2 is a corollary of Theorem 3.1.

Proof of Theorems 3.4. Fix any $\mu > 0$ if $p < 2 + 4/N$ or $\mu \in (0, 2\mu_N)$ if $p = 2 + 4/N$ (where μ_N is as in (3.4)), and let $\widetilde{\lambda}$ be a global minimizer of the function $f_\mu(\lambda) = \mathcal{J}_\Omega^{nod}(\lambda) - \frac{\mu}{2}\lambda$ (see Remark 3.21). Take $\widetilde{u}$ to be a nodal action ground state of mass μ corresponding to $\widetilde{\lambda}$ (when $p = 2 + \frac{4}{N}$, the fact that such a $\widetilde{u}$ exists for every $\mu < 2\mu_N$ is guaranteed by Remark 3.25). Let then w be any other nodal solution of (3.1), for some $\lambda_w \in \mathbb{R}$. Then

$$\begin{aligned} E(w,\Omega) &= J_{\lambda_w}(w,\Omega) - \frac{\mu}{2}\lambda_w \geq \mathcal{J}_\Omega^{nod}(\lambda_w) - \frac{\mu}{2}\lambda_w \geq \min_{\lambda \in \mathbb{R}} \Big(\mathcal{J}_\Omega^{nod}(\lambda) - \frac{\mu}{2}\lambda\Big) \\ &= \mathcal{J}_\Omega^{nod}(\widetilde{\lambda}) - \frac{\mu}{2}\widetilde{\lambda} = J_{\widetilde{\lambda}}(\widetilde{u},\Omega) - \frac{\mu}{2}\widetilde{\lambda} = E(\widetilde{u},\Omega), \end{aligned}$$

that is the nodal action ground state $\widetilde{u}$ is a least energy normalized nodal solution with mass μ. Moreover, if w is another least energy normalized nodal solution with mass μ, the previous lines become a chain of equalities, showing in particular that $J_{\lambda_w}(w,\Omega) = \mathcal{J}_\Omega^{nod}(\lambda_w)$, i.e. w is a nodal action ground state in $\mathcal{N}_{\lambda_w}^{nod}(\Omega)$. □

We are thus left to prove Theorem 3.7. To this end, in what follows we take

$$p > p_c := 2 + \frac{4}{N}, \qquad \Omega \text{ star-shaped with respect to } 0.$$

As is quite common in the literature, working on star-shaped domains allows one to profit of the Pohožaev identity. In particular, we will use the following fact.

Proposition 3.26. *Assume that $\Omega \subset \mathbb{R}^N$ is bounded, smooth and star-shaped. Then, for every solution u of* (3.2),

$$E(u,\Omega) \geq \frac{N(p-p_c)}{4p}\|u\|_p^p.$$

Proof. Up to translating the domain, we may assume that Ω is star-shaped with respect to 0. As Ω is regular, an H^1 solution u of (3.2) is in $\mathcal{C}^2(\overline{\Omega})$ and the Pohožaev identity (see e.g. [310, Chapter III, Lemma 1.4]) implies that

$$\frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N}{p}\|u\|_p^p + \frac{\lambda N}{2}\|u\|_2^2 + \frac{1}{2}\int_{\partial\Omega} |\partial_\nu u|^2 x \cdot \nu \, d\sigma = 0,$$

where ν is the exterior unit normal. Since Ω is star-shaped, $x \cdot \nu \geq 0$ for every $x \in \partial\Omega$, so that

$$\frac{N-2}{2}\|\nabla u\|_2^2 - \frac{N}{p}\|u\|_p^p + \frac{\lambda N}{2}\|u\|_2^2 \leq 0.$$

Recalling that $\|\nabla u\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p$ because $u \in \mathcal{N}_\lambda(\Omega)$, we see that

$$\Big(\frac{N-2}{2} - \frac{N}{p}\Big)\|u\|_p^p + \lambda\|u\|_2^2 \leq 0,$$

entailing

$$\begin{aligned}E(u,\Omega) &= J_\lambda(u,\Omega) - \frac{\lambda\|u\|_2^2}{2}\\ &\geq \Big(\frac{p-2}{2p} + \frac{p(N-2)-2N}{4p}\Big)\|u\|_p^p\\ &= \frac{N(p-p_c)}{4p}\|u\|_p^p.\end{aligned}$$ □

Proof of Theorem 3.7. The argument is divided in two steps.

Step 1. Letting μ_p the threshold given by Theorem 3.2(iii), we prove that there exists a least energy normalized solution for every $\mu \leq \mu_p$. To this end, let $\mu \leq \mu_p$ be fixed and

$$\mathcal{S}_\mu := \Big\{u \in H_0^1(\Omega) \;\Big|\; u \text{ solves (3.1) for some } \lambda \in \mathbb{R}\Big\}$$

be the set of all solutions of (3.1). Since $\mathcal{S}_\mu \neq \varnothing$ by Theorem 3.2(iii), let $(u_n)_n \subset \mathcal{S}_\mu$ be such that

$$-\Delta u_n + \lambda_n u_n = |u_n|^{p-2}u_n \qquad \text{and} \qquad E(u_n,\Omega) \xrightarrow[n\to\infty]{} \inf_{v\in\mathcal{S}_\mu} E(v,\Omega).$$

Observe first that, since for all n we have

$$E(u_n,\Omega) = J_{\lambda_n}(u_n,\Omega) - \frac{\lambda_n}{2}\mu \geq -\frac{\lambda_n}{2}\mu,$$

the sequence $(\lambda_n)_n$ is bounded from below. Furthermore, according to Proposition 3.26, as $u_n \in \mathcal{N}_{\lambda_n}(\Omega)$, we have

$$\begin{aligned}E(u_n,\Omega) &\geq \frac{N(p-p_c)}{2(p-2)} J_{\lambda_n}(u_n,\Omega)\\ &\geq \frac{N(p-p_c)}{2(p-2)} \inf_{v\in\mathcal{N}_{\lambda_n}(\mathbb{R}^N)} J(v,\mathbb{R}^N)\\ &= \frac{N(p-p_c)}{2(p-2)}\mathcal{J}_{\mathbb{R}^N}(\lambda_n).\end{aligned}$$

Recalling that $\mathcal{J}_{\mathbb{R}^N}(s) = s^{\frac{2N-p(N-2)}{2(p-2)}}\mathcal{J}_{\mathbb{R}^N}(1) \to +\infty$ as $s \to +\infty$, and since $p_c < p < 2^*$, this implies that $(\lambda_n)_n$ is also bounded from above and hence, up to subsequences, $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$.

Since $(E(u_n,\Omega))_n$ is bounded and $\|u_n\|_2 = \mu$, so is $(J_{\lambda_n}(u_n,\Omega))_n$. As usual, this implies that $(u_n)_n$ is bounded in $H^1(\Omega)$. Therefore, we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^q(\Omega)$ for every $q \in [2,2^*)$, showing that u is a solution to (3.2) of mass μ.

Moreover, by weak lower semi-continuity,

$$E(u,\Omega) \leq \liminf_{n\to\infty} E(u_n,\Omega) = \inf_{v\in\mathcal{S}_\mu} E(v,\Omega),$$

so that u is a least energy normalized solution of mass μ.

Arguing analogously one can prove that, taking μ_p^{nod} as in Theorem 3.2 and letting

$$\mathcal{S}_\mu^{nod} := \left\{ u \in H_0^1(\Omega) \;\middle|\; u \text{ solves (3.1) for some } \lambda \in \mathbb{R},\ u^\pm \not\equiv 0 \right\}$$

the set of all nodal solutions of (3.1), for every $\mu \leq \mu_p^{nod}$ there exists a least energy normalized nodal solution, that is $u \in \mathcal{S}_\mu^{nod}$ such that

$$E(u,\Omega) = \inf_{v\in\mathcal{S}_\mu^{nod}} E(v,\Omega).$$

Indeed, the argument above here shows again that, up to subsequences, a sequence $(u_n)_n \subset \mathcal{S}_\mu^{nod}$ such that $E(u_n,\Omega) \to \inf_{v\in\mathcal{S}_\mu^{nod}} E(v,\Omega)$ converges weakly in $H^1(\Omega)$ to a solution u of (3.1), and to conclude that u is a least energy normalized nodal solution it remains to show that u is still nodal. To see this, we notice that, by strong convergence in $L^2(\Omega)$, $u \not\equiv 0$. If it were $u \geq 0$, by Remark 3.9 we would have $\lambda > -\gamma_1$. But then,

$$\kappa \liminf_n \|u_n^\pm\|_p^p = \liminf_n J_{\lambda_n}(u_n^\pm,\Omega) \geq \liminf_n \mathcal{J}_\Omega(\lambda_n) = \mathcal{J}_\Omega(\lambda) > 0,$$

so that neither $(u_n^+)_n$ nor $(u_n^-)_n$ converge to zero in $L^p(\Omega)$, contradicting the assumption $u \geq 0$.

Step 2. We are left to prove that, when the mass is sufficiently small, least energy normalized solutions/least energy normalized nodal solutions are action ground states/nodal action ground states with frequency uniformly bounded from above. We give the details of the proof for least energy normalized nodal solutions, the signed case being analogous.

Set

$$\overline{\lambda}_p^{nod} := \frac{2p\gamma_2}{N(p-p_c)}, \qquad \overline{\mu}_p^{nod} := \frac{2\mathcal{J}_\Omega^{nod}(\overline{\lambda}_p^{nod})}{\overline{\lambda}_p^{nod} + \gamma_2}.$$

Note that, since $\mathcal{J}_\Omega^{nod}$ is locally Lipschitz by Proposition 3.17 and $\mathcal{J}_\Omega^{nod}(-\gamma_2) = 0$ by Proposition 3.11, then

$$\mathcal{J}_\Omega^{nod}\left(\overline{\lambda}_p^{nod}\right) = \int_{-\gamma_2}^{\overline{\lambda}_p^{nod}} (\mathcal{J}_\Omega^{nod})'(s)\,ds \leq (\overline{\lambda}_p^{nod} + \gamma_2) \sup_\lambda (\mathcal{J}_\Omega^{nod})'(\lambda),$$

so that $\overline{\mu}_p^{nod} \leq \mu_p^{nod}$, where μ_p^{nod} is the number defined in (3.19).

Moreover, by Proposition 3.26 and the definition of $\overline{\lambda}_p^{nod}$, if $u \in \mathcal{N}_\lambda^{nod}(\Omega)$ with $\lambda \geq \overline{\lambda}_p^{nod}$ and $\|u\|_2^2 = \mu$, then

$$\begin{aligned} E(u, \Omega) &\geq \frac{N(p - p_c)}{4p} \|u\|_p^p \\ &= \frac{N(p - p_c)}{4p} \left(\|\nabla u\|_2^2 + \lambda \|u\|_2^2\right) \\ &> \frac{N(p - p_c)}{4p} \lambda \mu \\ &\geq \frac{\gamma_2}{2} \mu. \end{aligned} \tag{3.22}$$

Let then $\mu \leq \overline{\mu}_p^{nod}$ be fixed. Since $\mu \leq \mu_p^{nod}$, the proof of Lemma 3.20 above guarantees that there exists a function $u \in \mathcal{S}_\mu^{nod}$ such that $E(u, \Omega) < \mathcal{J}_\Omega^{nod}(-\gamma_2) + \frac{\gamma_2}{2}\mu = \frac{\gamma_2}{2}\mu$, so that

$$\inf_{v \in \mathcal{S}_\mu^{nod}} E(v, \Omega) < \frac{\gamma_2}{2} \mu.$$

By (3.22), any least energy normalized nodal solution in $\mathcal{S}_\mu^{nod}$ (which exist by Step 1) must then belong to $\mathcal{N}_\lambda^{nod}(\Omega)$ for some $\lambda < \overline{\lambda}_p^{nod}$. Since $\overline{\lambda}_p^{nod}$ depends only on p and Ω, this proves that least energy normalized nodal solutions have uniformly bounded frequency, and

$$\inf_{v \in \mathcal{S}_\mu^{nod}} E(v, \Omega) = \inf_{\substack{v \in \mathcal{S}_\mu^{nod} \cap \mathcal{N}_\lambda^{nod}(\Omega) \\ \lambda < \overline{\lambda}_p^{nod}}} E(v, \Omega).$$

To conclude, it remains to show that least energy normalized nodal solutions are always nodal action ground states for $\mu \leq \overline{\mu}_p^{nod}$. This follows arguing exactly as in the proof of Lemma 3.20 and of Theorem 3.4, noting that the function

$$f_\mu(\lambda) := \mathcal{J}_\Omega^{nod}(\lambda) - \frac{\lambda}{2} \mu$$

has a minimum point in $(-\gamma_2, \overline{\lambda}_p^{nod})$, since it is continuous, $f_\mu(s) < f_\mu(-\gamma_2)$ for s in a right neighbourhood of $-\gamma_2$, and

$$f_\mu\left(\overline{\lambda}_p^{nod}\right) = \mathcal{J}_\Omega^{nod}\left(\overline{\lambda}_p^{nod}\right) - \frac{\overline{\lambda}_p^{nod}}{2} \mu \geq \frac{\gamma_2}{2} \mu = f_\mu(-\gamma_2). \qquad \square$$

Chapter 4

An infinity of normalized solutions of L^2-supercritical NLS equations on noncompact metric graphs with localized nonlinearities

4.1 Presentation of the chapter

Throughout this chapter we assume that $\mathcal{G}$ is a noncompact metric graph which belongs to the following class $\mathbf{G_4}$.

Definition 4.1. Let $\mathbf{G_4}$ be the class of metric graphs $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ having:

- a finite number of edges and vertices;
- a non trivial compact core $\mathcal{K}$ (defined[a] as the metric subgraph of $\mathcal{G}$ which consists in all the bounded edges of $\mathcal{G}$);
- at least one half-line.

[a]See e.g. [21, 297]. We noted the set of *bounded* edges $\mathcal{B}$ in Chapter 2 since on general metric graphs, $\mathcal{B}$ may be noncompact and may even be disconnected. Here, we use the notation $\mathcal{K}$ to insist that $\mathcal{K}$ is a *compact* metric graph included in $\mathcal{G}$.

The notion of metric graph is detailed in [68] and Appendix A.

The chapter is devoted to the existence of infinitely many normalized solutions, sometimes called *bound states*, of prescribed mass for the L^2-supercritical nonlinear Schrödinger (NLS) equation with localized nonlinearities on $\mathcal{G}$

$$-u'' + \lambda u = \kappa(x)|u|^{p-2}u, \tag{4.1}$$

coupled with the Kirchhoff conditions at the vertices, see ($\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G}}$) below. Here, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier associated to the mass constraint, $p > 6$ is a real number, $\mathcal{G}$ is a metric graph belonging to $\mathbf{G_4}$ and[1] κ is the characteristic function of the compact core $\mathcal{K}$ of $\mathcal{G}$.

[1]The reader should be warned not to confuse the truncation function κ with the constant $\varkappa = \frac{1}{2} - \frac{1}{p}$ used in the two first chapters. To avoid any confusions, we will not use both these symbols at the same time in the thesis.

We refer to the introduction for a discussion about the history and goals of the study of nonlinear Schrödinger equations on metric graphs, in particular to section II.8 for motivations for studying equations with a localized nonlinearity.

Solutions of the equation (4.1) satisfying the Kirchhoff conditions and having prescribed mass μ are often referred to as *normalized solutions of mass* μ. They correspond to critical points of the *energy functional* $E : H^1(\mathcal{G}) \to \mathbb{R}$ defined by

$$E(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathcal{K}} |u|^p \,\mathrm{d}x, \tag{4.2}$$

under the *mass constraint*

$$\int_{\mathcal{G}} |u|^2 \,\mathrm{d}x = \mu > 0. \tag{4.3}$$

It is standard to show that E is of class $\mathcal{C}^2$ on $H^1(\mathcal{G})$. Let us remark that solutions to (4.1) provide standing waves of the time-dependent focusing NLS on $\mathcal{G}$,

$$\boldsymbol{i}\partial_t\psi(t,x) = -\partial_{xx}\psi(t,x) - \kappa(x)|\psi(t,x)|^{p-2}\psi(t,x),$$

via the ansatz $\psi(t,x) = \mathrm{e}^{\boldsymbol{i}\lambda t}\,u(x)$. The constraint (4.3) is meaningful from a dynamics perspective as the mass (or charge), as well as the energy, is conserved by the NLS flow (see Appendix F for a discussion of conservation laws for the time-dependent focusing NLS equation on $\mathbb{R}^N$).

Recently, much effort has been devoted to establish the existence of normalized solutions of nonlinear Schrödinger equations on metric graphs, in the L^2-*subcritical* (i.e. $p \in (2,6)$) or L^2-*critical regimes* (i.e. $p = 6$). In these two regimes, the energy functional E is bounded from below and coercive on the mass constraint. A relevant notion is then the one of energy ground states, namely of solutions which minimize the energy functional on the constraint. For the existence of ground state solutions, the reader can consult [4, 6, 18, 19, 21, 257, 271] for noncompact graphs $\mathcal{G}$ and [89, 124] for compact ones. Some studies are also conducted on the existence of local minimizers, see e.g. [17, 272].

Regarding problems with a localized nonlinearity as in (4.1), existence and non-existence of ground state solutions is discussed in [322] and of bound state solutions in [298] for the L^2-subcritical case. We refer to [132, 133] for the same problem on the L^2-critical case.

Moreover, in the L^2-subcritical regime, one may obtain the existence of multiple bound states with negative energy levels by applying genus theory both in the compact case as in [124] and in the noncompact case with localised nonlinearities as in [297].

However, in the L^2-supercritical regime on general metric graphs, i.e. when $p > 6$, the energy functional E is always unbounded from below. Moreover, due to the fact that graphs are not scale invariant, the techniques based on scalings, usually employed in the Euclidean setting and related to the validity of a Pohožaev identity (see [183] or [48, 49, 181, 303, 304]), are not applicable.

The features mentioned above imply that the search for normalized solutions in the L^2-supercritical regime is delicate. Recently, in [102], this issue was considered on compact metric graphs for which the existence of a non-constant solution was proved for small values of $\mu > 0$. In [82], the case of a noncompact graph with a nonlinearity acting only on its compact core was considered. For any mass, the existence of at least one positive solution to (4.1) was obtained. Our aim here is to show that, under exactly the same assumptions as in [82], the existence of infinitely many, possibly sign-changing, solutions can be obtained for an arbitrary value of the mass μ.

Basic notations and main result

For any graph, we write $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, where $\mathbb{V}$ is the set of vertices and $\mathbb{E}$ is the set of edges. Each bounded edge e is identified with a closed bounded interval $I_e = [0, |e|]$ (where $|e|$ is the length of e), while each unbounded edge is identified with a closed half-line $I_e = [0, +\infty)$. The length of the shortest path between points provides a natural metric (whence a topology and a Borel structure) on $\mathcal{G}$. A function $u : \mathcal{G} \to \mathbb{R}$ is identified with a vector of functions $\{u_e\}_{e\in\mathbb{E}}$, where each u_e is defined on the corresponding interval I_e such that $u|_e = u_e$. Endowing each edge with the Lebesgue measure, one can define $\int_{\mathcal{G}} u(x)\,\mathrm{d}x$ and the space $L^p(\mathcal{G})$ in a natural way, with norm

$$\|u\|^p_{L^p(\mathcal{G})} = \sum_{e\in\mathbb{E}} \|u_e\|^p_{L^p(e)}.$$

All those notions are precised in Appendix A. We will consider the Sobolev space $H^1(\mathcal{G})$ made of continuous functions which are H^1 edge by edge (see section A.4). We recall that the norm in $H^1(\mathcal{G})$ is defined by

$$\|u\|^2_{H^1(\mathcal{G})} := \sum_{e\in\mathbb{E}} \|u'_e\|^2_{L^2(e)} + \|u_e\|^2_{L^2(e)}.$$

We shall study the existence of critical points of the functional $E : H^1(\mathcal{G}) \to \mathbb{R}$ constrained on the L^2-sphere

$$H^1_\mu(\mathcal{G}) := \Big\{ u \in H^1(\mathcal{G}) \ \Big|\ \int_{\mathcal{G}} |u|^2\,\mathrm{d}x = \mu \Big\}.$$

If $u \in H^1_\mu(\mathcal{G})$ is such a critical point, it is standard to show that there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that u satisfies the following problem:

$$\begin{cases} -u'' + \lambda u = \kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathbb{E}, \\ \displaystyle\sum_{e\succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \in \mathbb{V}, \end{cases} \tag{NLS$^{\text{loc}}_{\mathcal{G}}$}$$

where $e \succ v$ means that the edge e is incident at v, and the notation $\mathrm{d}u/\mathrm{d}x_e(v)$ stands for $u'_e(0)$ or $-u'_e(|e|)$, according to whether the vertex v is identified with 0 or $|e|$ (namely, the sum involves the derivatives away from the vertex v). The second equation is the so-called *Kirchhoff boundary condition*.

Our main result is the following.

Theorem 4.2. *Let $\mathcal{G} \in \mathbf{G_4}$ be a metric graph and $p > 6$ be a real number. Then, for any real number $\mu > 0$, Problem* ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) *with the mass constraint $\|u\|^2_{L^2(\mathcal{G})} = \mu$ has infinitely many distinct solutions.*

Moreover, these solutions are associated to positive Lagrange multipliers and correspond to a sequence of critical points of the functional E constrained to $H^1_\mu(\mathcal{G})$ whose levels go to $+\infty$.

In the proofs of [81, 102], a central difficulty was the of lack a priori bounds on Palais-Smale sequences for E constrained to $H^1_\mu(\mathcal{G})$. To overcome this difficulty, an approach by approximation (that we will also use in this chapter) was developed. It consists in considering the family of functionals $E_\rho : H^1(\mathcal{G}) \to \mathbb{R}$ given by

$$E_\rho(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x - \frac{\rho}{p}\int_{\mathcal{K}} |u|^p \,\mathrm{d}x, \qquad \forall u \in H^1(\mathcal{G}),\ \forall \rho \in \left[\frac{1}{2}, 1\right]. \tag{4.4}$$

Clearly, a critical point of E_ρ constrained to $H^1_\mu(\mathcal{G})$ is a solution to

$$\begin{cases} -u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathbb{E}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \in \mathbb{V}, \end{cases} \tag{$\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc}}$}$$

where λ is the associated Lagrange multiplier. Denoting by $\mathrm{m}(u)$ the Morse index of a solution $u \in H^1_\mu(\mathcal{G})$ to ($\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc}}$) (see Definition 4.11), we establish the following result.

Theorem 4.3. *Let $\mathcal{G} \in \mathbf{G_4}$ be a metric graph and $p > 6$. For any $\mu > 0$, there exists $N_0 \in \mathbb{Z}^{\geq 1}$ such that for almost every $\rho \in [1/2, 1]$, there exist sequences of Lagrange multipliers $(\lambda_\rho^N)_{N \geq N_0} \subseteq (0, +\infty)$ and solutions $(u_\rho^N)_{N \geq N_0} \subseteq H^1_\mu(\mathcal{G})$ to the problem*

$$\begin{cases} -(u_\rho^N)'' + \lambda_\rho^N u_\rho^N = \rho\kappa(x)|u_\rho^N|^{p-2}\, u_\rho^N & \textit{on every edge } e \in \mathbb{E}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u_\rho^N}{\mathrm{d}x_e}(v) = 0 & \textit{at every vertex } v \in \mathbb{V}. \end{cases} \tag{$\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc},N}$}$$

In addition, $c_\rho^N := E_\rho(u_\rho^N) \xrightarrow[N\to\infty]{} +\infty$ uniformly with respect to $\rho \in [1/2, 1]$ and $\mathrm{m}(u_\rho^N) \leq N + 1$.

To derive Theorem 4.2 from Theorem 4.3, one considers, for every fixed $\mu > 0$ and every fixed $N \geq N_0$, a sequence $(u_{\rho_n}^N)_{n \geq 1}$ of solutions to ($\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc},N}$) where $\rho_n \to 1^-$ and shows that it converges to some $u^N \in H^1_\mu(\mathcal{G})$. Such $u^N \in H^1_\mu(\mathcal{G})$ will be a solution to ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) such that $\|u\|^2_{L^2(\mathcal{G})} = \mu$.

The point here is to show that the sequence of solutions $(u^N_{\rho_n})_{n\geq 1}$ is bounded, which is itself equivalent to show that the sequence $(\lambda^N_{\rho_n})_{n\geq 1} \subseteq \mathbb{R}$ is bounded. In [81, 102], this step is performed using a blow-up analysis, taking advantage that $u^N_{\rho_n} \in H^1_\mu(\mathcal{G})$ are positive functions. This blow-up analysis can be generalized[2] to possibly sign-changing solutions. Using this generalized result, we can also deduce the boundedness of the sequence of $(\lambda^N_{\rho_n})_{n\geq 1} \subseteq \mathbb{R}$ from the boundedness of the Morse indices of the solutions $u^N_{\rho_n} \in H^1_\mu(\mathcal{G})$ (see Lemma 4.32).

Now, let us turn to the proof of Theorem 4.3. It relies on an abstract result, namely [81, Theorem 1.12]. We recall it here as Theorem 4.12. Applied to our family of functionals $E_\rho : H^1(\mathcal{G}) \to \mathbb{R}$, it guarantees that, for any $\mu > 0$ and any $N \in \mathbb{Z}^{\geq 1}$, under some geometric conditions, the functional E_ρ admits, for almost every $\rho \in [1/2, 1]$, a *bounded* Palais-Smale sequence $(u^N_{\rho,n})_{n\geq 1}$ at level c^N_ρ which has an "approximate constrained Morse index at most N".

To be more specific, our strategy to prove Theorem 4.3 is the following. First, we show that the geometrical assumptions on E_ρ, $\rho \in [1/2, 1]$ are satisfied. Second, we check that the Palais-Smale sequences provided by the application of Theorem 4.12 converge. Finally, we observe that this process guarantees the existence of infinitely many distinct solutions $u^N_\rho \in H^1_\mu(\mathcal{G})$ since $c^N_\rho \to +\infty$ as $N \to +\infty$.

Let us now provide more information on the first two steps.

The fact that the mentioned geometric assumptions hold is established in Proposition 4.24. Proving this proposition is a central part of this chapter and for this we are indebted to ideas from [45, 51, 273]. Our proof of Proposition 4.24 uses the assumption that $\mathcal{G}$ has at least one half-line, and it is unclear whether a similar result would hold for compact graphs. As a result, the noncompactness of $\mathcal{G}$ appears to be essential in the derivation of Theorem 4.3 (see also Remark 4.6).

Regarding the convergence of the bounded Palais-Smale sequences $(u^N_{\rho,n})_{n\geq 1}$ provided by the application of Theorem 4.12, an essential element of the argument is to establish that the associated sequence of *almost Lagrange multipliers* $(\lambda^N_{\rho,n})_{n\geq 1}$ (see page 262) converges, up to a subsequence, to a positive $\lambda^N_\rho \in \mathbb{R}$. This is done in two steps. First, making use of the Morse type information carried by the sequence $(u^N_{\rho,n})_{n\geq 1}$, we show that $\lambda^N_\rho < 0$ is impossible. Here again, we use the assumption that our graph contains one half-line, see the proof of Lemma 4.31. Second, to show that $\lambda^N_\rho \neq 0$ requires a specific treatment. In [81, 102], the Palais-Smale sequences consisted of non-negative functions and thus their weak limits (which are solutions to $(\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho})$ with possibly a smaller L^2 norm than $\sqrt{\mu}$) were also non-negative. It was then rather direct to show that $\lambda^N_\rho > 0$, see [102, Remark 1.2] in the case of a compact graph, and [81, Proof of Proposition 1.5] in the case of a noncompact graph with a localized nonlinearity. In the results presented in this chapter, the weak limits are likely to be sign-changing.

[2] This will be the object of a future joint paper with P. Carrillo, C. De Coster, L. Jeanjean and C. Troestler.

Let us remark that, in general, there may exist nonzero solutions of ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) whose Lagrange multipliers vanish, as was already observed in [298, Section 4].

A simple example of this phenomenon (taken from [298, Theorem 4.2 and Remark 4.6]), is given by the tadpole graph shown in Figure 4.1.

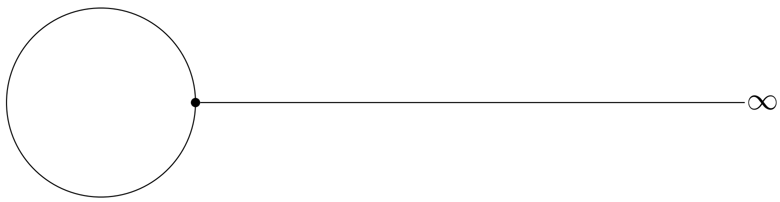


Figure 4.1: The tadpole graph

If the loop has a suitable length, one can put a sign-changing periodic solution of the equation $-u'' = |u|^{p-2}u$ on the loop and extend it by zero on the half-line to obtain a solution of the problem on the whole tadpole graph with a Lagrange multiplier equal to zero.

To treat general graphs, we make use of ODE techniques in a way which we believe new in this context. Assuming that $\lambda = 0$ in ($\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc}}$), we show that the L^2 norm of a solution $u \in H^1_\mu(\mathcal{G})$ goes to infinity as $E(u)$ goes to infinity, see Lemma 4.22 for a precise statement.

This observation leads to the conclusion that if the suspected energy level $c_\rho^N \in \mathbb{R}$ is sufficiently high, the case $\lambda_\rho^N = 0$ cannot happen. Having obtained that $\lambda_\rho^N > 0$ and using that the nonlinearity is compactly supported, we obtain the convergence of our Palais-Smale sequences and this proves Theorem 4.3.

Remark 4.4. Our multiplicity result (Theorem 4.2) is in sharp contrast to what has been observed in [297, 298] in the mass subcritical case $p < 6$.

Indeed, [298, Corollary 3.8] shows that for a *graph without cycles* (also called a *tree*), with at most one *pendant* (see [298] for the terminology), there are no solutions to ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) such that $\|u\|^2_{L^2(\mathcal{G})} = \mu$ when $p \in [4, 6)$ and $\mu > 0$ is small. Also, in [297, Theorem 1.2], to obtain $k \in \mathbb{Z}^{\geq 1}$ solutions it is necessary to assume that $\mu > \mu(k)$. We have no such limitations in Theorem 4.2.

Remark 4.5. As it was already observed in [297] in the mass subcritical case, the *localization* of the nonlinearity on the *non-trivial* compact core is essential to the multiplicity results.

Indeed, if the compact core is reduced to a point, $\mathcal{G}$ is a star graph, ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) is linear and the problem has no solution in $H^1(\mathcal{G})$ regardless of the value of $\mu > 0$.

Moreover, if $\mathcal{G}$ is an interval with two half-lines attached to its endpoints and the nonlinearity is not localized, then solutions to ($\mathrm{NLS}_{\mathcal{G}}^{\mathrm{loc}}$) with mass μ are the same as those on $\mathbb{R}$, namely the unique symmetric positive ground state and its opposite, along with their translations. Since all of those solutions have the same energy level, we do not obtain an analogue of Theorem 4.2.

Remark 4.6. Let us mention that the issue of multiplicity, even the existence of just two non-constant[3] solutions, is still open for a general compact graph $\mathcal{G}$ and a generic mass $\mu > 0$. In [102, Theorem 1.1], only one non-constant solution is obtained, assuming that μ is small enough.

In the compact case, one expects that, to obtain solutions with arbitrarily high mass, one must consider solutions with arbitrarily high Morse index (see [273, Theorem 1.2] for a result in that direction for bounded domains of $\mathbb{R}^N$).

The remainder of this chapter is organized as follows.

In section 4.2, we recall in Theorem 4.12 the contents of [81, Theorem 1.12] and explore some of its consequences. In particular, we show that second-order information on Palais-Smale sequences can be used to obtain uniform bounds from below on the associated sequences of almost Lagrange multipliers, see Lemma 4.14. We also derive results which provide abstract conditions allowing to check that the main assumptions of Theorem 4.12 hold (see Lemma 4.15 and Theorem 4.17).

Most of section 4.3 is devoted to show that solutions of the problem ($\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho}$) with $\lambda = 0$ have a L^2 norm going to infinity as their energy goes to infinity (see Proposition 4.23).

In section 4.4, we prove Proposition 4.24, which shows that our problem can indeed be treated by an application of Theorem 4.12.

In section 4.5, we give the proof of Theorem 4.3, showing the existence of infinitely many normalized solutions to the problems ($\mathrm{NLS}^{\mathrm{loc},N}_{\mathcal{G},\rho}$) for almost every $\rho \in [1/2, 1]$.

Finally, in section 4.6, we deduce our main result (Theorem 4.2) from Theorem 4.3, making use of the consequences of the previously mentioned generalized blow-up analysis.

4.2 An abstract multiplicity result

In this section, we recall in Theorem 4.12 the contents of [81, Theorem 1.12] and present some of its consequences. We also establish results which permit to check the two main hypotheses the set defined by (4.7) must satisfy. Indeed, Lemma 4.15 gives a procedure to prove that this set is non-void and Theorem 4.17 provides a tool to check the key strict inequality (4.8) appearing in Theorem 4.12.

In order to state [81, Theorem 1.12], we need to recall some definitions.

Let $(E, \langle\cdot,\cdot\rangle)$ and $(H, (\cdot,\cdot))$ be two *infinite-dimensional* Hilbert spaces. Let us assume that $E \hookrightarrow H \hookrightarrow E'$, with continuous injections. For simplicity, we assume that the injection $E \hookrightarrow H$ has norm at most 1 and we identify E with its image in H.

[3] Note that there always exists a constant solution of mass μ to ($\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G}}$) on a compact graph.

We set

$$\begin{cases} \|u\|^2 := \langle u, u\rangle, & u \in E, \\ |u|^2 := (u, u), & u \in H, \end{cases}$$

and we define for $\mu > 0$:

$$S_\mu := \Big\{ u \in E \,\Big|\, |u|^2 = \mu \Big\}.$$

In the context of this paper, we shall have $E = H^1(\mathcal{G})$ and $H = L^2(\mathcal{G})$. Clearly, S_μ is a smooth submanifold of E of codimension 1. Furthermore, its tangent space at a given point $u \in S_\mu$ is the closed subspace of codimension 1 of E given by:

$$T_u S_\mu = \Big\{ v \in E \,\Big|\, (u, v) = 0 \Big\}.$$

In the following definition, we denote respectively by $\|\cdot\|_*$ and $\|\cdot\|_{**}$ the operator norms of $\mathcal{L}(E, \mathbb{R})$ and of $\mathcal{L}(E, \mathcal{L}(E, \mathbb{R}))$.

Definition 4.7. Let $\phi : E \to \mathbb{R}$ be a $\mathcal{C}^2$-functional on E and $\alpha \in (0, 1]$. We say that ϕ' and ϕ'' are *α-Hölder continuous on bounded sets* if, for any $R > 0$, one can find $M = M(R) > 0$ such that, for any $u_1, u_2 \in B(0, R)$:

$$\|\phi'(u_1) - \phi'(u_2)\|_* \leq M\|u_1 - u_2\|^\alpha, \quad \|\phi''(u_1) - \phi''(u_2)\|_{**} \leq M\|u_1 - u_2\|^\alpha. \tag{4.5}$$

Remark 4.8. If ϕ'' is α-Hölder continuous on bounded sets, then ϕ' is Lipschitz continuous on bounded sets, whence also α-Hölder continuous on bounded sets.

Definition 4.9. Let ϕ be a $\mathcal{C}^2$-functional on E. For any $u \in E$, we define the continuous bilinear map $D^2\phi$ by:

$$D^2\phi(u) := \phi''(u) - \frac{\phi'(u)[u]}{|u|^2}(\cdot, \cdot).$$

Remark 4.10. If u is a critical point of ϕ restricted to the sphere S_μ, then $D^2\phi(u)$, seen as a bilinear form on $T_u S_\mu$, is the second derivative of $\phi|_{S_\mu}$ at u.

Definition 4.11. Let ϕ be a $\mathcal{C}^2$-functional on E. For any $u \in S_\mu$ and any $\theta \geq 0$, we define the *approximate Morse index* by

$$\tilde{\mathrm{m}}_\theta(u) := \sup\Big\{ \dim L \,\Big|\, L \text{ is a subspace of } T_u S_\mu \text{ such that } \forall \varphi \in L \setminus \{0\},\ D^2\phi(u)[\varphi, \varphi] < -\theta\|\varphi\|^2 \Big\}.$$

If u is a critical point for the constrained functional $\phi|_{S_\mu}$ and $\theta = 0$, we say that this is the *Morse index of u as a constrained critical point.*

We may now formulate [81, Theorem 1.12]. Its proof is based on a combination of ideas from [145, 186] implemented in a convenient geometric setting.

Theorem 4.12. *Let $I \subseteq (0,\infty)$ be an interval and let us consider a family of $\mathcal{C}^2$ functionals $\Phi_\rho : E \to \mathbb{R}$ of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \qquad \rho \in I,$$

where $B(u) \geq 0$ for all $u \in E$ and

$$A(u) \to +\infty \text{ or } B(u) \to +\infty \quad \text{as } u \in E \text{ and } \|u\| \to +\infty. \tag{4.6}$$

We assume that, for every $\rho \in I$, $\Phi_\rho|_{S_\mu}$ is even and moreover that Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets in the sense of Definition 4.7 for some $\alpha \in (0,1]$. Finally, we assume that there exists an integer $N \geq 2$ and two odd functions $\gamma_i : \mathbb{S}^{N-2} \to S_\mu$ where $i \in \{0,1\}$, such that the set

$$\Gamma_N := \Big\{\gamma \in \mathcal{C}([0,1]\times\mathbb{S}^{N-2}, S_\mu) \;\Big|\; \forall t \in [0,1],\ \gamma(t,\cdot) \text{ is odd}, \\ \gamma(0,\cdot) = \gamma_0 \text{ and } \gamma(1,\cdot) = \gamma_1\Big\} \tag{4.7}$$

is non void. We also assume that, for all $\rho \in I$, the inequality

$$c_\rho^N := \inf_{\gamma\in\Gamma_N} \max_{(t,a)\in[0,1]\times\mathbb{S}^{N-2}} \Phi_\rho(\gamma(t,a)) > \max_{a\in\mathbb{S}^{N-2}} \Big\{\Phi_\rho(\gamma_0(a)), \Phi_\rho(\gamma_1(a))\Big\} \tag{4.8}$$

holds. Then, for almost every $\rho \in I$, there exist sequences $(u_n)_{n\geq 1} \subseteq S_\mu$ and $(\zeta_n)_{n\geq 1} \subseteq \mathbb{R}$ such that, as $n \to +\infty$:

(i) $\Phi_\rho(u_n) \to c_\rho^N$;

(ii) $\Big\|\Phi'_\rho|_{S_\mu}(u_n)\Big\| \to 0$;

(iii) $(u_n)_n$ *is bounded in E;*

(iv) $\tilde{\mathrm{m}}_{\zeta_n}(u_n) \leq N$, *where $\zeta_n \to 0^+$.*

Remark 4.13. If the sequence $(u_n)_n \subseteq S_\mu$ provided by the Theorem above converges to some $u_\rho \in S_\mu$, then in view of points (i)–(ii), u_ρ is a critical point of $\Phi_\rho|_{S_\mu}$ at level c_ρ^N. Let us show that the Morse index of u_ρ, as a constrained critical point, satisfies $\tilde{\mathrm{m}}_0(u_\rho) \leq N$. Assume by contradiction that this is not the case. Then, in view of Definition 4.11, we may assume that there exists a subspace $W_0 \subset T_{u_\rho}S_\mu$ with $\dim W_0 = N+1$ such that $D^2\Phi_\rho(u_\rho)[w,w]$ is negative for all $w \in W_0 \setminus \{0\}$. Since $\dim W_0 < \infty$, its unit sphere is compact and there exists $\theta > 0$ such that

$$D^2\Phi_\rho(u_\rho)[w,w] < -\theta\|w\|^2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

Now, from [81, Corollary 1] or using directly that both Φ'_ρ and Φ''_ρ are α-Hölder continuous on bounded sets for some $\alpha \in (0,1]$, it follows that there exists $\delta > 0$ small enough such that, for any $v \in S_\mu$ satisfying $\|v - u_\rho\| \leq \delta$, we have

$$D^2\Phi_\rho(v)[w,w] < -\theta\|w\|^2/2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

In particular, for n large enough, $\|u_n - u_\rho\| \leq \delta$ and $\zeta_n < \theta/2$ (as $\zeta_n \to 0^+$), so the previous inequality implies

$$D^2\Phi_\rho(u_n)[w,w] < -\theta\|w\|^2/2 < -\zeta_n\|w\|^2 \quad \text{for all } w \in W_0 \setminus \{0\}.$$

Remembering that $\dim W_0 > N$ and observing that Theorem 4.12 (iv) directly implies that, if there exists a subspace $W_n \subset T_{u_n}S_\mu$ such that

$$D^2\Phi_\rho(u_n)[w,w] < -\zeta_n\|w\|^2, \quad \text{for all } w \in W_n \setminus \{0\}$$

then necessarily $\dim W_n \leq N$. Thus, we have reached a contradiction.

From Theorem 4.12 (ii)–(iii), we deduce in a standard way[4] that

$$\Phi'_\rho(u_n) + \lambda_n(u_n, \cdot) \to 0 \quad \text{in } E' \tag{4.9}$$

as $n \to +\infty$, where we have set

$$\lambda_n := -\Phi'_\rho(u_n)[u_n]/\mu. \tag{4.10}$$

We call the sequence $(\lambda_n)_n \subseteq \mathbb{R}$ defined in (4.10) the sequence of *almost Lagrange multipliers.* The following lemma allows to derive information on such sequences.

Lemma 4.14. *Let $(u_n)_{n\geq 1} \subseteq S_\mu$, $(\lambda_n)_{n\geq 1} \subseteq \mathbb{R}$ and $(\zeta_n)_{n\geq 1} \subseteq [0,+\infty)$ with $\zeta_n \to 0^+$. We assume that, for a given $M \in \mathbb{Z}^{\geq 1}$, the following conditions hold.*

(i) For large enough $n \in \mathbb{Z}^{\geq 1}$, all subspaces $W_n \subset E$ with the property

$$\Phi''_\rho(u_n)[\varphi,\varphi] + \lambda_n|\varphi|^2 < -\zeta_n\|\varphi\|^2 \qquad \text{for all } \varphi \in W_n \setminus \{0\}, \tag{4.11}$$

satisfy: $\dim(W_n) \leq M$.

(ii) There exist $\lambda \in \mathbb{R}$, a subspace Y of E with $\dim(Y) \geq M+1$ *and $\zeta > 0$ such that, for large enough $n \in \mathbb{Z}^{\geq 1}$,*

$$\Phi''_\rho(u_n)[\varphi,\varphi] + \lambda|\varphi|^2 \leq -\zeta\|\varphi\|^2 \qquad \text{for all } \varphi \in Y. \tag{4.12}$$

Then $\lambda_n > \lambda$ for all large enough $n \in \mathbb{Z}^{\geq 1}$. In particular, if (4.12) *holds for any $\lambda < 0$, then* $\liminf_{n\to\infty} \lambda_n \geq 0$.

[4] See [81, Remarks 1.3] or [66, Lemma 3].

Proof. Let us assume by contradiction that $\lambda_n \leq \lambda$ along a subsequence, still denoted $(\lambda_n)_n$. We keep denoting $(u_n)_n$ and $(\zeta_n)_n$ the corresponding subsequences. From (4.12) we have,

$$\Phi''_\rho(u_n)[\varphi,\varphi] + \lambda_n|\varphi|^2 \leq \Phi''_\rho(u_n)[\varphi,\varphi] + \lambda|\varphi|^2 \leq -\zeta\|\varphi\|^2 \qquad \text{for all } \varphi \in Y \setminus \{0\}.$$

Now, since $\zeta_n \to 0^+$, there exists $n_0 \in \mathbb{Z}^{\geq 1}$ such that: $\forall n \geq n_0$, $\zeta_n < \zeta$. Thus, for an arbitrary $n \geq n_0$, we obtain

$$\Phi''_\rho(u_n)[\varphi,\varphi] + \lambda_n|\varphi|^2 \leq -\zeta_n\|\varphi\|^2 \qquad \text{for all } \varphi \in Y \setminus \{0\},$$

in contradiction with (4.11) since $\dim(Y) \geq M+1$. □

Let us now focus on deriving sufficient conditions that will imply that the two conditions imposed on the class of paths Γ_N in Theorem 4.12 hold.

The following Lemma, directly inspired by [273, Remark 4.5], deals with the first hypothesis of showing that the set is non void.

Lemma 4.15. *Let $\{u_1,\ldots,u_{N-1}\} \subset S_\mu$ and $\{v_1,\ldots,v_{N-1}\} \subset S_\mu$, be two orthogonal families for the inner product $(\cdot,\cdot)$.*

If we define two odd functions $\gamma_0, \gamma_1 : \mathbb{S}^{N-2} \to S_\mu$ by

$$\gamma_0(a_1,\ldots,a_{N-1}) := \sum_{1\leq i<N} a_i u_i, \qquad \gamma_1(a_1,\ldots,a_{N-1}) := \sum_{1\leq i<N} a_i v_i,$$

then, the corresponding set Γ_N defined by (4.7) is non void.

Proof. We define the subspace $U := \operatorname{span}\{u_1,\ldots,u_{N-1},v_1,\ldots,v_{N-1}\}$ and we let $d := \dim(U) \leq 2(N-1)$. Let $R : U \to U$ be a linear operator such that $Ru_i = v_i$ for $i = 1,2,\ldots,N-1$. Possibly after permutation of the family $\{v_n\}$, we can choose R such that $R \in \mathrm{SO}(d)$ (there may be different choices of R). Now, since $SO(d)$ is pathwise-connected (see e.g. [300, Section 10.5]), there exists a continuous path $\tilde{\gamma} : [0,1] \to \mathrm{SO}(d)$ such that $\tilde{\gamma}(0) = \mathbb{1}$ and $\tilde{\gamma}(1) = R$. Let us define the map

$$\gamma : [0,1] \times \mathbb{S}^{N-2} \to S_\mu : (t,a_1,\ldots,a_{N-1}) \mapsto \sum_{1\leq i<N} a_i\,\tilde{\gamma}(t)(u_i).$$

By definition, γ is continuous and $\gamma(t,\cdot)$ is odd for all t. Since both equalities $\gamma(0,\cdot) = \gamma_0$ and $\gamma(1,\cdot) = \gamma_1$ hold, γ belongs to Γ_N, which is nonempty. □

We now turn to the second hypothesis, which requires to find conditions to ensure that the strict inequality (4.8) in Theorem 4.12 is satisfied. At this point, we shall rely on some results from [45]. In particular, the next Lemma is essentially [45, Lemma 3.2] and we refer to this reference for a proof.

Lemma 4.16. *Let L_1 and L be finite dimensional normed vector spaces such that* $\dim(L_1) < \dim(L)$. *Let $S := \{u \in L \mid \|u\| = 1\}$, $\alpha \in \mathbb{R}$ and*

$$H = (H_1, H_2) : [0,1] \times S \to \mathbb{R} \times L_1$$

be a continuous map such that, for all t, $u \mapsto H_1(t,u)$ is even, $u \mapsto H_2(t,u)$ is odd, and

$$H_1(0,u) < \alpha < H_1(1,u), \quad \text{for } u \in S.$$

Then, there exists $(t,u) \in [0,1] \times S$ such that $H(t,u) = (\alpha, 0)$.

The proof of the next result was inspired by [45, Lemma 3.3], see also [51, Lemma 2.3].

Theorem 4.17. *Let $\Phi : E \to \mathbb{R}$ be a continuous even functional, $d \in \mathbb{Z}^{\geq 1}$, and $\gamma_i : \mathbb{S}^d \to S_\mu$, $i \in \{1,2\}$, be two odd functions. Let us assume that the set*

$$\Gamma := \Big\{\gamma \in \mathcal{C}([0,1] \times \mathbb{S}^d, S_\mu) \;\Big|\; \forall t \in [0,1],\ \gamma(t,\cdot) \text{ is odd}, \\ \gamma(0,\cdot) = \gamma_0,\ \text{and } \gamma(1,\cdot) = \gamma_1\Big\}$$

is not empty. We assume further that there exists a continuous even functional $J : E \to \mathbb{R}$, $\beta \in \mathbb{R}$ and a subspace $W \subset E$ with $\dim W \leq d$ such that

(H1) $J(\gamma_0(s)) < \beta < J(\gamma_1(s))$ for all $s \in \mathbb{S}^d$;

(H2) the inequality

$$\max_{s \in \mathbb{S}^d} \max\Big\{\Phi(\gamma_0(s)), \Phi(\gamma_1(s))\Big\} < \inf_{u \in B} \Phi(u)$$

holds, where

$$B := \Big\{u \in S_\mu \cap W^\perp \mid J(u) = \beta\Big\}.$$

Then, the inequality

$$c := \inf_{\gamma \in \Gamma} \max_{(t,s) \in [0,1] \times \mathbb{S}^d} \Phi(\gamma(t,s)) \geq \inf_{u \in B} \Phi(u) \tag{4.13}$$

holds.

Proof. Let $\gamma \in \Gamma$ be arbitrary and $P : E \to W$ be the orthogonal projection. We define

$$h : S_\mu \to \mathbb{R} \times W : u \mapsto (J(u), Pu) \quad \text{and} \quad H := h \circ \gamma : [0,1] \times \mathbb{S}^d \to \mathbb{R} \times W.$$

Setting $L := \mathbb{R}^{d+1}$, $S := \mathbb{S}^d$, $L_1 := W$ and $\alpha := \beta$, we see that L, S, L_1, α and H satisfy all the conditions of Lemma 4.16.

Therefore, there exists $(t_0, s_0) \in [0,1] \times \mathbb{S}^d$ such that $H(t_0, s_0) = (\beta, 0)$, which means that $\gamma(t_0, s_0)$ belongs to B. Thus,

$$\max_{(t,s)\in[0,1]\times\mathbb{S}^d} \Phi(\gamma(t,s)) \geq \Phi(\gamma(t_0,s_0)) \geq \inf_{u\in B} \Phi(u),$$

which proves that (4.13) holds since $\gamma \in \Gamma$ is arbitrary. □

4.3 A Pohožaev type identity and its consequences

In this section we focus on deriving properties of solutions to $(\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho})$ when $\lambda = 0$. Observe that, we are assuming that the compact core is non trivial. Thus, $\mathcal{G}$ has at least one bounded edge so that there is at least one edge where the nonlinearity is acting. Some considerations in this section are slightly more general than what is needed to prove Theorem 4.2.

First, let us recall that if u is solution to $-u'' + \lambda u = \rho|u|^{p-2}u$ on some interval $I \subseteq \mathbb{R}$, then the function

$$H_u(x) := \frac{1}{2}\Big(u'(x)\Big)^2 + V_\lambda(u(x)) \quad \text{where} \quad V_\lambda(u) := \frac{\rho}{p}|u|^p - \frac{\lambda}{2}|u|^2$$

is constant on I. Indeed, $H_u'(x) := u'(x) \cdot (u''(x) + V_\lambda'(u(x))) = 0$. We call this constant H_u the *ODE energy*[5] of the solution u on I.

Proposition 4.18 (Pohožaev identity on metric graphs)**.** *Let $\mathcal{G}$ be a metric graph with finitely many edges (bounded or not). Let $p > 2$, $\lambda \in \mathbb{R}$, and $u \in H^1(\mathcal{G})$ be a solution to $(\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho})$. For each bounded edge e of $\mathcal{G}$, let the ODE energy of the solution u on e be given by*

$$H_u(e) := H_u(x) = \frac{1}{2}|u'(x)|^2 + \frac{\rho}{p}|u(x)|^p - \frac{\lambda}{2}|u(x)|^2, \tag{4.14}$$

where x is an arbitrary point of e. Finally, let us define

$$P_\rho(u) := \sum_{e \text{ is a bounded edge of } \mathcal{G}} |e|\, H_u(e) \tag{4.15}$$

where $|e|$ is the length of the edge e. Then, one has

$$\frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\rho}{p}\|\kappa u\|^p_{L^p(\mathcal{G})} = \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} + P_\rho(u).$$

[5]One needs to be careful not to confuse it with the energy functional E. Note that the quantities E and H are related through the Pohožaev identity.

Proof. Let e be a bounded edge of $\mathcal{G}$. We identify it with the interval $[0,|e|]$. Integrating (4.14) on e, we get

$$\frac{1}{2}\|u'\|^2_{L^2(e)} + \frac{\rho}{p}\|\kappa u\|^p_{L^p(e)} = \frac{\lambda}{2}\|u\|^2_{L^2(e)} + |e|\, H_u(e). \tag{4.16}$$

Note that (4.16) also holds when e is a half-line if in this case we set $|e|\, H_u(e) := 0$ since $\kappa|_e = 0$ and $u \in H^1(\mathcal{G})$. We end the proof by taking the sum of (4.16) over all edges of $\mathcal{G}$ (whether bounded or not). □

Lemma 4.19. *Let $\mathcal{G}$ be a metric graph with finitely many edges (bounded or not). Let $p > 2$ and $\lambda \in \mathbb{R}$. Let $u \in H^1(\mathcal{G})$ be a solution to* ($\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho}$). *Then, one has*

$$E_\rho(u) = \frac{(p-6)\lambda}{2(p+2)}\|u\|^2_{L^2(\mathcal{G})} + \frac{p-2}{p+2}P_\rho(u),$$

where $P_\rho(u)$ is defined by (4.14)–(4.15).

Proof. First, let us note that multiplying $-u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u$ by u and integrating over $\mathcal{G}$ (taking into account the Kirchhoff boundary conditions), we get

$$\|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \rho\|\kappa u\|^p_{L^p(\mathcal{G})}. \tag{4.17}$$

From Proposition 4.18 and (4.17), we obtain

$$\begin{aligned}\|u'\|^2_{L^2(\mathcal{G})} &= \frac{(p-2)\lambda}{p+2}\|u\|^2_{L^2(\mathcal{G})} + \frac{2p}{p+2}P_\rho(u),\\ \rho\|\kappa u\|^p_{L^p(\mathcal{G})} &= \frac{2p\lambda}{p+2}\|u\|^2_{L^2(\mathcal{G})} + \frac{2p}{p+2}P_\rho(u).\end{aligned}$$

Thus,

$$E_\rho(u) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{\rho}{p}\|\kappa u\|^p_{L^p(\mathcal{G})} = \frac{(p-6)\lambda}{2(p+2)}\|u\|^2_{L^2(\mathcal{G})} + \frac{p-2}{p+2}P_\rho(u). \qquad \square$$

Let us now establish some relationships between the ODE energy H_u of a solution u on an interval and its L^2-norm in the case $\lambda = 0$.

Lemma 4.20. *Let $\alpha > 0$ and $p \geq 2$. Let $u : \mathbb{R} \to \mathbb{R}$ be a τ-periodic*[a] *solution of $-u'' = \alpha|u|^{p-2}u$ for some $\tau > 0$. Let H_u be the ODE energy of u. Then,*

$$\frac{\tau}{8}\Big(\frac{pH_u}{\alpha}\Big)^{2/p} = \frac{\tau}{8}\|u\|^2_{L^\infty} \leq \int_0^\tau |u(x)|^2\,\mathrm{d}x \leq \tau\|u\|^2_{L^\infty} = \tau\Big(\frac{pH_u}{\alpha}\Big)^{2/p}. \tag{4.18}$$

[a] i.e. so that $u(x+\tau) = u(x)$ for all $x \in \mathbb{R}$; τ may not be the minimum period.

Proof. It is advantageous to consider that we are studying periodic solutions of the equation of motion in the potential well, defined by

$$V(u) := \frac{\alpha}{p}|u|^p,$$

since the equation reads $u'' = -V'(u)$. The *ODE energy* of u, given here by

$$H_u := \tfrac{1}{2}(u')^2 + V(u),$$

is constant with respect to time. We immediately obtain that

$$\tfrac{1}{2}(u'(x))^2 \begin{cases} \leq H_u - V(\|u\|_{L^\infty}/2) & \text{for all } x \in [0,\tau] \text{ such that } |u(x)| \geq \|u\|_{L^\infty}/2, \\ \geq H_u - V(\|u\|_{L^\infty}/2) & \text{for all } x \in [0,\tau] \text{ such that } |u(x)| \leq \|u\|_{L^\infty}/2. \end{cases}$$

Therefore, a particle in the potential well always has a smaller speed (in absolute value) when going through the region $[-|u|_\infty, -|u|_\infty/2] \cup [|u|_\infty/2, |u|_\infty]$ than when going through the region $[-|u|_\infty/2, |u|_\infty/2]$. Since both those regions have the same length, we deduce that

$$|A| \geq \tfrac{1}{2}\tau \qquad \text{where } A := \left\{x \in [0,\tau] \;\middle|\; |u(x)| \geq \tfrac{1}{2}\|u\|_{L^\infty}\right\}$$

as the particle spends at least half its time in the zone where it has a slower speed.

Regarding the inequalities in (4.18), the upper bound is trivial and the lower bound follows from the inequalities

$$\int_0^\tau |u(x)|^2 \, \mathrm{d}x \geq \int_A |u(x)|^2 \, \mathrm{d}x \geq \tfrac{1}{4}\|u\|_{L^\infty}^2 \, |A| \geq \tfrac{1}{8}\tau \, \|u\|_{L^\infty}^2.$$

The equalities in (4.18) follow from the fact that, for periodic solutions, one has

$$H_u = V(\|u\|_{L^\infty}) = \frac{\alpha\|u\|_{L^\infty}^p}{p},$$

since the derivative of the solution vanishes at maximum or minimum points. □

Lemma 4.21. *Let $\alpha > 0$ and $p \geq 2$. The solution of $-u'' = \alpha|u|^{p-2}u$ with initial conditions $u'(0) = 0$ and $u(0) = u_0 > 0$ is $\tau(u_0)$-periodic, where*

$$\tau(u_0) := \frac{C(p)}{\sqrt{\alpha}} u_0^{(2-p)/2}$$

for some constant $C(p) > 0$. Its ODE energy is given by $H_u = V(u_0) = \frac{\alpha}{p}u_0^p$. Moreover, it is (up to time translations) the unique solution of the ODE with this energy, and the unique solution of the ODE with this period.

Proof. It is a standard fact (see e.g. [33, p. 18]) that the period is given by

$$\begin{aligned}\tau(u_0) &= 2\int_{-u_0}^{u_0} \frac{\mathrm{d}u}{\sqrt{2(V(u_0)-V(u))}}\\ &= \sqrt{\frac{8p}{\alpha}}\int_0^{u_0}\frac{\mathrm{d}u}{\sqrt{u_0^p-u^p}}\\ &= \left(\sqrt{\frac{8p}{\alpha}}\int_0^1\frac{\mathrm{d}t}{\sqrt{1-t^p}}\right)u_0^{1-p/2}.\end{aligned}$$

The claim about the energy follows from the definitions. The fact that the set

$$\left\{(u,v)\in\mathbb{R}^2 \;\middle|\; \tfrac{1}{2}v^2+V(u)=h\right\}$$

is empty for $h<0$, is $\{(0,0)\}$ for $h=0$, and is a simple closed curve for $h>0$ implies that no other solutions have this same energy since by a phase plane analysis we obtain that there is a unique orbit of energy h for every $h>0$. We then deduce from the previous computations that this orbit corresponds to a solution of period $C(\alpha,p)u_0^{1-p/2}$, which ends the proof as the map

$$(0,+\infty)\to(0,+\infty): u_0\mapsto C(\alpha,p)u_0^{1-p/2}$$

is decreasing, so all orbits correspond to solutions with different periods. □

From here we may deduce that functions with a high ODE energy necessarily have a high L^2 norm.

Corollary 4.22. *Let $\ell>0$, $0<\underline{\alpha}<\overline{\alpha}<\infty$ and $p>2$. For every $\underline{\mu}>0$, there exists $\underline{H}>0$ such that if $u:[0,\ell]\to\mathbb{R}$ is a solution to*

$$-u''=\alpha|u|^{p-2}u \qquad \text{with } \alpha\in[\underline{\alpha},\overline{\alpha}] \text{ and } H_u\geq\underline{H},$$

then

$$\int_0^\ell |u(x)|^2\,\mathrm{d}x\geq\underline{\mu}.$$

Proof. Lemma 4.21 implies that if the ODE energy $H_u\geq\underline{H}$, then

$$u_0\geq(p\underline{H}/\overline{\alpha})^{1/p},$$

so

$$\tau(u_0)\leq\frac{C(p)}{\sqrt{\underline{\alpha}}}\left(\frac{p\,\underline{H}}{\overline{\alpha}}\right)^{(2-p)/(2p)}.$$

Thus, if $\underline{H}$ is large enough, u is periodic with a period τ less then $\ell/2$. There thus exists some interval $[0,k\tau]\subseteq[0,\ell]$, with $k\in\mathbb{Z}^{\geq1}$ and $k\tau\geq\ell/2$. Thus, u is $k\tau$-periodic and its ODE energy is at least $\underline{H}$. Lemma 4.20 implies that the L^2-norm of u on $[0,k\tau]$ can be made arbitrarily high taking $\underline{H}$ large enough. □

The last result of this section will be crucial to rule out the possibility that the Lagrange multiplier associated to a weak limit of some Palais-Smale sequence is 0.

Proposition 4.23. *Let $\mathcal{G}$ be a metric graph with finitely many edges (bounded or not) and $p > 2$. Let $(u_n)_{n\geq 1} \subseteq H^1(\mathcal{G})$ and $\{\rho_n\}_{n\geq 1} \subseteq [1/2, 1]$ be sequences such that u_n is a solution to $(\mathrm{NLS}^{\mathrm{loc}}_{\mathcal{G},\rho})$ with $\rho = \rho_n$ and $\lambda = 0$. If*

$$E_{\rho_n}(u_n) \to +\infty,$$

then

$$\|u_n\|_{L^2(\mathcal{G})} \to \infty.$$

Proof. It is sufficient to prove that, up to a subsequence, $\|u_n\|_{L^2(\mathcal{G})} \to \infty$. Indeed, replaying the argument on an arbitrary subsequence of $(u_n)_n$ will give a sub-subsequence which converges to infinity, which is equivalent to the claim.

Since $E_{\rho_n}(u_n) \to +\infty$ and $\lambda = 0$, Lemma 4.19 implies that $P_{\rho_n}(u_n) \to +\infty$. Let $e_0 \in \mathbb{E}$ be a bounded edge such that $\ell_{e_0} H_{u_n}(e_0) \geq \ell_e H_{u_n}(e)$ for all bounded edges $e \in \mathbb{E}$. Given that $\mathbb{E}$ is finite, it is possible to select e_0 independent of n, taking subsequences of $(u_n)_n$ and $(\rho_n)_n$ if necessary. Since

$$P_{\rho_n}(u_n) \leq \operatorname{card}(\mathbb{E})\, \ell_{e_0} H_{u_n}(e_0),$$

we deduce that $H_{u_n}(e_0) \to +\infty$. Now, using Corollary 4.22 with $u = u_n$, $\alpha = \rho_n$ and $[0, \ell] = e_0$, we deduce that $\|u_n\|_{L^2(e_0)} \to \infty$, thus $\|u_n\|_{L^2(\mathcal{G})} \to \infty$. □

4.4 Infinitely many minimax levels for E_ρ for almost every $\rho \in [\frac{1}{2}, 1]$

This aim of this section is to prove the following result.

Proposition 4.24. *For any $\mu > 0$ and $p > 2$, there exists $N_0 \in \mathbb{Z}^{\geq 1}$ so that if $N \geq N_0$, there exist functions $\gamma_{0,N}$ and $\gamma_{1,N}$ such that the family of functionals*

$$E_\rho : H^1(\mathcal{G}) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_{\mathcal{G}} |u'|^2 - \frac{\rho}{p}\int_{\mathcal{K}} |u|^p, \qquad \rho \in \Big[\frac{1}{2}, 1\Big]$$

satisfies the assumptions of Theorem 4.12. In particular, the set

$$\Gamma_N := \Big\{\gamma \in \mathcal{C}([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \;\Big|\; \forall t \in [0,1],\ \gamma(t,\cdot) \text{ is odd, } \gamma(0,\cdot) = \gamma_{0,N} \text{ and } \gamma(1,\cdot) = \gamma_{1,N}\Big\}$$

is non void.

Moreover, defining

$$c_\rho^N := \inf_{\gamma\in\Gamma_N} \max_{(t,s)\in[0,1]\times\mathbb{S}^{N-2}} E_\rho(\gamma(t,s)),$$

the inequality

$$c_\rho^N > \max_{s\in\mathbb{S}^{N-2}} \max\Big\{E_\rho(\gamma_0(s))), E_\rho(\gamma_1(s))\Big\}, \tag{4.19}$$

holds for all $\rho \in [\frac{1}{2}, 1]$. *Furthermore,* $c_\rho^N \xrightarrow[N\to+\infty]{} +\infty$ *uniformly with respect to* $\rho \in [1/2, 1]$. *In particular, there are infinitely many distinct values of* c_ρ^N.

Remark 4.25. Let us note that the levels c_ρ^N are real numbers for every $N \geq N_0$ and every $\rho \in [\frac{1}{2}, 1]$ since they are defined by infima over nonempty sets (thus $c_\rho^N < +\infty$) and that inequality (4.19) implies that $c_\rho^N > -\infty$.

We consider Theorem 4.12 with $\Phi_\rho = E_\rho$, $E = H^1(\mathcal{G})$, $H = L^2(\mathcal{G})$, $S_\mu = H^1_\mu(\mathcal{G})$. If we set

$$A(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,\mathrm{d}x \quad\text{and}\quad B(u) := \frac{\rho}{p}\int_{\mathcal{K}} |u|^p\,\mathrm{d}x,$$

then assumption (4.6) holds, since we have that

$$u \in H^1_\mu(\mathcal{G}),\ \|u\|_{H^1(\mathcal{G})} \to +\infty \quad\Longrightarrow\quad A(u) \to +\infty.$$

Let E'_ρ and E''_ρ denote respectively the free first and second Fréchet derivatives of E_ρ. Note that B'', whence E''_ρ, is $\min\{p-2, 1\}$-Hölder continuous on bounded sets of $H^1(\mathcal{G})$. In view of Remark 4.8, this implies that assumption (4.5) holds. As such, it only remains to show that the two hypothesis posed on Γ_N hold. This is where Lemma 4.15 and Theorem 4.17 will come into play.

The next lemmas provide orthogonal families that will be used in Lemma 4.15.

Lemma 4.26. *Let* $\mathcal{G} \in \mathbf{G_4}$ *be a metric graph,* $p > 2$ *and* $\mu > 0$. *For any* $\beta > 0$, *there exists a sequence of functions* $(\varphi_i)_{i\geq 1}$ *such that for any* $i, j \in \mathbb{Z}^{\geq 1}$ *and any* $\rho \in [\frac{1}{2}, 1]$:

(i) $\varphi_i \in S_\mu$, $\quad \|\varphi_i'\|_{L^2(\mathcal{G})} = \beta$, $\quad E_\rho(\varphi_i) = \beta^2/2$;

(ii) φ_i *has compact support and* $\operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j) = \varnothing$ *for* $i \neq j$;

(iii) for any $N \geq 2$ *and* $a \in \mathbb{S}^{N-2}$, *we have* $\Big\|\Big(\sum_{1\leq i<N} a_i\varphi_i\Big)'\Big\|_{L^2(\mathcal{G})} = \beta$ *and* $E_\rho\Big(\sum_{1\leq i<N} a_i\varphi_i, \mathcal{G}\Big) = \beta^2/2$.

Proof. Let $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ be a function supported on the interval $(0, 1)$ such that $\|\varphi\|^2_{L^2(\mathbb{R})} = \mu$. For $t \in [0, +\infty)$, we define a function φ^t by

$$\varphi^t(x) := t^{1/2}\varphi(tx). \tag{4.20}$$

If we now view φ as a function in $H^1(\mathcal{G})$ whose support is contained in a half-line which we identify with $[0, \infty)$, we can define

$$\varphi_1 := \varphi^\tau \quad \text{with} \quad \tau := \frac{\beta}{\|\varphi'\|_{L^2(\mathcal{G})}}.$$

The function φ_1 satisfies (i). Indeed, for any $t > 0$, $\|\varphi^t\|_{L^2(\mathcal{G})} = \|\varphi\|_{L^2(\mathcal{G})} = \mu$, and a direct calculation yields

$$\|\varphi_1'\|^2_{L^2(\mathcal{G})} = \tau^2 \|\varphi'\|^2_{L^2(\mathcal{G})} = \beta^2. \tag{4.21}$$

Finally, since φ_1 is supported in the half-line, we have

$$E_\rho(\varphi_1) = \tfrac{1}{2}\|\varphi_1'\|^2_{L^2(\mathcal{G})} = \frac{\beta^2}{2}.$$

Define now, for $i \geq 2$,

$$\varphi_i(x) := \varphi_1\left(x - \frac{i-1}{\tau}\right).$$

Since the φ_i are translations of φ_1 they still satisfy (i). Also, we observe that by definition $\operatorname{supp}(\varphi_i) \subset \left(\frac{i-1}{\tau}, \frac{i}{\tau}\right)$ and so they all have disjoint compact supports. This is (ii). Finally, we observe that for $a \in \mathbb{S}^{N-2}$

$$\left\| \left(\sum_{1 \leq i < N} a_i \varphi_i \right)' \right\|^2_{L^2(\mathcal{G})} = \sum_{1 \leq i < N} a_i^2 \, \|\varphi_i'\|^2_{L^2(\mathcal{G})} = \beta^2,$$

from which (iii) follows, ending the proof. □

Lemma 4.27. *Let $\mathcal{G} \in \mathbf{G_4}$ be a metric graph, $p > 6$ and $\mu > 0$. For any fixed integer $N \geq 2$ and any given values of $\tilde{\beta} > 0$, $\tilde{b} > 0$, there exist functions $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_N$, compactly supported in $\mathcal{K}$, such that for all $i, j \in \{1, \ldots, N\}$ and all $\rho \in [1/2, 1]$:*

(i) $\tilde{\varphi}_i \in S_\mu, \quad \|\tilde{\varphi}_i'\|_{L^2(\mathcal{G})} \geq \tilde{\beta}$;

(ii) $\operatorname{supp}(\tilde{\varphi}_i) \cap \operatorname{supp}(\tilde{\varphi}_j) = \varnothing$ *for* $i \neq j$;

(iii) if $a \in \mathbb{S}^{N-2}$ *then* $\left\|\left(\sum_{1\leq i<N} a_i \tilde{\varphi}_i\right)'\right\|_{L^2(\mathcal{G})} \geq \tilde{\beta}$ *and* $E_\rho\left(\sum_{1\leq i<N} a_i\tilde{\varphi}_i, \mathcal{G}\right) \leq \tilde{b}$.

Proof. Let $e = [0, |e|]$ be any bounded edge of $\mathcal{G}$. Let $\varphi \in \mathcal{C}_c^\infty\big((0, |e|/N)\big)$ be any function such that $\|\varphi\|_{L^2(\mathbb{R})} = \mu$. Using the notation (4.20), we notice that $\operatorname{supp}(\varphi^t) \subset (0, |e|/N)$ whenever $t \geq 1$. Let us define the functions

$$\tilde{\varphi}_i := \varphi^t\left(x - \frac{(i-1)|e|}{N}\right), \qquad i = 1, \ldots, N,$$

where $t \geq 1$ will be chosen later.

Let us note that

$$\operatorname{supp}(\tilde{\varphi}_i) \subseteq \left(\frac{(i-1)|e|}{N}, \frac{i|e|}{N}\right),$$

so the functions $\tilde{\varphi}_i$ have disjoint supports and point (ii) is satisfied. Viewing now $\tilde{\varphi}_i$ as functions in $H^1(\mathcal{G})$ supported in e, we may compute the energy of the function $\sum_{1\le i<N} a_i\tilde{\varphi}_i$ with $a \in \mathbb{S}^{N-2}$ as follows:

$$\begin{aligned} E_\rho\Bigg(\sum_{1\le i<N} a_i\tilde{\varphi}_i\Bigg) &= \frac{1}{2}\int_e \Bigg|\sum_{i=1}^{N} a_i\tilde{\varphi}_i'\Bigg|^2 \mathrm{d}x - \frac{\rho}{p}\int_e \Bigg|\sum_{i=1}^{N-1} a_i\tilde{\varphi}_i\Bigg|^p \mathrm{d}x \\ &= \frac{t^2}{2}\sum_{1\le i<N} a_i^2 \int_0^{|e|/N} |\varphi'|^2\,\mathrm{d}x - \frac{\rho\, t^{(p-2)/2}}{p}\sum_{1\le i<N} |a_i|^p \int_0^{|e|/N} |\varphi|^p \\ &\le \frac{t^2\|\varphi'\|^2_{L^2(\mathcal{G})}}{2} - \frac{Ct^{(p-2)/2}\,\|\varphi\|^p_{L^p(\mathcal{K})}}{2p} \xrightarrow[t\to+\infty]{} -\infty \end{aligned}$$

where $C := \min_{a\in\mathbb{S}^{N-2}} \sum_{1\le i<N} |a_i|^p$. Thus, for all $\tilde{b} \in \mathbb{R}$, there exists $T_0 > 0$ such that for all $t > T_0$, we have $E_\rho\Big(\sum_{1\le i\le N} a_i\tilde{\varphi}_i\Big) < \tilde{b}$. As a result, if we choose

$$t := \max\left\{1, \frac{\tilde{\beta}}{\|\varphi'\|_{L^2(\mathcal{G})}}, T_0\right\},$$

the functions $\tilde{\varphi}_i$ satisfy all of the desired properties. Indeed, $\tilde{\varphi}_i \in H^1_\mu(\mathcal{G})$ and from (4.21) we have

$$\|\tilde{\varphi}_i'\|_{L^2(\mathcal{G})} = t\|\varphi'\|_{L^2(\mathcal{G})} \ge \tilde{\beta}, \tag{4.22}$$

which implies point (i). Finally, the choice of t, (4.22) and the equality

$$\Bigg\|\sum_{1\le i<N} a_i\tilde{\varphi}_i'\Bigg\|^2_{L^2(\mathcal{G})} = \sum_{1\le i<N} a_i^2\|\tilde{\varphi}_i'\|^2_{L^2(\mathcal{G})}$$

show point (iii), ending the proof. □

Now, let $(V_N)_{N\ge1}$ be a sequence of linear subspaces of $H^1(\mathcal{G})$ with $\dim(V_N) = N$ which exhausts $H^1(\mathcal{G})$ in the sense that

$$\bigcup_{N\ge1} V_N$$

is dense in $H^1(\mathcal{G})$. We remark that for separable Hilbert spaces, such as $H^1(\mathcal{G})$, such a sequence always exists.

The next lemma is an adaptation of [51, Lemma 2.1].

Lemma 4.28. *For any $p > 2$, there holds:*

$$S_N := \inf_{u \in V_{N-2}^\perp} \frac{\int_\mathcal{G} |u'|^2 + |u|^2}{\left(\int_\mathcal{K} |u|^p\right)^{2/p}} \xrightarrow[N\to\infty]{} +\infty.$$

Proof. We assume by contradiction that there exists a sequence $(u_N)_{N\geq 1} \subseteq V_{N-2}^\perp$ such that $\|u_N\|_{L^p(\mathcal{K})} = 1$ and that $\|u_N\|_{H^1(\mathcal{G})}$ is bounded. In particular, up to a subsequence, there exists $u \in H^1(\mathcal{G})$ such that $u_N \rightharpoonup u$ in $H^1(\mathcal{G})$ (and thus in $H^1(\mathcal{K})$) and therefore $u_N \to u$ in $L^p(\mathcal{K})$. Let $v \in H^1(\mathcal{G})$. Because $\{V_N\}_N$ exhausts $H^1(\mathcal{G})$, there exists a sequence $(v_N)_N \subseteq H^1_\mu(\mathcal{G})$ such that, for all $N \in \mathbb{Z}^{\geq 1}$, $v_N \in V_{N-2}$ and $v_N \to v$ in $H^1(\mathcal{G})$. Taking the scalar product in $H^1(\mathcal{G})$ we have

$$\begin{aligned}
|\langle u_N, v\rangle| &\leq |\langle u_N, v - v_N\rangle| + |\langle u_N, v_N\rangle| \\
&= |\langle u_N, v - v_N\rangle| \\
&\leq \|u_N\|_{H^1(\mathcal{G})}\|v - v_N\|_{H^1(\mathcal{G})} \\
&\xrightarrow[N\to\infty]{} 0.
\end{aligned}$$

It follows that $u_N \rightharpoonup 0 = u$, in contradiction with $\|u_N\|_{L^p(\mathcal{K})} = 1$. □

We now define

$$\beta_N := \left(\frac{S_N^{p/2}}{L}\right)^{1/(p-2)} \quad \text{where} \quad L = L(p) := \frac{3}{p}\max_{x>0}\frac{(\mu + x^2)^{p/2}}{\mu + x^p}. \tag{4.23}$$

Lemma 4.28 implies that $\beta_N \to \infty$. Thus, if we define

$$b_\rho^N := \inf_{u\in B_N} E_\rho(u) \quad \text{where} \quad B_N := \left\{u \in V_{N-2}^\perp \cap H^1_\mu(\mathcal{G}) \;\middle|\; \|u'\|_{L^2(\mathcal{G})} = \beta_N\right\}, \tag{4.24}$$

we obtain that

Lemma 4.29. *$b_\rho^N \to +\infty$ as $N \to +\infty$, uniformly in $\rho \in [1/2, 1]$.*

Proof. For every $u \in B_N$, we have

$$\begin{aligned}
E_\rho(u) &= \frac{1}{2}\int_\mathcal{G} |u'|^2 - \frac{\rho}{p}\left(\int_\mathcal{K} |u|^p\right)^{\frac{2}{p}\cdot\frac{p}{2}} \\
&\geq \frac{1}{2}\int_\mathcal{G} |u'|^2 - \frac{1}{p}\left(\frac{\mu + \int_\mathcal{G} |u'|^2}{S_N}\right)^{p/2} \\
&\geq \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{L}{3S_N^{p/2}}\left(\mu + \|u'\|^p_{L^2(\mathcal{G})}\right) \\
&= \frac{1}{2}\beta_N^2 - \frac{1}{3}\beta_N^{2-p}(\mu + \beta_N^p) \\
&= \frac{1}{6}\beta_N^2 + o(1).
\end{aligned}$$

The proof is completed by taking the infimum over B_N. □

We are finally in position to prove Proposition 4.24.

Proof of Proposition 4.24. We have already proved that (4.5) and (4.6) hold. Let $(V_N)_{N\geq 1}$ with $\dim(V_N) = N$ be an exhausting sequence of $H^1(\mathcal{G})$ and, for each $N \geq 2$, let us define the values β_N and b_ρ^N respectively by (4.23) and (4.24). By Lemma 4.28 and Lemma 4.29, both sequences $(\beta_N)_N$ and $(b_\rho^N)_N$ diverge.

Let us consider now a sequence of functions $(\varphi_i)_{i\geq 1}$ as given by Lemma 4.26 taking $\beta = 1$ and a set of N functions $(\tilde{\varphi}_i)_{i\geq 1}$ given by Lemma 4.27, taking $\tilde{\beta} = 2\beta_N$ and $\tilde{b} = 1$. Moreover, let us define the functions

$$\gamma_{0,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G}) : (a_1, \dots, a_{N-1}) \mapsto \sum_{1\leq i<N} a_i \varphi_i$$

and

$$\gamma_{1,N} : \mathbb{S}^{N-2} \to H^1_\mu(\mathcal{G}) : (a_1, \dots, a_{N-1}) \mapsto \sum_{1\leq i<N} a_i \tilde{\varphi}_i$$

which satisfy, for every $N \geq 2$ and $a \in \mathbb{S}^{N-2}$,

$$\begin{cases} \|\gamma_{0,N}(a)'\|_{L^2(\mathcal{G})} = 1, \\ E_\rho\Big(\gamma_{0,N}(a)\Big) = \frac{1}{2}, \end{cases}$$

and

$$\begin{cases} \|\gamma_{1,N}(a)'\|_{L^2(\mathcal{G})} \geq 2\beta_N, \\ E_\rho\Big(\gamma_{1,N}(a)\Big) \leq 1. \end{cases}$$

From Lemma 4.15, we know that the set

$$\Gamma_N = \Big\{\gamma \in \mathcal{C}([0,1] \times \mathbb{S}^{N-2}, H^1_\mu(\mathcal{G})) \;\Big|\; \forall t \in [0,1],\ \gamma(t,\cdot) \text{ is odd}, \\ \gamma(0,\cdot) = \gamma_{0,N},\ \text{and } \gamma(1,\cdot) = \gamma_{1,N}\Big\}$$

is not empty.

Now, we use Theorem 4.17 with $\Phi = E_\rho$, $d = N - 2$, $J(u) = \|u'\|_{L^2(\mathcal{G})}$, $\beta = \beta_N$ and $W = V_{N-2}$. We easily check that its assumptions ((H1)) and ((H2)) are satisfied for any N sufficiently large (uniformly in ρ), thus (4.19) also holds. Finally, using $b_\rho^N \to +\infty$ as $N \to \infty$ and (4.13), we get that $c_\rho^N \to +\infty$ as $N \to \infty$. □

4.5 Proof of Theorem 4.3

This section is devoted to the proof of Theorem 4.3.

As a consequence of Proposition 4.24, we may apply Theorem 4.12 to the family of functionals given by (4.4).

From Theorem 4.12 and the considerations just after it (see in particular (4.9)–(4.10)), for all $N \in \mathbb{Z}^{\geq 1}$ large enough and for almost every $\rho \in [1/2, 1]$, we deduce the existence of a bounded sequence $(u^N_{\rho,n})_{n\geq 1} \subseteq H^1_\mu(\mathcal{G})$, that we shall simply denote $(u_n)_n$, such that

$$E_\rho(u_n) \to c^N_\rho \tag{4.25}$$

and

$$E'_\rho(u_n) + \lambda_n(u_n, \cdot) \to 0 \quad \text{in the dual of } H^1_\mu(\mathcal{G}), \tag{4.26}$$

where

$$\lambda_n := -\frac{1}{\mu} E'_\rho(u_n)[u_n]. \tag{4.27}$$

Finally, there exists a sequence $(\zeta_n)_n \subseteq [0, +\infty)$ with $\zeta_n \to 0^+$ such that, if the inequality

$$\begin{aligned} \int_{\mathcal{G}} |\varphi'|^2 + \Big(\lambda_n - (p-1)\rho\kappa(x)|u_n|^{p-2}\Big)\varphi^2 \, \mathrm{d}x &= E''_\rho(u_n)[\varphi,\varphi] + \lambda_n \|\varphi\|^2_{L^2(\mathcal{G})} \\ &< -\zeta_n \|\varphi\|^2_{H^1(\mathcal{G})} \end{aligned} \tag{4.28}$$

holds for any $\varphi \in W_n \setminus \{0\}$ in a subspace W_n of $T_{u_n} H^1_\mu(\mathcal{G})$, then the dimension of W_n is at most N.

Since the sequence $(u_n)_n \subseteq H^1(\mathcal{G})$ is bounded, passing to a subsequence we may assume that there exists $u^N_\rho \in H^1(\mathcal{G})$ such that

$$u_n \rightharpoonup u^N_\rho \quad \text{in } H^1(\mathcal{G}), \tag{4.29}$$

$$u_n \to u^N_\rho \quad \text{in } L^r_{\mathrm{loc}}(\mathcal{G}) \text{ for all } r \geq 2. \tag{4.30}$$

Observe also that, since $(u_n)_n \subseteq H^1(\mathcal{G})$ is a bounded sequence, it follows from (4.27) that $(\lambda_n)_n \subseteq \mathbb{R}$ is bounded. As before, passing to a subsequence, there exists $\lambda^N_\rho \in \mathbb{R}$ such that $\lim_{n\to+\infty} \lambda_n = \lambda^N_\rho$.

The sequences $(\lambda^N_\rho)_{N\geq 1} \subset \mathbb{R}$ and $(u^N_\rho)_{N\geq 1} \subset H^1_\mu(\mathcal{G})$ are the candidates to prove Theorem 4.3. We begin by verifying that the limit $u^N_\rho \in H^1(\mathcal{G})$ solves $(\mathrm{NLS}^{\mathrm{loc},N}_{\mathcal{G},\rho})$. Indeed, using (4.26) and the fact that $\lim_{n\to+\infty} \lambda_n = \lambda^N_\rho$, we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big(E'_\rho(u_n) + \lambda_n(u_n,\cdot)\Big)[\eta] \\ &= \lim_{n\to\infty} \left[\int_{\mathcal{G}} u'_n \eta' + \lambda_n \int_{\mathcal{G}} u_n \eta - \rho \int_{\mathcal{K}} |u_n|^{p-2} u_n \eta\right] \\ &= \int_{\mathcal{G}} (u^N_\rho)'\eta' + \lambda^N_\rho \int_{\mathcal{G}} u^N_\rho \eta - \rho \int_{\mathcal{K}} |u^N_\rho|^{p-2} u^N_\rho \eta \end{aligned} \tag{4.31}$$

for every $\eta \in H^1(\mathcal{G})$. We have thus proved the claim.

We now focus on proving the strong H^1-convergence of the sequences $(u_n)_n$ to ensure that the limits u^N_ρ belong to the mass constraint $H^1_\mu(\mathcal{G})$.

Proposition 4.30. *The following convergence holds:*

$$\int_{\mathcal{G}} |(u_n - u_\rho^N)'|^2 + \lambda_\rho^N \int_{\mathcal{G}} |u_n - u_\rho^N|^2 \xrightarrow[n\to\infty]{} 0.$$

In particular, if $\lambda_\rho^N > 0$, the sequence $(u_n)_n$ converges strongly in $H^1(\mathcal{G})$.

Proof. First, rewriting (4.26) as

$$\begin{aligned} o(1)\|\eta\|_{H^1(\mathcal{G})} &= \int_{\mathcal{G}} u_n'\eta' - \rho \int_{\mathcal{K}} |u_n|^{p-2}u_n\eta + \lambda_n \int_{\mathcal{G}} u_n\eta \\ &= \int_{\mathcal{G}} u_n'\eta' - \rho \int_{\mathcal{K}} |u_n|^{p-2}u_n\eta + \lambda_\rho^N \int_{\mathcal{G}} u_n\eta + (\lambda_n - \lambda_\rho^N) \int_{\mathcal{G}} u_n\eta, \end{aligned}$$

we obtain

$$\int_{\mathcal{G}} u_n'\eta' - \rho \int_{\mathcal{K}} |u_n|^{p-2}u_n\eta + \lambda_\rho^N \int_{\mathcal{G}} u_n\eta = o(1)\|\eta\|_{H^1(\mathcal{G})}. \tag{4.32}$$

Now, taking the difference between (4.32) and (4.31), choosing $\eta = \eta_n := u_n - u_\rho^N$ and taking into account (4.30) and that $(\eta_n)_n$ is bounded, we deduce that

$$\begin{aligned} o(1) &= o(1)\|\eta_n\|_{H^1(\mathcal{G})} \\ &= \int_{\mathcal{G}} \Big(u_n' - (u_\rho^N)'\Big)\eta_n' - \rho \int_{\mathcal{K}} \Big(|u_n|^{p-2}u_n - |u_\rho^N|^{p-2}u_\rho^N\Big)\eta_n + \lambda_\rho^N \int_{\mathcal{G}} (u_n - u_\rho^N)\eta_n \\ &= \int_{\mathcal{G}} \Big(u_n' - (u_\rho^N)'\Big)\eta_n' + \lambda_\rho^N \int_{\mathcal{G}} (u_n - u_\rho^N)\eta_n + o(1)\|\eta_n\|_{H^1(\mathcal{G})} \\ &= \int_{\mathcal{G}} \Big|(u_n - u_\rho^N)'\Big|^2 + \lambda_\rho^N \int_{\mathcal{G}} |u_n - u_\rho^N|^2 + o(1), \end{aligned}$$

which proves the claim. □

In order to apply Proposition 4.30 we need to show that the assumption $\lambda_\rho^N > 0$ holds. We will do this in two steps. In first place, we will show that $\lambda_\rho^N < 0$ is not possible by making use of Lemma 4.14. The following result will aid to check that its assumptions hold.

Lemma 4.31. *For any $\lambda < 0$ and $d \in \mathbb{Z}^{\geq 1}$, there exists a subspace Y of $H^1(\mathcal{G})$ with $\dim(Y) = d$ such that, for all $w \in Y$,*

$$E_\rho''(u_n)[w, w] + \lambda\|w\|_{L^2(\mathcal{G})}^2 = \int_{\mathcal{G}} |w'|^2 \,\mathrm{d}x + \lambda \int_{\mathcal{G}} |w|^2 \,\mathrm{d}x \leq \frac{\lambda}{2}\|w\|_{H^1(\mathcal{G})}^2.$$

Proof. We proceed similarly to the proof of Lemma 4.26. Let us take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (0, 1)$ and such that $\int_0^{+\infty} |\varphi|^2 \,\mathrm{d}x = 1$. Viewing φ as a function in $H^1(\mathcal{G})$ whose support is contained in a half-line which we identify with $[0, \infty)$, we define (using the notation of (4.20)) $\varphi_1 := \varphi^\tau$, where $\tau > 0$ is taken small enough so that

$$\tau^2\|\varphi'\|_{L^2(\mathbb{R})^2} + \lambda \leq \frac{\lambda}{2}\Big(\tau^2\|\varphi'\|_{L^2(\mathbb{R})^2} + 1\Big). \tag{4.33}$$

A computation shows that

$$\|\varphi_1\|_{L^2(\mathcal{G})} = 1, \qquad \|\varphi_1'\|_{L^2(\mathcal{G})} = \tau^2 \|\varphi'\|_{L^2(\mathcal{G})}.$$

We define now, for $i \geq 2$,

$$\varphi_i(x) := \varphi_1\left(x - \frac{i-1}{\tau}\right).$$

Since $\operatorname{supp}(\varphi_i) \subseteq \left(\frac{i-1}{\tau}, \frac{i}{\tau}\right)$, all the φ_i have disjoint supports. Let $Y \subset H^1(\mathcal{G})$ be the subspace generated by $\varphi_1, \ldots, \varphi_d$. Any element $w \in Y$ can be written as

$$w := \sum_{1 \leq i \leq d} \theta_i \varphi_i$$

where $\theta_1, \ldots, \theta_d \in \mathbb{R}$. By direct calculations, we have

$$\begin{aligned} \int_{\mathcal{G}} |w'|^2 \, \mathrm{d}x + \lambda \int_{\mathcal{G}} |w|^2 \, \mathrm{d}x &= \tau^2 \left(\sum_{1 \leq i \leq d} \theta_i^2 \|\varphi'\|^2_{L^2(\mathbb{R})} \right) + \lambda \left(\sum_{1 \leq i \leq d} \theta_i^2 \right) \\ &= (\tau^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + \lambda) \sum_{1 \leq i \leq d} \theta_i^2. \end{aligned}$$

Similarly, $\|w\|^2_{H^1(\mathcal{G})} = (\tau^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + 1) \sum_{1 \leq i \leq d} \theta_i^2$. Therefore, (4.33) implies that

$$\begin{aligned} \int_{\mathcal{G}} |w'|^2 \, \mathrm{d}x + \lambda \int_{\mathcal{G}} |w|^2 \, \mathrm{d}x &= (\tau^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + \lambda) \sum_{1 \leq i \leq d} \theta_i^2 \\ &\leq \frac{\lambda}{2} (\tau^2 \|\varphi'\|^2_{L^2(\mathbb{R})} + 1) \sum_{1 \leq i \leq d} \theta_i^2 \\ &= \frac{\lambda}{2} \|w\|^2_{H^1(\mathcal{G})}. \end{aligned}$$

The fact that w vanishes outside the half-line justifies the equality in the claim, ending the proof. □

Let us observe that the codimension of $T_{u_n} H^1_\mu(\mathcal{G})$ in $H^1(\mathcal{G})$ is one. Thus, if inequality (4.28) holds for every $\varphi \in W_n \setminus \{0\}$ for a subspace W_n of $H^1(\mathcal{G})$, then the dimension of W_n is at most $N+1$. Let $\lambda < 0$. Let Y be the space of dimension $d = N + 2$ provided by Lemma 4.31. We may thus apply Lemma 4.14 to obtain that

$$\lambda_\rho^N \geq 0. \tag{4.34}$$

Combining Proposition 4.30 and (4.34), we get that

$$\int_{\mathcal{G}} |(u_n - u_\rho^N)'|^2 \xrightarrow[n \to \infty]{} 0.$$

Using in addition (4.30) and recalling that the nonlinearity acts only on $\mathcal{K}$, the compact core of the graph, we obtain that $E_\rho(u_n) \to E_\rho(u_\rho^N)$. In particular, in view of (4.25), it follows that

$$E_\rho(u_\rho^N) = c_\rho^N. \tag{4.35}$$

Let us now prove that $\lambda_\rho^N = 0$ is not possible either, assuming that $N \in \mathbb{Z}^{\geq 1}$ is large enough uniformly in $\rho \in [1/2, 1]$. It is here that we will use what has been developed in section 4.3. For the sake of contradiction, assume that there exists a subsequence $(u_{\rho_k}^{N_k})_{k\geq 1}$, with $N_k \to +\infty$ and $\rho_k \in [1/2, 1]$ for all k, such that the weak limits $u_{\rho_k}^{N_k} \in H^1(\mathcal{G})$ have an associated $\lambda_{\rho_k}^{N_k}$ which is 0. By Proposition 4.24, c_ρ^N converges to $+\infty$ as $N \to \infty$, uniformly with respect to ρ. Thus, we deduce from (4.35) that $E_{\rho_k}(u_{\rho_k}^{N_k}) \to +\infty$ as $k \to \infty$. This is in contradiction with Proposition 4.23, since $(u_{\rho_k}^{N_k})_{k\geq 1} \subseteq H_\mu^1(\mathcal{G})$. In conclusion, we have $\lambda_\rho^N > 0$.

Finally, let us show that the Morse index $\mathrm{m}(u_\rho^N)$ of u_ρ^N as a solution to $(\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc},N})$ satisfies $\mathrm{m}(u_\rho^N) \leq N+1$. We recall that the Morse index of a solution $u \in H^1(\mathcal{G})$ of $(\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc}})$ is defined as the maximal dimension of a subspace $W \subset H^1(\mathcal{G})$ such that $Q_u(\varphi) < 0$ for all $\varphi \in W \setminus \{0\}$, where

$$Q_u(\varphi) := \int_{\mathcal{G}} |\varphi'|^2 + \Big(\lambda - \kappa(x)(p-1)\rho|u|^{p-2}\Big)\varphi^2 \,\mathrm{d}x.$$

We also note the relationship between the Morse index of a solution to $(\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc}})$ and the Morse index as a constrained critical point (refer to Definition 4.11) via the equality

$$\begin{aligned} D^2E_\rho(u_\rho^N)[w,w] &:= E_\rho''(u_\rho^N)[w,w] + \lambda_\rho^N(w,w) \\ &= \int_{\mathcal{G}} \Big[|w'|^2 + \Big(\lambda_\rho^N - (p-1)\kappa(x)|u_\rho^N|^{p-2}\Big)w^2\Big]\,\mathrm{d}x, \end{aligned} \tag{4.36}$$

valid for all $w \in H^1(\mathcal{G})$. Since $u_{\rho,n}^N \to u_\rho^N$ as $n \to \infty$, we know from Remark 4.13 that the Morse index of $u_\rho^N \in H_\mu^1(\mathcal{G})$ as a constrained critical point is less than N. In view of (4.36) and recalling that $H_\mu^1(\mathcal{G})$ is of codimension 1 in $H^1(\mathcal{G})$, we deduce

$$\mathrm{m}(u_\rho^N) \leq N+1. \tag{4.37}$$

Summarizing what has been observed so far, we can finally prove 4.3.

Proof of Theorem 4.3. For any $\mu > 0$ and any $N \in \mathbb{Z}^{\geq 1}$ sufficiently large, we have shown that the particular bounded Palais-Smale sequence, satisfying (4.25)–(4.28), provided for almost every $\rho \in [1/2, 1]$ by Theorem 4.12 is converging. Thus, there exists a sequence of couples $\{(\lambda_\rho^N, u_\rho^N)\} \subseteq (0, +\infty) \times H_\mu^1(\mathcal{G})$ which are solutions to $(\mathrm{NLS}_{\mathcal{G},\rho}^{\mathrm{loc},N})$. By (4.35), we also know that $E(u_\rho^N) = c_\rho^N$ converges to $+\infty$ as $N \to \infty$. The estimate (4.37) completes the proof. □

4.6 Proof of Theorem 4.2

Let $\mu > 0$ and $N \in \mathbb{Z}^{\geq 1}$ be sufficiently large. By Theorem 4.3, it is possible to choose a sequence $\rho_n \to 1^-$, and a corresponding sequence of critical points $u^N_{\rho_n} \in H^1_\mu(\mathcal{G})$ of E_{ρ_n} constrained to $H^1_\mu(\mathcal{G})$, at the level $c^N_{\rho_n}$ and having a Morse index $\mathrm{m}(u^N_{\rho_n}) \leq N+1$. Additionally, the Lagrange multipliers satisfy $\lambda^N_{\rho_n} > 0$.

To prove Theorem 4.2, it suffices to show that the sequence $(u^N_{\rho_n})_n \subseteq H^1_\mu(\mathcal{G})$ converges. For this, the key point is to show that $(u^N_{\rho_n})_n \subseteq H^1(\mathcal{G})$ is bounded. The monotonicity of c^N_ρ as a function of $\rho \in [1/2, 1]$ implies that $\{c^N_{\rho_n})_n$ is bounded since it belongs to $[c^N_1, c^N_{1/2}]$ with $c^N_1, c^N_{1/2} \in \mathbb{R}$ (see Remark 4.25). In addition, the Kirchhoff boundary condition

$$\int_{\mathcal{G}} \left|(u^N_{\rho_n})'\right|^2 + \lambda^N_{\rho_n} \left(u^N_{\rho_n}\right)^2 \mathrm{d}x = \rho_n \int_{\mathcal{K}} \left|u^N_{\rho_n}\right|^p \mathrm{d}x$$

implies that

$$c^N_{\rho_n} = E_{\rho_n}(u^N_{\rho_n}) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |(u^N_{\rho_n})'|^2 \,\mathrm{d}x - \frac{\lambda^N_{\rho_n}\mu}{p}$$

so that

$$\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathcal{G}} |(u^N_{\rho_n})'|^2 \,\mathrm{d}x = c^N_{\rho_n} + \frac{\lambda^N_{\rho_n}\mu}{p}.$$

Thus, if $(\lambda^N_{\rho_n})_n \subseteq (0, +\infty)$ is bounded, then $(u^N_{\rho_n})_n \subset H^1(\mathcal{G})$ is bounded as well. To conclude the proof of Theorem 4.2, it suffices to make use of the following result.

Lemma 4.32. *Let $\mathcal{G} \in \mathbf{G_4}$ be a metric graph, and $p > 6$. Let us assume that $(\rho_n)_n \subseteq [\frac{1}{2}, 1]$ is a sequence converging to 1. Let $(\lambda_n, u_n)_{n \geq 1} \subseteq \mathbb{R} \times H^1(\mathcal{G})$ be a sequence of solutions to*

$$\begin{cases} -u'' + \lambda u = \rho\kappa(x)|u|^{p-2}u & \text{on every edge } e \in \mathbb{E}, \\ \sum_{e \succ v} u'_e(v) = 0 & \text{at every vertex } v \in \mathbb{V}, \end{cases}$$

and satisfying additionally, for some $\mu > 0$,

$$\int_{\mathcal{G}} |u_n|^2 \,\mathrm{d}x = \mu, \quad \text{for all } n \geq 1$$

and whose Morse indices $\mathrm{m}(u_n)$ are bounded. Then, the sequence $(\lambda_n)_n \subseteq \mathbb{R}$ is bounded from above.

The previous lemma is a consequence of the generalized blow-up analysis that we mentioned in section 4.1 (see footnote 2 on page 257).

Chapter 5

The near-linear regime $p \approx 2$ and solutions vanishing identically on edges of compact metric graphs

5.1 Presentation of the chapter

Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a *compact* metric graph, namely a graph having a finite number of edges, all bounded. As in Chapter 2, we are interested in solutions of the nonlinear Schrödinger equation, with mixed Kirchhoff and Dirichlet conditions.

Given two real numbers $\lambda \geq 0$ and $p > 2$ as well a (possibly empty) set Z of vertices having degree one, we consider the nonlinear problem

$$\begin{cases} -\tilde{u}'' + \lambda\tilde{u} = |\tilde{u}|^{p-2}\tilde{u} & \text{on every edge of } \mathcal{G}, \\ \tilde{u} \text{ is continuous} & \text{on } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}\tilde{u}}{\mathrm{d}x_e}(v) = 0 & \text{for every } v \in \mathbb{V} \setminus Z, \\ \tilde{u}(v) = 0 & \text{for every } v \in Z, \end{cases} \tag{NLS$_{\mathcal{G},Z}$}$$

where $\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x_e}(v)$ is the outgoing derivative along the edge e incident at the vertex v, and $e \succ v$ means that the sum is extended to all such edges. If $Z = \varnothing$, we assume that $\lambda > 0$. Let us recall that solutions of (NLS$_{\mathcal{G},Z}$) correspond to critical points of the action functional $\mathcal{J}_\lambda$ on the Sobolev space $H^1_Z(\mathcal{G})$ (see its definition in equation (2.3) and the following discussion on page 182).

We will consider the "near-linear regime", namely the asymptotic study of the problem when $p \to 2$. In this case, we expect that the sequence $(\gamma_k)_{k\geq 1}$ of eigenvalues[1] of the spectral problem

$$\begin{cases} -u'' + \lambda u = \gamma u & \text{on every edge of } \mathcal{G}, \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{for every } v \in \mathbb{V} \setminus Z, \\ u(v) = 0 & \text{for every } v \in Z \end{cases} \tag{$\mathcal{P}_2$}$$

will play a significant role.

[1] As for bounded domains of $\mathbb{R}^N$, the spectrum of a quantum graph consists only of isolated eigenvalues having finite multiplicity, see e.g. [68, Theorem 3.1.1].

For every integer $k \geq 1$ and every real number $p > 2$ close to 2, we want to relate solutions of the nonlinear problem ($\mathrm{NLS}_{\mathcal{G},Z}$) to the eigenfunctions of the spectral problem ($\mathcal{P}_2$) with eigenvalue γ_k. In order to have a better view of the behavior of the solutions as $p \to 2$, let us consider the rescaling $u := \gamma_k^{-1/(p-2)}\tilde{u}$, corresponding to solutions of the nonlinear problem

$$\begin{cases} -u'' + \lambda u = \gamma_k |u|^{p-2}u & \text{on every edge of } \mathcal{G}, \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{for every } v \in \mathbb{V} \setminus Z, \\ u(v) = 0 & \text{for every } v \in Z. \end{cases} \tag{$\mathcal{P}_{p,k}$}$$

Remark 5.1. The solutions of ($\mathrm{NLS}_{\mathcal{G},Z}$) and of ($\mathcal{P}_{p,k}$) are in bijection by a simple rescaling, so that any (existence, multiplicity, symmetry, etc.) results obtained on ($\mathcal{P}_{p,k}$) translates to a result on ($\mathrm{NLS}_{\mathcal{G},Z}$).

We now need to introduce some terminology and notation. Let $(\varphi_i)_{1 \leq i \leq n_k}$ be an L^2-orthonormal basis of the eigenspace E_k associated to γ_k, so that

$$E_k = \operatorname{span}\{\varphi_1, \dots, \varphi_{n_k}\}.$$

We define the two orthogonal projections

$$P_{E_k} : L^2(\mathcal{G}) \to E_k : u \mapsto \sum_{1 \leq i \leq n_k} (u \mid \varphi_i)_{L^2(\mathcal{G})} \varphi_i$$

and

$$P_{E_k^\perp} : L^2(\mathcal{G}) \to E_k^\perp : u \mapsto u - P_{E_k}(u),$$

where

$$E_k^\perp := \Big\{ w \in L^2(\mathcal{G}) \;\Big|\; \forall \varphi \in E_k, \int_{\mathcal{G}} w\varphi = 0 \Big\}.$$

Definition 5.2. A function $\varphi_* \in E_k$ is a *solution of the reduced problem on* E_k if and only if

$$P_{E_k}\big(\varphi_* \ln |\varphi_*|\big) = 0.$$

Remark 5.3. For this whole chapter, we will (implicitly) extend all expressions of the type "$|u|^\alpha \ln |u|$", $\alpha > 0$, by 0 when $u = 0$, i.e. $|u|^\alpha \ln |u|$ will be understood as an evaluation of the continuous function

$$\mathbb{R} \to \mathbb{R} : u \mapsto \begin{cases} 0 & \text{if } u = 0, \\ |u|^\alpha \ln |u| & \text{if } u \neq 0. \end{cases}$$

A first link between ($\mathcal{P}_2$) and ($\mathcal{P}_{p,k}$) is given by the following result.

Proposition 5.4. *Let $k \geq 1$ be an integer, $(p_n)_{n\geq 1} \subseteq (2, +\infty)$ be a sequence of exponents which converges to 2 and $(u_{p_n})_{n\geq 1} \subseteq H^1_Z(\mathcal{G})$ be a sequence of nonzero solutions to the problems ($\mathcal{P}_{p_n,k}$). Assume that $(u_{p_n})_n$ converges weakly in $H^1_Z(\mathcal{G})$ to a function $u_* \in H^1_Z(\mathcal{G})$. Then, u_* belongs to E_k, u_* is a solution of the reduced problem, and the convergence holds strongly in H^2 on every edge.*

Two main questions arise.

(Q1) *Given a solution of the reduced problem, under which conditions is it a limit of a sequence of solutions to ($\mathcal{P}_{p,k}$) as $p \to 2$?*

(Q2) *Can one obtain local uniqueness results around a solution of the reduced problem in the asymptotic regime $p \approx 2$?*

Before going any further, let us specify our functional setting. We will consider H, the function space given by

$$H := \left\{ u \in H^1_Z(\mathcal{G}) \;\middle|\; u_{|e} \in H^2(e) \text{ for all } e \in \mathbb{E}; \sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 \text{ for all } v \in \mathbb{V} \setminus Z \right\}.$$

We will endow H with the norm $\|u\|_H := \sqrt{\sum_{e\in\mathbb{E}} \|u\|^2_{H^2(e)}}$ for $u \in H$. The space H is a Hilbert space when equipped with this norm, since H^2 convergence implies[2] pointwise convergence of $u_e(v)$ and of $\frac{\mathrm{d}u}{\mathrm{d}x_e}(v)$ for all $v \in \mathbb{V} \setminus Z$.

Given a fixed integer $k \geq 1$, we define the map

$$F : \begin{cases} [2, +\infty) \times H & \to L^2(\mathcal{G}), \\ (p, u) & \mapsto -u'' + au - \gamma_k |u|^{p-2} u. \end{cases} \tag{5.1}$$

For any $p \geq 2$, the solutions of problem ($\mathcal{P}_{p,k}$) are given exactly by the functions $u \in H$ such that $F(p, u) = 0$. Note that the image of F is included in $L^\infty(\mathcal{G})$, thus into $L^2(\mathcal{G})$, since u and $|u|^{p-2}u$ are continuous on $\mathcal{G}$, which is compact.

Using the previous notation, our goal can be stated easily.

We want to relate the roots of $u \mapsto F(p, u)$ with those of $u \mapsto F(2, u)$.

Implicit function theorems[3] are natural tools to study the dependence of roots of a function on a parameter.

[2]Remark that convergence with respect to the norm of H implies convergence in $\mathcal{C}^1$, and in particular in H^1 and L^p (for any $p \in [1, +\infty]$) on all edges of the graph (identified with closed intervals).

[3]See e.g. [206] to know more about the use of implicit function theorems and Appendix E for statements of such results under minimal regularity assumptions.

However, we will not be able to use them directly to study the dependence of roots of F with respect to p. Indeed, the map $u \mapsto F(2,u)$ "vanishes too much" since its set of roots is the whole vector space E_k, thus none of its roots are such that the linear map $u \mapsto \partial_u F(2,u)$ is invertible.

Such a situation is quite frequently encountered in nonlinear analysis. A well-known strategy to solve the issue is to use a *Lyapunov-Schmidt reduction*. Roughly stated, this method consists in studying higher order terms while applying implicit function theorems in order to treat situations where the linearized equations have a nontrivial kernel. We refer to [206, Chapter 1] for a presentation of this method (see also [277] for an application of this method to the study of concentrated solutions). Such a method was already considered to study the $p \approx 2$ regime on bounded domains of $\mathbb{R}^N$, see [174, 294].

A quite serious inconvenience occurs.

⚠ **Implicit function theorems require regularity!** ⚠

Checking that the regularity properties required by the Lyapunov-Schmidt method are satisfied will be done in Section 5.2.

The argument relies in an important way on the fact that we are working in dimension one and that the convergence in the norm of H implies $\mathcal{C}^1$ convergence on every edge.

As we will see, functions vanishing identically on some edges will cause a lot of trouble and will often need to be excluded in our "general results".

More precisely, an important set in our study is given by[4]

$$S := \left\{ u \in H \;\middle|\; \min_{e \in \mathbb{E}} \inf_{x \in e} \left(|u_{|e}(x)| + |u'_{|e}(x)| \right) > 0 \right\}, \tag{5.2}$$

where $u_{|e}$ is the restriction of u to edge e, parametrized[5] by the interval $[0, |e|]$. Namely, a function $u \in H$ belongs to S if and only if all its roots are simple[6]. We observe that the set S is open[7] in H.

Let us remark that an eigenfunction in some eigenspace E_k belongs to S if and only if it does not vanish identically on any edge of the graph.

[4]We write a double infimum over edges $e \in \mathbb{E}$ then over $x \in e$ and not directly over $x \in \mathcal{G}$ because there are several derivatives associated with the same point x if x is a vertex.

[5]So that $u'_{|e}(x)$ is a "standard" derivative for a function defined on a real interval.

[6]Since u is $\mathcal{C}^1$ on each edge and $\mathcal{G}$ is compact.

[7]Because the map $u \mapsto \min_{e \in \mathbb{E}} \inf_{x \in e}(|u_{|e}(x)| + |u'_{|e}(x)|)$ is continuous with respect to the convergence in $\mathcal{C}^1$ edge by edge, in particular with respect to the one in H.

For solutions belonging to S, one may consider the following definition.

Definition 5.5. A solution $\varphi_* \in E_k \cap S$ of the reduced problem on E_k is *nondegenerate* if and only if the map

$$E_k \to E_k : \psi \mapsto P_{E_k}\Big((1 + \ln|\varphi_*|)\psi\Big)$$

is invertible.

Remark 5.6. A more “practical” way to formulate the previous condition is to check whether the $n_k \times n_k$ matrix

$$\left(\int_{\mathcal{G}} (1 + \ln|\varphi_*|)\varphi_i\varphi_j\right)_{1\le i,j\le n_k}$$

is invertible. Let us note that the integrals make sense since $\ln|\varphi_*|$ belongs to $L^2(\mathcal{G})$ (see Lemma 5.12 below) and since the eigenfunctions φ_i are bounded.

Remark 5.7. If $\dim E_k = 1$, then all solutions of the reduced problem in $E_k \cap S$ are nondegenerate. Indeed, in this case, a function $\varphi_* \in E_k \cap S \setminus \{0\}$ is a solution of the reduced problem if and only if

$$\int_{\mathcal{G}} |\varphi_*|^2 \ln|\varphi_*| \,\mathrm{d}x = 0.$$

It is nondegenerate if and only if

$$\int_{\mathcal{G}} |\varphi_*|^2(1 + \ln|\varphi_*|) \,\mathrm{d}x \neq 0.$$

However, using the previous equality, this last integral is equal to $\int_{\mathcal{G}} |\varphi_*|^2 \,\mathrm{d}x$, which is positive (as $0 \notin S$), showing that φ_* is nondegenerate.

We will prove the following theorem in Section 5.2. It provides answers to the questions (Q1) and (Q2) for eigenfunctions belonging to S.

Theorem 5.8. *Let $k \ge 1$ be an integer and let $\varphi_* \in E_k \cap S$.*

(i) ***Non-existence**. If the eigenfunction φ_* is not a solution of the reduced problem, then there exists a neighborhood U of $(2, \varphi_*)$ in $[2, +\infty) \times H$ so that problem $(\mathcal{P}_{p,k})$ has no solution in U with $p > 2$.*

(ii) ***Existence, uniqueness and non-degeneracy**. If the function φ_* is a nondegenerate solution of the reduced problem, then there exists a neighborhood U of $(2, \varphi_*)$ in $[2, +\infty) \times H$ and a number $\varepsilon > 0$ so that for all $p \in (2, 2 + \varepsilon]$, there exists a unique $u_p \in H$ so that (p, u_p) belongs to U and so that u_p is a solution of problem $(\mathcal{P}_{p,k})$. Moreover, the map $(2, 2 + \varepsilon) \to H : p \mapsto u_p$ is continuous.*

Several questions arise.

(Q3) *How restrictive is the hypothesis "$\varphi_* \in S$" in applications?*

(Q4) *Are there solutions which do not belong to S?*

Regarding question (Q3), a result of G. Berkolaiko and W. Liu implies that, on compact graphs, eigenfunctions are simple and nonvanishing on edges[8] for a *generic choice of lengths of edges*, in the sense of the Baire category (see [69, Theorem 3.6]).

However, such an answer is somewhat disappointing. Indeed:

- a specific example is *never generic*;
- examples enjoying some symmetries will have some edges of the same length and thus are *not generic*. This is in particular the case for *equilateral graphs*, those whose edges all have the same length.

In what follows, we will not provide an "abstract" alternative to Theorem 5.8 for eigenfunctions vanishing on some edges. Instead, we will focus on question (Q4) and on solutions of $(\mathcal{P}_{p,k})$ vanishing identically on edges, that we already encountered in Chapter 2 (see Theorem 2.9). To this end, we will study the cases in which nodal ground states vanish on edges of *compact star graphs* in detail. Section 5.3 is dedicated to the study of such graphs.

All the preceding chapters use the variational approach in a way or another. This naturally leads us to the following question.

(Q5) *How do action ground states and nodal ground states behave as $p \approx 2$?*

Regarding ground states, there is a general uniqueness result[9] for the positive solutions in the asymptotic regime $p \approx 2$.

Theorem 5.9. *Assume that $\lambda > 0$. Then, there exists $\varepsilon > 0$ so that for all $p \in (2, 2+\varepsilon)$, positive solutions of $(\mathcal{P}_{p,1})$ are unique.*

The previous result will be obtained combining Theorem 5.8 with suitable *a priori bounds* for positive solutions of $(\mathcal{P}_{p,1})$.

Regarding nodal ground states, we will show (see Theorem 5.25) that they converge to global minima[10] of the *reduced functional* $\mathcal{J}_{*,2} : E_2 \to \mathbb{R}$ defined by

$$\mathcal{J}_{*,2}(\varphi) := \frac{1}{4}\int_{\mathcal{G}} \varphi^2(x)\Big(1 - 2\ln|\varphi(x)|\Big)\,\mathrm{d}x \tag{5.3}$$

[8] With the possible exception of eigenfunctions supported in a single loop of the graph.

[9] Similar results are true for solutions of nonlinear elliptic partial differential equations on bounded domains, see e.g. [117, Lemma 1].

[10] Let us remark that critical points of the reduced functional $\mathcal{J}_{*,2}$ correspond exactly to the solutions of the reduced problem on E_2, see Section 5.2.5.

on the *reduced Nehari manifold*

$$\mathcal{N}_{*,2} := \Big\{ u \in E_k \setminus \{0\} \;\Big|\; \mathcal{J}'_{*,2}(u)[u] = 0 \Big\}.$$

This adapts to the graph setting a similar result for bounded domains in $\mathbb{R}^N$, proved by D. Bonheure, V. Bouchez, C. Grumiau and J. Van Schaftingen in [74, Theorem 4]. Although there is "no surprise" here, it is interesting to understand the relation between nodal ground states and solutions vanishing on edges, when those exist. This leads us to the following question.

(Q6) *If there are eigenfunctions vanishing identically on edges of the graph in the second eigenspace E_2, is it possible that nodal ground states converge to an eigenfunction which belongs to S as $p \to 2$?*

It turns out that the answer to question (Q6) is *positive*[11].

Indeed, in Section 5.4, we will consider the *tetrahedron graph*[12], the equilateral graph $\mathcal{G}_t$ depicted hereunder.

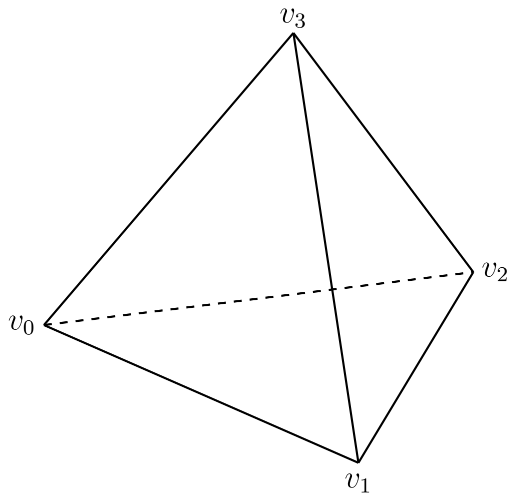


Figure 5.1: The tetrahedron graph $\mathcal{G}_t$, an equilateral graph made of four vertices v_0, v_1, v_2, v_3 and six edges of length one.

For the tetrahedron graph, it turns out that E_2 has dimension three, that some eigenfunctions in E_2 vanish identically on edges but that *the minimizer of the reduced functional $\mathcal{J}_{*,2}$ on the reduced Nehari manifold $\mathcal{N}_{*,2}$ does not vanish identically on any edge.* This implies that nodal ground states are unique (up to sign and symmetries) as $p \approx 2$ and nonvanishing on edges (i.e. belong to S).

[11] Contrasting what will be observed for compact star graphs in section 5.3.

[12] We decided to call this graph the "tetrahedron" due to its natural embedding into $\mathbb{R}^3$ depicted in Figure 5.1. In graph theoretical terms, this is the equilateral complete graph K_4 on four vertices.

In Section 5.4, we will more generally be able to characterize all critical points of $\mathcal{J}_{*,2}$ and prove that there are *four* families of critical points, where elements of the same family are obtained from one another using the symmetries of the graph:

- *global minima* of $\mathcal{J}_{*,2}$ on the reduced Nehari manifold $\mathcal{N}_{*,2}$, corresponding to the limits of nodal ground states and nonvanishing on edges;
- *local minima* of $\mathcal{J}_{*,2}$ on $\mathcal{N}_{*,2}$, vanishing on one edge of the graph[13];
- *saddle points* of $\mathcal{J}_{*,2}$ on $\mathcal{N}_{*,2}$;
- *global maxima* of $\mathcal{J}_{*,2}$ on $\mathcal{N}_{*,2}$.

We will see that all those critical points are generated by suitable symmetries of the tetrahedron via the *principle of symmetric criticality*.

Classifying the critical points of $\mathcal{J}_{*,2}$ is not an easy task. Even though all the expressions that have to be considered are explicit, this appears to be difficult to do "by hand". Therefore, we will perform this step via a *computer-assisted proof*, using *interval arithmetic* to produce certified numerical computations.

Now, let us turn to the proofs of our "general" results.

5.2 The near-linear regime: general theory

5.2.1 Why do solutions of the reduced problem show up?

First of all, let us see how the reduced problem shows up naturally.

Proof of Proposition 5.4. For all n and all $v \in H^1_Z(\mathcal{G})$, we have that

$$\left(u'_{p_n} \mid v'\right)_{L^2(\mathcal{G})} + \lambda\left(u_{p_n} \mid v\right)_{L^2(\mathcal{G})} = \gamma_k \int_{\mathcal{G}} |u_{p_n}|^{p_n-2} u_{p_n} v \,\mathrm{d}x. \tag{5.4}$$

Since weak H^1 convergence implies strong L^∞ convergence (since $\mathcal{G}$ is compact), the right hand side of the previous equality converges to $(u^* \mid v)_{L^2(\mathcal{G})}$. By weak convergence, the left hand side converges to $(u'_* \mid v')_{L^2(\mathcal{G})} + \lambda(u_* \mid v)_{L^2(\mathcal{G})}$. Since this holds for every v, u_* belongs to E_k. Writing

$$-u''_{p_n} = \gamma_k |u_{p_n}|^{p_n-2} u_{p_n} - \lambda u_{p_n},$$

and using that the right hand side converges to $(\gamma_k - \lambda)u_* = -u''_*$ in $L^2(\mathcal{G})$, we deduce that u_{p_n} converges in H to u_*. Now, let us show that u_* is a solution of the reduced problem. To this end, let us use specifically $v = \psi \in E_k$ as a test function in (5.4). We obtain

$$\int_{\mathcal{G}} (u'_{p_n}\psi' + \lambda u_{p_n}\psi) \,\mathrm{d}x = \gamma_k \int_{\mathcal{G}} |u_{p_n}|^{p_n-2} u_{p_n} \psi \,\mathrm{d}x.$$

[13] If $\varphi_* \notin S$ is a solution of the reduced problem, the property that it is a local minimum of $\mathcal{J}_{*,2}$ on $\mathcal{N}_{*,2}$ holds rather generally, as we will see in Section 5.2.6 where we will prove Theorem 5.24.

Using u_{p_n} as a test function in the eigenvalue equation $-\psi'' + \lambda\psi = \gamma_k\psi$, we get

$$\int_{\mathcal{G}} (u'_{p_n}\psi' + \lambda u_{p_n}\psi)\,\mathrm{d}x = \gamma_k \int_{\mathcal{G}} u_{p_n}\psi\,\mathrm{d}x.$$

Combining the two previous equalities, we obtain

$$\int_{\mathcal{G}} \Big(|u_{p_n}|^{p_n-2} - 1\Big) u_{p_n}\psi\,\mathrm{d}x = 0.$$

Dividing by $p_n - 2$ gives

$$\int_{\mathcal{G}} \frac{|u_{p_n}|^{p_n-2} - 1}{p_n - 2} u_{p_n}\psi\,\mathrm{d}x = \int_{\mathcal{G}\cap u_{p_n}^{-1}(\mathbb{R}\setminus\{0\})} \frac{\mathrm{e}^{(p_n-2)\ln|u_{p_n}|} - 1}{p_n - 2} u_{p_n}\psi\,\mathrm{d}x = 0,$$

and taking $n \to \infty$ leads to

$$\int_{\mathcal{G}} \Big(u_* \ln|u_*|\Big)\psi\,\mathrm{d}x = 0,$$

showing that u_* is a solution of the reduced problem. □

5.2.2 Regularity

We want to use methods based on implicit function theorems to study the roots of the map F defined by (5.1). To do so, we need to consider derivatives of F with respect to p and u. The only problematic term with respect to regularity is the nonlinear one. In this section, we will thus study the derivatives of the map M defined by

$$M : \begin{cases} [2,+\infty) \times H & \to L^2(\mathcal{G}), \\ (p,u) & \mapsto |u|^{p-2}u. \end{cases} \tag{5.5}$$

As we will see, the set S defined in (5.2) plays an important role in the investigation of the regularity of M.

We summarize our results in the following proposition.

Proposition 5.10. *Let M be defined as in* (5.5)*. Then,*

- *the function M is continuous on $[2,+\infty) \times H$.*
- *the derivative $\partial_p M$ exists on $[2,+\infty) \times H$, is given by*

$$\partial_p M(p,u) = |u|^{p-2}u \ln|u|$$

and the map

$$\partial_p M : \begin{cases} [2,+\infty) \times H & \to L^2(\mathcal{G}), \\ (p,u) & \mapsto |u|^{p-2}u \ln|u| \end{cases}$$

is continuous on $[2,+\infty) \times H$.

- *the differential* $\partial_u M$ *exists on* $[2,+\infty)\times H$*, is given by*[a]

$$\partial_u M(p,u)[v] = (p-1)|u|^{p-2}v$$

for every $v \in H$ *and the map*

$$\partial_u M : \begin{cases} [2,+\infty)\times H & \to \mathcal{L}(H; L^2(\mathcal{G})) \\ (p,u) & \mapsto \partial_u M(p,u) \end{cases}$$

is continuous on $\big((2,+\infty)\times H\big) \cup (\{2\}\times S)$*.*

[a] Recalling that $0^0 = 1$.

A classical lemma

The following lemma is classical in nonlinear analysis, see e.g. [154, Theorem 2.6].

Lemma 5.11. *Let* $q \geq 1$ *and* $f \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$*. Then, the Nemytskii mapping*

$$N_f : \begin{cases} H & \to L^q(\mathcal{G}) \\ u & \mapsto \big(x \mapsto f(u(x))\big) \end{cases}$$

is of class $\mathcal{C}^1$ *and its differential is given by*

$$\partial_u N_f(u)[v] = \big(x \mapsto f'(u(x))v(x)\big).$$

The role of the set S

The lemma hereunder will turn out to be crucial and will explain the role played by the set S defined by (5.2).

Lemma 5.12. *Let* $q \geq 1$*. The map*

$$N_{\ln} : \begin{cases} S & \to L^q(\mathcal{G}) \\ u & \mapsto \ln|u| \end{cases}$$

is well-defined and is continuous.

Remark 5.13. We adopt the convention $\ln 0 = -\infty$. Let us recall that a function $u \in S$ has only simple roots, so that it is nonzero except at a finite number of points and that $\ln|u|$ is finite almost everywhere. The previous lemma claims that $\ln|u|$ belongs to $L^2(\mathcal{G})$.

To prove Lemma 5.12, we will need the next lemma.

Lemma 5.14. *Let* $u \in \mathcal{C}^1([0,l])$ *and* $c > 0$ *be so that, for all* $x \in [0,l]$, $|u(x)| \leq 1$ *and* $|u'(x)| \geq c$. *Then, for all* $q \geq 1$,

$$\int_0^l \left|\ln|u(x)|\right|^q \,\mathrm{d}x \leq \frac{2}{c}\int_0^{cl} \left|\ln(s)\right|^q \,\mathrm{d}s < +\infty.$$

Proof. Up to replacing u by $-u$, one may assume that $u'(x) \geq c$ so that u is increasing in $[0,l]$.

We first consider the case where u has a root $\xi \in [0,l]$ (necessarily unique as u is increasing). Then, the bound on the derivative of u implies that, for all $x \in [0,l]$,

$$1 \geq |u(x)| \geq c|x-\xi|. \tag{5.6}$$

Therefore, one has that

$$0 \geq \ln|u(x)| \geq \ln(c|x-\xi|),$$

thus

$$\left|\ln|u(x)|\right|^q \leq \left|\ln(c|x-\xi|)\right|^q.$$

Integrating the previous inequality gives

$$\begin{aligned}\int_0^l \left|\ln|u(x)|\right|^q \,\mathrm{d}x &\leq \int_0^\xi \left|\ln(ct)\right|^q \,\mathrm{d}t + \int_0^{l-\xi} \left|\ln(ct)\right|^q \,\mathrm{d}t \\ &\leq 2\int_0^l \left|\ln(ct)\right|^q \,\mathrm{d}t \\ &= \frac{2}{c}\int_0^{cl} \left|\ln(s)\right|^q \,\mathrm{d}s.\end{aligned}$$

If u does not vanish in $[0,l]$ then, either $u(0) > 0$ in which case one has that $1 \geq |u(x)| \geq u(0) + cx \geq cx$, and one concludes as above by using $\xi = 0$ in (5.6), or $u(l) < 0$ and one has $1 \geq |u(x)| \geq |u(l)| + c|x-l| \geq c|x-l|$ and concludes by using $\xi = l$. □

Now, let us turn to the proof of Lemma 5.12.

Proof of Lemma 5.12. Let $u \in S$. As $\mathcal{G}$ is compact, we may reason edge by edge. We thus assume that u is defined on some interval $[0,l]$. Using the assumption that $\inf(|u| + |u'|) > 0$, one has that u has a finite number of roots $\xi_1, \dots, \xi_k \in [0,l]$, which we assume to be ordered. We denote $R := \{\xi_1, \dots, \xi_k\}$. Then, there exist some numbers $c > 0$ and $\gamma > 0$ so that all intervals $[\xi_i - \gamma, \xi_i + \gamma]$ are disjoint and one has, for all $x \in [0,l]$ at distance at most γ from R, $|u(x)| \leq 1$ and $|u'(x)| \geq c$. Lemma 5.14 implies that

$$\int_{\substack{0 \leq x \leq l \\ d(x,R) \leq \gamma}} \left|\ln|u(x)|\right|^q \,\mathrm{d}x \leq \frac{2k}{c}\int_0^{2c\gamma} \left|\ln(s)\right|^q \,\mathrm{d}s.$$

Since the map $x \mapsto \left|\ln|u(x)|\right|^q$ is continuous and bounded (thus integrable) on the set $\{x \in [0,l] \mid d(x,R) > \gamma\}$, this shows that $\ln|u|$ belongs to $L^q(0,l)$.

The continuity is very similar. Given $\varepsilon > 0$, we want to show the existence of $\delta > 0$ such that for all $v \in H$ with $\|u - v\|_H \leq \delta$, the inequality

$$\| \ln|u| - \ln|v| \|_{L^q} \leq \varepsilon$$

holds. Let us reason as above. If $\delta > 0$ is small enough, one may find[14] $c > 0$ and $\gamma > 0$ so that all intervals $[\xi_i - \gamma, \xi_i + \gamma]$ are disjoint and, for all $v \in B_H(u,\delta)$ and all $x \in \cup_{1\leq i\leq k}[\xi_i - \gamma, \xi_i + \gamma]$, we have $|v(x)| \leq 1$ and $|v'(x)| \geq c$. Thus,

$$\begin{aligned}\int_{\substack{0\leq x\leq l\\ d(x,R)\leq\gamma}} \left|\ln|u(x)| - \ln|v(x)|\right|^q \mathrm{d}x &\leq 2\int_{\substack{0\leq x\leq l\\ d(x,R)\leq\gamma}} \left(\left|\ln|u(x)|\right|^q + \left|\ln|v(x)|\right|^q\right) \mathrm{d}x \\ &\leq \frac{4k}{c}\int_0^{2c\gamma} \left|\ln(s)\right|^q \mathrm{d}s, \end{aligned} \tag{5.7}$$

which can be made smaller than $\frac{\varepsilon}{2}$ up to taking γ smaller if required. Now, let us fix the value of γ. Taking δ small enough, there exists $d > 0$ so that

$$\text{for all } v \in B_H(u,\delta), \qquad \inf_{\substack{0\leq x\leq l\\ d(x,R)\geq\gamma}} |v(x)| \geq d,$$

and thus we obtain (still reducing δ if needed) that for all $v \in B_H(u,\delta)$,

$$\sup_{\substack{0\leq x\leq l\\ d(x,R)\geq\gamma}} \left|\ln|v| - \ln|u|\right|^q \leq \frac{\varepsilon}{2l}, \tag{5.8}$$

as convergence in H implies uniform convergence. Using (5.7) and (5.8), we obtain

$$\int_0^l \left|\ln|u(x)| - \ln|v(x)|\right|^q \mathrm{d}x \leq \frac{\varepsilon}{2} + l\frac{\varepsilon}{2l} = \varepsilon,$$

which ends the proof. □

Another important Lemma is the following.

Lemma 5.15. *Let $h \in \mathcal{C}^1(\mathbb{R})$ be such that $h(0) = 0$. Then, the map*

$$\Gamma_h : \begin{cases} S & \to L^2(\mathcal{G}) \\ u & \mapsto h(u)\ln|u| \end{cases}$$

is of class $\mathcal{C}^1$ and its differential is given, for all $v \in H$, by

$$\partial_u \Gamma_h(u)[v] = (q_h(u) + h'(u)\ln|u|)v,$$

where $q_h(s) := h(s)/s$ if $s \neq 0$ and $q_h(0) := h'(0)$.

[14] Recalling that convergence in H implies convergence in $\mathcal{C}^1$.

Proof. We first define $f : \mathbb{R} \to \mathbb{R}$ by $f(s) := h(s) \ln |s|$ if $s \neq 0$ and $f(0) = 0$. This function is continuous since

$$\lim_{\substack{s \to 0 \\ s \neq 0}} f(s) = \lim_{\substack{s \to 0 \\ s \neq 0}} \frac{h(s)}{s} \cdot (s \ln |s|) = h'(0) \cdot 0 = 0$$

and is continuously differentiable on $\mathbb{R} \setminus \{0\}$ with $f'(s) = q_h(s) + h'(s) \ln |s|$. We note that q_h is continuous on $\mathbb{R}$, as can be seen by writing

$$q_h(s) = \frac{1}{s} \int_0^s h'(t)\, \mathrm{d}t.$$

We remark that for all $u, v \in \mathbb{R}$, the function $g_{u,v} : [0,1] \to \mathbb{R}$ defined by

$$g_{u,v}(t) := \begin{cases} (q_h(u+tv) + h'(u+tv) \ln |u+tv|)v & \text{if } v \neq 0 \text{ and } u + tv \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is integrable, and one has that

$$f(u+v) - f(u) = \int_0^1 g_{u,v}(t)\, \mathrm{d}t. \tag{5.9}$$

Indeed, this may be proved using the fundamental theorem of differential and integral calculus extended to open intervals as in [233, Proposition p. 401].

Now, let us consider $u \in S$. We need to show that

$$\sup_{v \in B_H(0,r)} \frac{\left\| f(u+v) - f(u) - (q_h(u) + h'(u) \ln |u|)v \right\|_{L^2}}{\|v\|_H} \xrightarrow[r \to 0^+]{} 0.$$

Using (5.9), for any $v \in H$, we have

$$\begin{aligned} &\left| \, f(u+v) - f(u) - (q_h(u) + h'(u) \ln |u|)v \, \right| \\ = \quad & \left| \int_0^1 \Big(\big(q_h(u+tv) - q_h(u) \big) \right. \\ & \qquad + \big(h'(u+tv) - h'(u) \big) \ln |u+tv| \\ & \qquad \left. + h'(u) \big(\ln |u+tv| - \ln |u| \big) \Big) v \, \mathrm{d}t \right| \\ \leq & \ \|v\|_{L^\infty(\mathcal{G})} \int_0^1 \Big[\big| q_h(u+tv) - q_h(u) \big| \\ & \qquad + \big| h'(u+tv) - h'(u) \big| \big| \ln |u+tv| \big| \\ & \qquad + |h'(u)| \big| \ln |u+tv| - \ln |u| \big| \Big] \, \mathrm{d}t. \end{aligned} \tag{5.10}$$

Now, we notice that

$$\sup_{v\in B_H(0,r)}\left\|\int_0^1 q_h(u+tv)-q_h(u)\,\mathrm{d}t\right\|_{L^2(\mathcal{G})}\leq|\mathcal{G}|^{\frac{1}{2}}\sup_{v\in B_H(0,r)}\|q_h(u+v)-q_h(u)\|_{L^\infty(\mathcal{G})}\xrightarrow[r\to 0^+]{}0 \tag{5.11}$$

since q_h is continuous (hence uniformly continuous on compact sets) and since convergence in H implies convergence in L^∞. Moreover,

$$\begin{aligned}&\sup_{v\in B_H(0,r)}\left\|\int_0^1 (h'(u+tv)-h'(u))\ln|u+tv|\,\mathrm{d}t\right\|_{L^2(\mathcal{G})}\\ &\leq\Big(\sup_{v\in B_H(0,r)}\|h'(u+v)-h'(u)\|_{L^\infty(\mathcal{G})}\Big)\Big(\sup_{v\in B_H(0,r)}\|\ln|u+v|\|_{L^2(\mathcal{G})}\Big)\\ &\xrightarrow[r\to 0^+]{}0\end{aligned} \tag{5.12}$$

since the first supremum converges to 0 as above (see (5.11)) and the second one is bounded for r small enough according to Lemma 5.12. Finally, we have that

$$\begin{aligned}&\sup_{v\in B_H(0,r)}\left\|\int_0^1 |h'(u)|\Big|\ln|u+tv|-\ln|u|\Big|\,\mathrm{d}t\right\|_{L^2(\mathcal{G})}\\ &\leq\|h'(u)\|_{L^\infty(\mathcal{G})}\sup_{v\in B_H(0,r)}\|\ln|u+tv|-\ln|u|\|_{L^2(\mathcal{G})}\\ &\xrightarrow[r\to 0^+]{}0\end{aligned} \tag{5.13}$$

where we used Lemma 5.12 again. Combining the bound (5.10) with (5.11)–(5.13) proves the differentiability claim.

To show that Γ is $\mathcal{C}^1$ on S, if suffices to check that the map

$$S\to L^2(\mathcal{G}):u\mapsto q_h(u)+h'(u)\ln|u|$$

is continuous, which is true according to Lemma 5.12 since q_h and h' are continuous. □

Regularity of M: proof of Proposition 5.10

Proof of Proposition 5.10. We prove all claimed properties one by one.

Step 1. Continuity of M.

We want to prove that if $(p_n)_{n\geq 1}\subseteq[2,+\infty)$ converges to $p\in[2,+\infty)$ and if $(u_n)_{n\geq 1}\subseteq H$ converges to $u\in H$, then

$$\int_{\mathcal{G}}\Big||u_n|^{p_n-2}u_n-|u|^{p-2}u\Big|^2\,\mathrm{d}x\xrightarrow[n\to\infty]{}0. \tag{5.14}$$

Since convergence in H implies convergence in $\mathcal{C}(\mathcal{G})$, one has that, for all $x\in\mathcal{G}$,

$$|u_n(x)|^{p_n-2}u_n(x)\xrightarrow[n\to\infty]{}|u(x)|^{p-2}u(x).$$

Indeed,

- if $u(x) \neq 0$, then it suffices to write
$$|u_n(x)|^{p_n-2}u_n(x) = \exp\Big((p_n-2)\ln|u_n(x)|\Big)u_n(x),$$
since $u_n(x)$ converges to $u(x) \neq 0$;
- if $u(x) = 0$, we deduce that $|u_n(x)|^{p_n-2}u_n(x)$ converges to 0 since $|u_n(x)|^{p_n-2}$ is smaller than 1 for n large and that $u_n(x)$ converges to 0.

We also check that
$$\Big\| |u_n|^{p_n-2}u_n \Big\|_{L^\infty(\mathcal{G})} \leq \Big(\|u\|_\infty + 1\Big)^p$$
for every n large enough (since convergence in H implies L^∞ convergence).

Combining the pointwise convergence result with the previous L^∞ bound proves (5.14), using the dominated convergence theorem (as $\mathcal{G}$ is compact).

Step 2. Existence of $\partial_p M$.

Let $(p,u) \in [2,+\infty) \times H$. Let $h \in \mathbb{R}\setminus\{0\}$, assuming that $h > 0$ if $p = 2$. Then,
$$\begin{aligned}
&\left\| \frac{M(p+h,u) - M(p,u)}{h} - |u|^{p-2}u\ln|u| \right\|^2_{L^2(\mathcal{G})} \\
&= \int_{x\in\mathcal{G},u(x)\neq 0} |u(x)|^{2(p-1)} \cdot \left| \frac{\exp(h\ln|u(x)|) - 1}{h} - \ln|u(x)| \right|^2 \mathrm{d}x \\
&= \int_{x\in\mathcal{G},u(x)\neq 0} |u(x)|^{2(p-1)} \cdot \Big|\ln|u(x)|\Big|^2 \cdot \left| \int_0^1 \exp(hs\ln|u(x)|)\,\mathrm{d}s - 1 \right|^2 \mathrm{d}x \\
&= \int_{x\in\mathcal{G},u(x)\neq 0} \left| \int_0^1 |u(x)|^{p-1} \cdot \Big|\ln|u(x)|\Big| \cdot \Big(\exp(hs\ln|u(x)|) - 1\Big)\,\mathrm{d}s \right|^2 \mathrm{d}x \\
&\leq \int_{x\in\mathcal{G},u(x)\neq 0} \int_0^1 \left| |u(x)|^{p-1} \cdot \Big|\ln|u(x)|\Big| \cdot \Big(\exp(hs\ln|u(x)|) - 1\Big) \right|^2 \mathrm{d}s\,\mathrm{d}x
\end{aligned}$$
where the last step follows from the Cauchy-Schwarz inequality. Now, for every $(x,s) \in \mathcal{G} \times [0,1]$ so that $u(x) \neq 0$, one has that
$$|u(x)|^{hs} = \exp(hs\ln|u(x)|) \xrightarrow[h\to 0]{} 1.$$
Moreover for every $(x,s) \in \mathcal{G} \times [0,1]$ and every $h \in \mathbb{R}$ with $|h| \leq 1$, we have
$$\Big|\exp(hs\ln|u(x)|) - 1\Big| \leq \|u\|_\infty + 2.$$
Since $x \mapsto |u(x)|^{p-1}\Big|\ln|u(x)|\Big|$ is bounded on $\mathcal{G}$, we deduce that
$$\int_{x\in\mathcal{G},u(x)\neq 0} \int_0^1 \Big| |u(x)|^{p-1}\Big|\ln|u(x)|\Big|(\exp(hs\ln|u(x)|) - 1)\Big|^2 \mathrm{d}s\,\mathrm{d}x \xrightarrow[h\to 0]{} 0,$$
showing that $\partial_p M(p,u) = |u|^{p-2}u\ln|u|$.

Step 3. Continuity of $\partial_p M$.

We want to prove that if $(p_n)_{n\geq 1} \subseteq [2, +\infty)$ converges to $p \in [2, +\infty)$ and if $(u_n)_{n\geq 1} \subseteq H$ converges to $u \in H$, then

$$\int_{\mathcal{G}} \Big| |u_n|^{p_n-2} u_n \ln |u_n| - |u|^{p-2} u \ln |u| \Big|^2 \, \mathrm{d}x \xrightarrow[n\to\infty]{} 0.$$

Reasoning as for the continuity of M, we can show the pointwise convergence of the integrand to 0. We can also bound the integrand uniformly on $\mathcal{G}$ since the function $s \mapsto s\ln(s)$ is continuous at 0 and that $(u_n)_n$ converges uniformly to u.

Step 4. Existence of $\partial_u M$.

For any $p \geq 2$, it suffices to apply Lemma 5.11 with $f(u) = |u|^{p-2}u$, which is of class $\mathcal{C}^1(\mathbb{R}; \mathbb{R})$ with derivative given by $f'(u) = |u|^{p-2}$.

Step 5. Continuity of $\partial_u M$.

We want to prove that if $(p_n)_{n\geq 1} \subseteq [2, +\infty)$ converges to $p \in [2, +\infty)$ and if $(u_n)_{n\geq 1} \subseteq H$ converges to $u \in H$ with $u \in S$ in case $p = 2$, then

$$\sup_{\|v\|_H \leq 1} \Big\| (p_n - 1)|u_n|^{p-2} v - (p-1)|u|^{p-2} v \Big\|_{L^2(\mathcal{G})} \xrightarrow[n\to\infty]{} 0.$$

As H is continuously embedded in $\mathcal{C}(\mathcal{G})$ (thus $L^\infty(\mathcal{G})$), it suffices to prove that

$$\int_{\mathcal{G}} \Big| (p_n - 1)|u_n(x)|^{p_n-2} - (p-1)|u(x)|^{p-2} \Big|^2 \, \mathrm{d}x \xrightarrow[n\to\infty]{} 0.$$

As $(u_n)_n$ converges uniformly to u, the sequence of integrated functions is bounded in $L^\infty(\mathcal{G})$ (reasoning as above), thus it only remains to check almost everywhere convergence in order to apply the dominated convergence theorem.

For any $x \in \mathcal{G}$ so that $u(x) \neq 0$, one has that

$$(p_n - 1)|u_n(x)|^{p_n-2} = (p_n - 1)\exp((p_n - 2)\ln |u_n(x)|) \xrightarrow[n\to\infty]{} (p-1)|u(x)|^{p-2}.$$

Now, we observe that:

- if $p = 2$, we assume $u \in S$. This implies that u has only finitely many roots and we have the convergence a.e. by the above argument. This ends the proof that $\partial_u M$ is continuous at $(2, u)$ if $u \in S$;
- if $p > 2$ and $x \in \mathcal{G}$ is such that $u(x) = 0$, we have

$$(p_n - 1)|u_n(x)|^{p_n-2} = (p_n - 1)|u_n(x)|^{p_n-p}|u_n(x)|^{p-2}$$

  and, for n large enough, $|u_n(x)|^{p_n-p} \leq 1$ while $|u_n(x)|^{p-2} \xrightarrow[n\to\infty]{} 0 = |u(x)|^{p-2}$. This ends the proof of the continuity of $\partial_u M$ at (p, u) with $p > 2$. □

5.2.3 Lyapunov-Schmidt reduction

Recalling that

$$E_k^\perp := \Big\{ w \in L^2(\mathcal{G}) \;\Big|\; \forall \varphi \in E_k, \int_{\mathcal{G}} w\varphi = 0 \Big\},$$

we define

$$Y := E_k^\perp \cap H.$$

We remark that Y is a closed subspace of H.

The equation $F(p,u) = 0$ is equivalent to the system of equations

$$\begin{cases} P_{E_k} F(p,u) = 0, \\ P_{E_k^\perp} F(p,u) = 0. \end{cases}$$

Now, we define

$$\tilde{F} : \begin{cases} [2,+\infty) \times E_k \times Y & \to E_K^\perp, \\ (p,\varphi,w) & \mapsto P_{E_k^\perp} F(p,\varphi + w). \end{cases}$$

Explicitly, one has that

$$\tilde{F}(p,\varphi,w) = -w'' + aw - \gamma_k w + \gamma_k P_{E_K^\perp}\Big((1 - |\varphi + w|^{p-2})(\varphi + w) \Big),$$

since, using the Fredholm theory (see e.g. [85, Section 6.2]), the linear operator $H \to L^2(\mathcal{G}) : u \mapsto -u'' + au - \gamma_k u$ is a bijection between Y and $E_k^\perp$.

According to the expression of $\tilde{F}$, we have that

$$\tilde{F}(2,\varphi,0) = \gamma_k P_{E_K^\perp}\Big((1 - |\varphi|^0)\varphi \Big) = 0$$

for all $\varphi \in E_k$ (since $s^0 = 1$ for all $s \in \mathbb{R}$).

Now, we consider $\varphi_* \in E_k \cap S$. By Proposition 5.10, M is $\mathcal{C}^1$ in a neighborhood of $(2,\varphi_*)$, so that $\tilde{F}$ is $\mathcal{C}^1$ in a neighborhood of $(2,\varphi_*,0)$. Moreover, one has that

$$\partial_w \tilde{F}(2,\varphi_*,0) = \Big(w \mapsto -w'' + aw - \gamma_k w \Big)$$

since the derivative of the nonlinear term is given by

$$w \mapsto \gamma_k P_{E_K^\perp}\Big(w - \partial_u N(2,\varphi_*)[w] \Big) = 0.$$

Therefore, recalling the Fredholm theory, one has that $\partial_w F(2,\varphi_*,0)$ is a linear isomorphism between Y and $E_k^\perp$. Using the implicit function theorem (we refer to Appendix E for precise statements), we obtain the following result.

Proposition 5.16. *Let $\varphi_* \in E_k \cap S$. There exists a neighborhood V of $(2, \varphi_*)$ in $[2, +\infty) \times E_k$, a neighborhood W of 0 in $Y = E_k^\perp \cap H$, and a $\mathcal{C}^1$ map $\eta : V \to W$ so that for all $(p, \varphi, w) \in V \times W$,*

$$\tilde{F}(p, \varphi, w) = 0 \iff w = \eta(p, \varphi)$$

and so that, for all $(p, \varphi) \in V$, $\varphi + \eta(p, \varphi)$ belongs to S.

We remark that $\eta(2, \varphi) = 0$ for all $\varphi \in E_k$, since $\tilde{F}(2, \varphi, 0) = 0$.

Now, for (p, u) close to $(2, \varphi_*)$, solving the equation $F(p, u) = 0$ is equivalent to finding functions φ so that

$$G(p, \varphi) := P_{E_k} F\Big(p, \varphi + \eta(p, \varphi)\Big)$$

vanishes. Since η is $\mathcal{C}^1$, F is $\mathcal{C}^1$ on $[2, +\infty) \times S$ and $\varphi + \eta(p, \varphi)$ belongs to S for all $(p, \varphi) \in V$, we deduce that G is a $\mathcal{C}^1$ map from V to E_k. We remark that, for all $\varphi \in E_k$, the equalities

$$G(2, \varphi) = P_{E_k} F(2, \varphi) = 0$$

hold. Now, let us define $Q : V \subset [2, +\infty[\times E_k \to E_k$ by

$$Q(p, \varphi) := \begin{cases} \frac{G(p,\varphi)}{p-2} & \text{if } p > 2, \\ \partial_p G(2, \varphi) & \text{if } p = 2. \end{cases} \tag{5.15}$$

Clearly, when $p > 2$, finding the roots of $\varphi \mapsto G(p, \varphi)$ is equivalent to finding the roots of $\varphi \mapsto Q(p, \varphi)$. This implies that, for $(p, \varphi) \in V$ and $w \in W$ with $p > 2$,

$$F(p, \varphi + w) = 0 \iff \begin{cases} w = \eta(p, \varphi), \\ G(p, \varphi) = 0 \end{cases} \iff \begin{cases} w = \eta(p, \varphi), \\ Q(p, \varphi) = 0. \end{cases} \tag{5.16}$$

Observe that

$$\partial_p G(2, \varphi) = P_{E_k} \partial_p F(2, \varphi) + P_{E_k} \partial_u F(2, \varphi)[\partial_p \eta(2, \varphi)] = -\gamma_k P_{E_k} \Big(\varphi \ln |\varphi|\Big) \tag{5.17}$$

using Proposition 5.10 and since $\partial_u F(2, \varphi)$ is the linear map $u \mapsto -u'' + au - \gamma_k u$ whose image is included in $E_k^\perp$ so that $P_{E_k} \partial_u F(2, \varphi)$ vanishes. On the other hand, for all $p > 2$, we may write

$$\frac{G(p, \varphi)}{p-2} = \frac{G(p, \varphi) - G(2, \varphi)}{p-2} = \int_0^1 \partial_p G(2 + s(p-2), \varphi)\, \mathrm{d}s.$$

Hence, for all $p \geq 2$, we have

$$Q(p, \varphi) = \int_0^1 \partial_p G(2 + s(p-2), \varphi)\, \mathrm{d}s. \tag{5.18}$$

Using (5.18) and recalling that G is $\mathcal{C}^1(V, E_k)$, we deduce that Q is continuous.

Now, let us prove the nonexistence result from Theorem 5.8.

Proof of Theorem 5.8, (i). We consider couples (p, u) given by $(p, \varphi + w)$ with $(p, \varphi) \in V$ and $w \in W$ (using the notations of Proposition 5.16). Then, if $p > 2$, up to taking V smaller, we have that $Q(p, \varphi) \neq 0$ since Q is continuous and $Q(2, \varphi_*) \neq 0$ by hypothesis (as φ_* is not a solution of the reduced problem on E_k and recalling (5.15) and (5.17)). This contradicts (5.16). $\square$

Our goal is now to prove *existence and uniqueness results.*

To obtain uniqueness results, we will use the implicit function theorem on the function Q. We therefore need to differentiate Q again.

Proposition 5.17. *The function $Q : V \to W$ is differentiable with respect to φ on V and its differential is given, for all $\psi \in E_k$, by*

$$\partial_\varphi Q(p,\varphi)[\psi] - \frac{\gamma_k}{p-2} P_{E_k}\Big(\Big[(p-1)|\varphi + \eta(p,\varphi)|^{p-2} - 1\Big](1 + \partial_\varphi \eta(p,\varphi))\,\psi\Big)$$

if $p > 2$ and by

$$\partial_\varphi Q(2,\varphi)[\psi] = -\gamma_k P_{E_k}\Big(\big(1 + \ln|\varphi|\big)\psi\Big)$$

if $p = 2$. Moreover, the map

$$\partial_\varphi Q : \begin{cases} V & \to \mathcal{L}(E_k; E_k), \\ (p,\varphi) & \mapsto \big(\psi \mapsto \partial_\varphi Q(p,\varphi)[\psi]\big) \end{cases}$$

is continuous at $(2, \varphi_*)$.

Proof. We proceed step by step.

Step 1. Existence and expression of $\partial_\varphi Q(2, \varphi)$.

For all φ, one has that

$$Q(2,\varphi) = \partial_p G(2,\varphi) = -\gamma_k P_{E_k}\big(\varphi \ln|\varphi|\big)$$

according to (5.17). Therefore,

$$\partial_\varphi Q(2,\varphi)[\psi] = -\gamma_k P_{E_k}\Big(\big(1 + \ln|\varphi|\big)\psi\Big),$$

using Lemma 5.15 with $h(s) = s$ (note that, since $(2, \varphi) \in V$, we have $\varphi \in S$à.

Step 2. Existence and expression of $\partial_\varphi Q(p, \varphi)$ for $p > 2$.

This follows from the chain rule.

Step 3. Continuity of $\partial_\varphi Q(p,\varphi)$ at $(2,\varphi_*)$.

To prove this, it is enough to prove that

$$\lim_{(p,\varphi)\to(2,\varphi_*)} \left\| \frac{\left[(p-1)|\varphi+\eta(p,\varphi)|^{p-2}-1\right]\Big(1+\partial_\varphi\eta(p,\varphi)\Big)}{p-2} - \Big(1+\ln|\varphi_*|\Big) \right\|_{L^2(\mathcal{G})} = 0.$$

Observe that

$$\begin{aligned}&\frac{\left[(p-1)|\varphi+\eta(p,\varphi)|^{p-2}-1\right]\Big(1+\partial_\varphi\eta(p,\varphi)\Big)}{p-2} - \Big(1+\ln|\varphi_*|\Big)\\ &= \left(\frac{(p-1)|\varphi+\eta(p,\varphi)|^{p-2}-1}{p-2} - \Big(1+\ln|\varphi_*|\Big)\right)\Big(1+\partial_\varphi\eta(p,\varphi)\Big)\\ &\quad + \Big(1+\ln|\varphi_*|\Big)\partial_\varphi\eta(p,\varphi).\end{aligned}$$

Let us recall that $\partial_\varphi\eta$ is continuous and that $\partial_\varphi\eta(2,\varphi)=0$. Hence, denoting $\tilde\varphi := \varphi+\eta(p,\varphi)$ and observing that, by continuity of η, we have

$$\lim_{(p,\varphi)\to(2,\varphi_*)} \|\tilde\varphi-\varphi_*\|_H = 0,$$

the result will follow if we prove that

$$\lim_{(p,\tilde\varphi)\to(2,\varphi_*)} \int_{\mathcal{G}} \left| \frac{(p-1)|\tilde\varphi|^{p-2}-1}{p-2} - \Big(1+\ln|\varphi_*|\Big)\right|^2 \mathrm{d}x = 0.$$

In order to do this, we will prove that

$$\lim_{p\to2} \int_{\mathcal{G}} \left| \frac{(p-1)|\varphi_*|^{p-2}-1}{p-2} - \Big(1+\ln|\varphi_*|\Big)\right|^2 \mathrm{d}x = 0 \tag{5.19}$$

and that

$$\lim_{(p,\tilde\varphi)\to(2,\varphi_*)} \int_{\mathcal{G}} \left| \frac{|\tilde\varphi|^{p-2}-|\varphi_*|^{p-2}}{p-2}\right|^2 \mathrm{d}x = 0. \tag{5.20}$$

Let us start with the proof of (5.19). Define $g_x(s) := (1+s(p-2))|\varphi_*(x)|^{s(p-2)}$. For all $x\in\mathcal{G}$ with $\varphi_*(x)\neq0$, (i.e. for a.e. $x\in\mathcal{G}$ as $\varphi_*\in S$), this function satisfies $g_x(1)=(p-1)|\varphi_*(x)|^{p-2}$, $g_x(0)=1$, and its derivative is given by

$$g_x'(s) = (p-2)|\varphi_*(x)|^{s(p-2)}\Big((1+s(p-2))\ln|\varphi_*(x)|+1\Big).$$

Hence,

$$\begin{aligned}&\int_{\mathcal{G}} \left| \frac{(p-1)|\varphi_*(x)|^{p-2}-1}{p-2} - \Big(1+\ln|\varphi_*(x)|\Big)\right|^2 \mathrm{d}x\\ &= \int_{\mathcal{G}} \left| \int_0^1 \left[|\varphi_*(x)|^{s(p-2)}\Big((1+s(p-2))\ln|\varphi_*(x)|+1\Big) - \Big(1+\ln|\varphi_*(x)|\Big)\right]\mathrm{d}s\right|^2 \mathrm{d}x.\end{aligned}$$

Using the Cauchy-Schwarz inequality, (5.19) will follow if we prove that

$$\lim_{p\to 2}\int_{\mathcal{G}}\int_0^1\Big||\varphi_*(x)|^{s(p-2)}\Big((1+s(p-2))\ln|\varphi_*(x)|+1\Big)-\Big(1+\ln|\varphi_*(x)|\Big)\Big|^2\,\mathrm{d}s\,\mathrm{d}x=0.$$

The pointwise convergence of

$$\Big||\varphi_*(x)|^{s(p-2)}\Big((1+s(p-2))\ln|\varphi_*(x)|+1\Big)-\Big(1+\ln|\varphi_*(x)|\Big)\Big|^2$$

to 0 as $p\to 2$ for all $(x,s)\in\mathcal{G}\times[0,1]$ such that $\varphi_*(x)\neq 0$ is easily obtained. We can also prove that this function is bounded by an integrable function using Lemma 5.12. This proves (5.19).

In order to prove (5.20), let us define for $\psi\in S$ and for all $x\in\mathcal{G}$ such that $\psi(x)\neq 0$, $g(s):=|\psi(x)|^{s(p-2)}$. Observe that $g(1)=|\psi(x)|^{p-2}$, $g(0)=1$ and $g'(s)=|\psi(x)|^{s(p-2)}(p-2)\ln|\psi(x)|$. Hence,

$$\begin{aligned}\lim_{(p,\tilde\varphi)\to(2,\varphi_*)}\int_{\mathcal{G}}\Big|\frac{|\tilde\varphi|^{p-2}-|\varphi_*|^{p-2}}{p-2}\Big|^2\mathrm{d}x&=\lim_{(p,\tilde\varphi)\to(2,\varphi_*)}\int_{\mathcal{G}}\Big|\frac{|\tilde\varphi|^{p-2}-1+1-|\varphi_*|^{p-2}}{p-2}\Big|^2\mathrm{d}x\\&=\lim_{(p,\tilde\varphi)\to(2,\varphi_*)}\int_{\mathcal{G}}\Big|\int_0^1\Big[|\tilde\varphi(x)|^{s(p-2)}\ln|\tilde\varphi(x)|-|\varphi_*(x)|^{s(p-2)}\ln|\varphi_*(x)|\Big]\,\mathrm{d}s\Big|^2\,\mathrm{d}x.\end{aligned}$$

Using the Cauchy-Schwarz inequality, it remains to prove that

$$\lim_{(p,\tilde\varphi)\to(2,\varphi_*)}\int_{\mathcal{G}}\int_0^1\Big|\Big(|\tilde\varphi(x)|^{s(p-2)}-|\varphi_*(x)|^{s(p-2)}\Big)\ln|\tilde\varphi(x)|\Big|^2\,\mathrm{d}s\,\mathrm{d}x=0\tag{5.21}$$

and

$$\lim_{(p,\tilde\varphi)\to(2,\varphi_*)}\int_{\mathcal{G}}\int_0^1\Big||\varphi_*(x)|^{s(p-2)}\Big(\ln|\tilde\varphi(x)|-\ln|\varphi_*(x)|\Big)\Big|^2\,\mathrm{d}s\,\mathrm{d}x=0.\tag{5.22}$$

So, applying again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\Big[\int_{\mathcal{G}}\int_0^1\Big|\Big(|\tilde\varphi(x)|^{s(p-2)}-|\varphi_*(x)|^{s(p-2)}\Big)\ln|\tilde\varphi(x)|\Big|^2\,\mathrm{d}s\,\mathrm{d}x\Big]^2\\&\leq\Big[\int_{\mathcal{G}}\int_0^1\Big(|\tilde\varphi(x)|^{s(p-2)}-|\varphi_*(x)|^{s(p-2)}\Big)^4\,\mathrm{d}s\,\mathrm{d}x\Big]\cdot\Big[\int_{\mathcal{G}}\int_0^1\Big(\ln|\tilde\varphi(x)|\Big)^4\,\mathrm{d}s\,\mathrm{d}x\Big].\end{aligned}$$

By Lemma 5.12, the second factor in the product is bounded and the first one tends to 0 by the dominated convergence theorem since $\varphi_*(x)\neq 0$ a.e. in $\mathcal{G}$. Thus, we have shown that (5.21) holds. In order to prove (5.22), let us apply one more time the Cauchy-Schwarz inequality. We have

$$\begin{aligned}&\Big[\int_{\mathcal{G}}\int_0^1\Big||\varphi_*(x)|^{s(p-2)}\Big(\ln|\tilde\varphi(x)|-\ln|\varphi_*(x)|\Big)\Big|^2\,\mathrm{d}s\,\mathrm{d}x\Big]^2\\&\leq\Big[\int_{\mathcal{G}}\int_0^1|\varphi_*(x)|^{4s(p-2)}\,\mathrm{d}s\,\mathrm{d}x\Big]\cdot\Big[\int_{\mathcal{G}}\int_0^1\Big(\ln|\tilde\varphi(x)|-\ln|\varphi_*(x)|\Big)^4\,\mathrm{d}s\,\mathrm{d}x\Big],\end{aligned}$$

and we conclude by a new application of Lemma 5.12. □

We now have all the tools required to prove the existence and uniqueness part of Theorem 5.8.

Proof of Theorem 5.8, (ii). Let $\varphi_* \in E_k \cap S$ be a nondegenerate solution of the reduced problem. Then (by definition[15]), the map $E_k \to E_k : \psi \mapsto \partial_\varphi Q(2, \varphi_*)[\psi]$ is invertible. Hence, by Proposition 5.17, we may apply the implicit function theorem (see Appendix E) to the function Q at the point $(2, \varphi_*)$, ending the proof. □

5.2.4 Asymptotic behavior of positive solutions as $p \to 2$

As we mentioned in the beginning of this chapter, we will prove that positive solutions of ($\mathrm{NLS}_{\mathcal{G},Z}$) are unique when p is close to 2. The convenient way to see this is to consider the equivalent rescaled problem ($\mathcal{P}_{p,1}$) and to show the uniqueness of its positive solutions.

Before proving the uniqueness result, let us prove the following a priori bound[16] on positive solutions of ($\mathcal{P}_{p,1}$).

Theorem 5.18. *Assume that $\lambda > 0$. Then, for all $\bar{p} > 2$, there exists $C_{\bar{p}} > 0$ so that for all $p \in (2, \bar{p}]$, all positive solutions of ($\mathcal{P}_{p,1}$) satisfy*

$$\|u\|_{C^1(\mathcal{G})} \leq C_{\bar{p}}$$

where

$$\|u\|_{C^1(\mathcal{G})} := \max\Big\{\|u\|_{L^\infty(\mathcal{G})}, \|u'\|_{L^\infty(\mathcal{G})}\Big\}.$$

Remark 5.19. If $\lambda = 0$ and if there are no Dirichlet vertices, then the problem has no positive solutions as can be seen by integrating the equation over $\mathcal{G}$, leading to

$$0 = \int_{\mathcal{G}} (-u'') \,\mathrm{d}x = \gamma_1 \int_{\mathcal{G}} u^{p-1} \,\mathrm{d}x.$$

To obtain the a priori bound, we will use the following lemma.

Lemma 5.20. *Let $\varphi_1 \in E_1$ be the (unique) positive eigenfunction such that $\|\varphi_1\|_{L^\infty(\mathcal{G})} = 1$. Then, there exists $E > 0$ so that for all $u \in H^1_Z(\mathcal{G})$, we have*

$$\left\|\frac{u}{\varphi_1}\right\|_{L^\infty(\mathcal{G})} \leq E\|u\|_{C^1(\mathcal{G})}.$$

Proof. This follows easily noting that $\varphi_1 > 0$ on $\mathcal{G} \setminus Z$, and that, for all $v \in Z$, one has $u(v) = \varphi_1(v) = 0$ and $\varphi_1'(v) \neq 0$. □

[15] Up to a multiplicative constant irrelevant for invertibility.

[16] Similar results may be found for instance in [116] where the quasi-linear limit for another differential problem is studied.

Proof of Theorem 5.18. Let u be a positive solution of $(\mathcal{P}_{p,1})$. Then, considering φ_1, the positive eigenfunction such that $\|\varphi_1\|_{L^\infty} = 1$, as a test function in $(\mathcal{P}_{p,1})$, we obtain

$$\int_{\mathcal{G}} u^{p-1}\varphi_1 \,\mathrm{d}x = \int_{\mathcal{G}} u\varphi_1 \,\mathrm{d}x.$$

Since $u\varphi_1^{\frac{1}{p-1}}$ belongs to L^{p-1} and since $\varphi_1^{\frac{p-2}{p-1}}$ belongs to $L^{\frac{p-1}{p-2}}$, Hölder's inequality implies that

$$\int_{\mathcal{G}} u^{p-1}\varphi_1 \,\mathrm{d}x = \int_{\mathcal{G}} u\varphi_1^{\frac{1}{p-1}} \cdot \varphi_1^{\frac{p-2}{p-1}} \,\mathrm{d}x \le \Big(\int_{\mathcal{G}} u^{p-1}\varphi_1 \,\mathrm{d}x\Big)^{\frac{1}{p-1}} \Big(\int_{\mathcal{G}} \varphi_1 \,\mathrm{d}x\Big)^{\frac{p-2}{p-1}}$$

so that

$$\int_{\mathcal{G}} u^{p-1}\varphi_1 \,\mathrm{d}x \le \int_{\mathcal{G}} \varphi_1 \,\mathrm{d}x. \tag{5.23}$$

Now, let us define $(p_i)_{i\ge 0}$ by $p_i := 2+\frac{i}{2}$. Let us now prove that for all $i \ge 0$, there exists $C_i > 0$ so that for all $p \in (p_i, p_{i+1}]$ and for all solutions u of $(\mathcal{P}_{p,1})$, we have $\|u\|_{H^1(\mathcal{G})} \le C_i$. If we do so, the theorem will be proved because for any $\bar{p} > 2$, there exists $\bar{i} \ge 0$ so that $\bar{p} \in (p_{\bar{i}}, p_{\bar{i}+1}]$ and it suffices to take $C_{\bar{p}} := \max\{C_i \mid 0 \le i \le \bar{i}\}$.

Let us show that for every $i \ge 0$, there exists $D_i > 0$ so that for all $p \in (p_i, p_{i+1}]$,

$$\|u^{p-1}\|_{L^{q_i}} \le D_i \|u\|_{\mathcal{C}^1(\mathcal{G})}^{\frac{(p-1)(q_i-1)}{q_i}} \quad \text{where} \quad q_i := \frac{p_i}{p_i - 1}.$$

Indeed,

$$\int_{\mathcal{G}} (u^{p-1})^{q_i} \,\mathrm{d}x = \int_{\mathcal{G}} u^{p-1}\varphi_1 \Big(\frac{u}{\varphi_1}\Big)^{(p-1)(q_i-1)} \varphi_1^{(p-1)(q_i-1)-1} \,\mathrm{d}x.$$

Using Lemma 5.20, noting that $(p-1)(q_i-1) \ge (p_i-1)/(p_i-1) = 1$ since $p \ge p_i$ and recalling that $\|\varphi_1\|_{L^\infty} = 1$, we obtain

$$\begin{aligned}\int_{\mathcal{G}} (u^{p-1})^{q_i} \,\mathrm{d}x &\le \Big(\|u\|_{\mathcal{C}^1(\mathcal{G})} E\Big)^{(p-1)(q_i-1)} \int_{\mathcal{G}} u^{p-1}\varphi_1 \,\mathrm{d}x \\ &\le \Big(\|u\|_{\mathcal{C}^1(\mathcal{G})} E\Big)^{(p-1)(q_i-1)} \int_{\mathcal{G}} \varphi_1 \,\mathrm{d}x,\end{aligned}$$

where the second inequality follows from (5.23). Therefore, we have

$$\|u^{p-1}\|_{L^{q_i}} \le \Big(\|u\|_{\mathcal{C}^1(\mathcal{G})} E\Big)^{\frac{(p-1)(q_i-1)}{q_i}} \Big(\int_{\mathcal{G}} \varphi_1 \,\mathrm{d}x\Big)^{\frac{1}{q_i}}.$$

We remark that, since $p_i < p \le p_{i+1}$, we have

$$\frac{(p-1)(q_i-1)}{q_i} \le \frac{(p_i - 1/2)(q_i-1)}{q_i} = 1 - \frac{1}{2p_i} < 1.$$

Therefore,

$$\|u^{p-1}\|_{L^{q_i}} \le \Big(\int_{\mathcal{G}} \varphi_1 \,\mathrm{d}x\Big)^{\frac{1}{q_i}} \max(1, E) \|u\|_{\mathcal{C}^1(\mathcal{G})}^{\frac{(p-1)(q_i-1)}{q_i}}.$$

Multiplying $(\mathcal{P}_{p,1})$ by u, integrating and using Hölder's inequality, we obtain

$$\int_{\mathcal{G}} |u'|^2 + \lambda |u|^2 \, dx = \gamma_1 \int_{\mathcal{G}} u^{p-1} u \, dx \leq \gamma_1 \|u^{p-1}\|_{L^{q_i}} \|u\|_{L^{p_i}},$$

since $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Now, there exists $C_i > 0$ so we have $\|u\|_{L^{p_i}} \leq C_i \|u\|_{H^1}$. Since $\lambda > 0$, $\min(1, \lambda)\|u\|_{H^1}^2 \leq \int_{\mathcal{G}} |u'|^2 + \lambda |u|^2 \, dx$, so that

$$\|u\|_{H^1} \leq C_i \gamma_1 \|u^{p-1}\|_{L^{q_i}} \leq \max(1, \|u\|_{\mathcal{C}^1(\mathcal{G})})^{1 - \frac{1}{2p_i}}.$$

Since $\|u\|_{L^{q_i}} \leq \tilde{C}_i \|u\|_{H^1}$ and that $-u'' = u^{p-1} - \lambda u$, there exists $D_i > 0$ so that

$$\|u\|_{W^{2,q_i}} \leq D_i \max(1, \|u\|_{\mathcal{C}^1(\mathcal{G})})^{1 - \frac{1}{2p_i}}.$$

Since $\mathcal{C}^1$ is continuously embedded in W^{2,q_i}, there exists $E_i > 0$ such that

$$\|u\|_{\mathcal{C}^1(\mathcal{G})} \leq E_i \max(1, \|u\|_{\mathcal{C}^1(\mathcal{G})})^{1 - \frac{1}{2p_i}},$$

which proves that $\|u\|_{\mathcal{C}^1(\mathcal{G})}$ is bounded. □

Now, we remark the following easy fact.

Lemma 5.21. *All nonzero positive solutions of* $(\mathcal{P}_{p,1})$ *with* $p > 2$ *satisfy* $\|u\|_{L^\infty(\mathcal{G})} \geq 1$.

Proof. It suffices to note that

$$\gamma_1 \|u\|_{L^2}^2 \leq \|u'\|_{L^2}^2 + \lambda \|u\|_{L^2}^2 = \gamma_1 \|u\|_{L^p}^p \leq \gamma_1 \|u\|_{L^2}^2 \|u\|_{L^\infty}^{p-2}.$$ □

We now turn to the proof of Theorem 5.9.

Proof of Theorem 5.9. Let $(u_{p_n})_n \subseteq H_Z^1(\mathcal{G})$ be a sequence of positive solutions associated to $(\mathcal{P}_{p,1})$ with $p = p_n$ where $(p_n)_n \subseteq (2,3)$ converges to 2.

According to Theorem 5.18, $(u_{p_n})_n$ is bounded in $H_Z^1(\mathcal{G})$. Up to extracting a subsequence, we may assume that $u_{p_n} \rightharpoonup u_*$ for some $u_* \in H_Z^1(\mathcal{G})$.

Using Proposition 5.4, we deduce that $u_* \in E_1$ is a solution of the reduced problem. According to Lemma 5.21 and since weak H^1 convergence implies L^∞ convergence, we deduce that $\|u_*\|_{L^\infty} \geq 1$. In particular, u_* is nonzero.

Since $\dim E_1 = 1$, u_* is the unique positive function in $\mathcal{N}_{*,1}$. In conclusion, the weak limit is independent of the sequence of solutions $(u_{p_n})_n \subseteq H_Z^1(\mathcal{G})$. Using Proposition 5.4 again, we deduce that u_{p_n} converges to u_* in H.

Moreover, u_* belongs to S since the first eigenfunction is positive at the interior of edges of $\mathcal{G}$. Thus, u_* is a nondegenerate critical point of $\mathcal{J}_*$.

Now, if there exists another sequence $(\tilde{u}_{p_n})_n \subseteq H_Z^1(\mathcal{G})$ of positive solutions with $\tilde{u}_{p_n} \neq u_{p_n}$ for all n, it would also converge to u_* in H, contradicting the local uniqueness of solutions in the neighborhood of $(2, u_*)$ shown by Theorem 5.8. □

Remark 5.22. If the set Z of Dirichlet vertices is empty and $\lambda > 0$, then $\gamma_1 = \lambda$ and the unique positive solution of $(\mathcal{P}_{p,1})$ is the constant function equal to one.

5.2.5 Variational formulation of the reduced problem

First considerations

Given $k \geq 1$, we consider the *reduced functional* $\mathcal{J}_{*,k} : E_k \to \mathbb{R}$ defined by

$$\mathcal{J}_{*,k}(\varphi) := \frac{1}{4} \int_{\mathcal{G}} \varphi^2(x)\Big(1 - 2\ln|\varphi(x)|\Big)\,\mathrm{d}x. \tag{5.24}$$

Using Lemma 5.11 with the $\mathcal{C}^1(\mathbb{R})$ function $f(s) := s^2(1 - 2\ln|s|)$, we deduce that $\mathcal{J}_{*,k}$ is $\mathcal{C}^1$ on[17] E_k with

$$\mathcal{J}'_{*,k}(\varphi)[\psi] = -\int_{\mathcal{G}} \varphi(x)\ln|\varphi(x)|\psi(x)\,\mathrm{d}x \tag{5.25}$$

for all $\varphi, \psi \in E_k$.

From this expression, we deduce that an eigenfunction $\varphi_* \in E_k$ is a solution of the reduced problem on E_k if and only if it is a critical point of $\mathcal{J}_{*,k}$.

Using Lemma 5.15 with $h(s) = s$ and the previous expression for $\partial_\varphi \mathcal{J}_{*,k}$, we deduce that $\mathcal{J}_{*,k}$ is of class $\mathcal{C}^2$ on $E_k \cap S$, with second differential given by

$$\mathcal{J}''_{*,k}(\varphi)[\psi_1, \psi_2] = -\int_{\mathcal{G}} (1 + \ln|\varphi(x)|)\psi_1(x)\psi_2(x)\,\mathrm{d}x.$$

From this, we deduce (recalling Remark 5.6) that an eigenfunction $\varphi_* \in E_k \cap S$ is a nondegenerate solution of the reduced problem on E_k if and only if it is a nondegenerate critical point of the reduced functional $\mathcal{J}_{*,k}$.

The reduced Nehari manifold and its geometry[18]

All critical points of $\mathcal{J}_{*,k}$ belong to *the reduced Nehari manifold* $\mathcal{N}_{*,k}$ defined by

$$\mathcal{N}_{*,k} := \Big\{\varphi \in E_k \setminus \{0\} \;\Big|\; \mathcal{J}'_{*,k}(\varphi)[\varphi] = 0\Big\} = \Big\{u \in E_k \setminus \{0\} \;\Big|\; \mathcal{S}_{*,k}(\varphi) = 0\Big\} \tag{5.26}$$

where

$$\mathcal{S}_{*,k} : E_k \to \mathbb{R} : \varphi \mapsto -\int_{\mathcal{G}} \varphi^2 \ln|\varphi|\,\mathrm{d}x. \tag{5.27}$$

Using Lemma 5.11 with $f(s) = -s^2\ln|s|$, we deduce that $\mathcal{S}_{*,k}$ is $\mathcal{C}^1$ on E_k, with differential given by

$$\mathcal{S}'_{*,k}(\varphi)[\psi] = -\int_{\mathcal{G}} (2\ln|\varphi| + 1)\varphi\psi\,\mathrm{d}x. \tag{5.28}$$

[17]Note that E_k is finite dimensional, so that all norms on E_k are equivalent. Here, one may for instance use the norm from H and apply Lemma 5.11.

[18]This section is inspired on [294, Section 2.2] and reuses its notations.

Moreover, for all $\varphi \in \mathcal{N}_{*,k}$, we have that

$$\mathcal{S}'_{*,k}(\varphi)[\varphi] = -\int_{\mathcal{G}} \varphi^2 \,\mathrm{d}x < 0.$$

Thus, $\nabla \mathcal{S}_{*,k}$ does not vanish on $\mathcal{N}_{*,k}$, so that $\mathcal{N}_{*,k}$ is a $\mathcal{C}^1$-manifold in E_k. Moreover, the tangent space to $\mathcal{N}_{*,k}$ at a point φ is given by

$$T_\varphi \mathcal{N}_{*,k} = \Big\{\psi \in E_k \mid \mathcal{S}'_{*,k}(\varphi)[\psi] = 0\Big\}. \tag{5.29}$$

Using Lemma 5.15 with $h(s) = s^2$, we deduce that $\mathcal{S}_{*,k}$ is $\mathcal{C}^2$ on S. Therefore, if $\varphi \in \mathcal{N}_{*,k}$ belongs to S, $\mathcal{N}_{*,k}$ is locally given by a $\mathcal{C}^2$-manifold around φ. This will allow to compute Hessian matrices for critical points belonging to S, which we will do when studying the tetrahedron graph in section 5.4.

For any $\varphi \in E_k \setminus \{0\}$, the ray $\{t\varphi | t > 0\}$ intersects $\mathcal{N}_{*,k}$ at a single point $n_*(\varphi)\varphi$ maximizing $t \mapsto \mathcal{J}_{*,k}(t\varphi)$. This value can be computed explicitly and is given by

$$n_*(\varphi) = \exp\Big(-\frac{\int_{\mathcal{G}} \varphi^2 \ln|\varphi| \,\mathrm{d}x}{\int_{\mathcal{G}} \varphi^2 \,\mathrm{d}x}\Big). \tag{5.30}$$

We define a natural projection map by

$$\pi_{\mathcal{N}_{*,k}} : E_k \setminus \{0\} \to \mathcal{N}_{*,k} : \varphi \mapsto n_*(\varphi)\varphi. \tag{5.31}$$

Then, we can show that

$$\pi_{\mathcal{N}_{*,k}} : \Big\{\psi \in E_k \mid |\varphi|_{E_k} = 1\Big\} \to \mathcal{N}_{*,k} : \varphi \mapsto \pi_{\mathcal{N}_{*,k}}(\varphi)$$

is a $\mathcal{C}^1$-diffeomorphism between the unit sphere of E_k and $\mathcal{N}_{*,k}$.

The reduced functional $\mathcal{J}_{*,k}$ takes a particularly simple form when restricted to the reduced Nehari manifold. Indeed, for all $\varphi \in \mathcal{N}_{*,k}$, equations (5.24) and (5.26) imply that

$$\mathcal{J}_{*,k}(\varphi) = \frac{1}{4}\int_{\mathcal{G}} \varphi^2(x) \,\mathrm{d}x. \tag{5.32}$$

Finally, let us mention a result about tangent spaces to $\mathcal{N}_{*,k}$ at critical points of $\mathcal{J}_{*,k}$.

Proposition 5.23. *If $\varphi \in \mathcal{N}_{*,k}$ is a critical point of $\mathcal{J}_{*,k}$, then*

$$T_\varphi \mathcal{N}_{*,k} = \Big\{\psi \in E_k \mid (\psi \mid \varphi)_{L^2} = 0\Big\}.$$

Proof. This follows directly from the expressions of $\mathcal{J}'_{*,k}$ and of $\mathcal{N}_{*,k}$, see the equations (5.25), (5.29) and (5.28). □

5.2.6 A result about eigenfunctions vanishing on edges

Even if we have to exclude solutions vanishing identically on some edges when using the Lyapunov-Schmidt reduction (see the hypotheses of Theorem 5.8), we are able to prove that they are *local minima of $\mathcal{J}_{*,k}$ on $\mathcal{N}_{*,k}$* in many situations.

Theorem 5.24. *Let $\varphi_* \in \mathcal{N}_{*,k} \setminus S$ be a critical point of $\mathcal{J}_{*,k}$.*

Let $E_0 \subseteq \mathcal{G}$ be the subset of $\mathcal{G}$ on which φ_* vanishes identically. Let us assume that if $\varphi \in E_k$ is not a multiple of φ_*, then the restriction of φ to E_0 does not vanish identically.

Then, φ_* is a strict local minimum of $\mathcal{J}_{*,k}$ on $\mathcal{N}_{*,k}$.

Proof. Using Proposition 5.23, we deduce that

$$T_{\varphi_*} \mathcal{N}_{*,k} = \Big\{\psi \in E_k \;\Big|\; (\psi \mid \varphi_*)_{L^2(\mathcal{G})} = 0\Big\} =: \langle\varphi_*\rangle^\perp.$$

We have to show that for every $\psi \in \langle\varphi_*\rangle^\perp$, small enough[19], we have

$$\mathcal{J}_*\Big(n_*(\varphi_* + \psi)\,(\varphi_* + \psi)\Big) > \mathcal{J}_*(\varphi_*),$$

where n_* is defined by (5.30). Using equation (5.32), we have

$$\mathcal{J}_*\Big(n_*(\varphi_* + \psi)(\varphi_* + \psi)\Big) = \Big(n_*(\varphi_* + \psi)\Big)^2\Big(\|\varphi_*\|^2_{L^2(\mathcal{G})} + \|\psi\|^2_{L^2(\mathcal{G})}\Big).$$

The claim follows if we prove that $n_*(\varphi_* + \psi) \geq 1$ for every $\psi \in \langle\varphi_*\rangle^\perp$ small enough. Recalling (5.30), this amounts to show that

$$\mathcal{S}(\varphi_* + \psi) = -\int_{\mathcal{G}} (\varphi_* + \psi)^2 \ln|\varphi_* + \psi| \,\mathrm{d}x$$

satisfies $\mathcal{S}(\varphi_* + \psi) \geq 0$ when ψ is small enough.

[19] Where the choice of the norm does not matter since E_k is finite dimensional.

Since φ_* vanishes identically on E_0, we deduce that for all $\psi \in \langle\varphi_*\rangle^\perp$, we have $\mathcal{S}(\varphi_* + \psi) = \mathcal{S}_1(\psi) + \mathcal{S}_2(\psi)$, where

$$\mathcal{S}_1(\psi) := -\int_{\mathcal{G}\setminus E_0} (\varphi_* + \psi)^2 \ln|\varphi_* + \psi| \, \mathrm{d}x,$$
$$\mathcal{S}_2(\psi) := -\int_{E_0} \psi^2 \ln|\psi| \, \mathrm{d}x.$$

As φ_* belongs to $\mathcal{N}_{*,k}$ and vanishes identically on E_0, we have

$$\mathcal{S}_1(0) = -\int_{\mathcal{G}\setminus E_0} (\varphi_*)^2 \ln|\varphi_*| \, \mathrm{d}x = -\int_{\mathcal{G}} (\varphi_*)^2 \ln|\varphi_*| \, \mathrm{d}x = \mathcal{S}(\varphi_*) = 0. \tag{5.33}$$

Moreover, using (5.28) and recalling that $\mathcal{J}'_{*,k}(\varphi_*) = 0$, we have

$$\mathcal{S}_1'(0)[\psi] = -\int_{\mathcal{G}\setminus E_0} \varphi_*(2\ln|\varphi_*| + 1)\psi \, \mathrm{d}x = -\int_{\mathcal{G}} \varphi_*\psi \, \mathrm{d}x = 0. \tag{5.34}$$

Since φ_* does not vanish identically on any edge of $\mathcal{G} \setminus E_0$, using Lemma 5.15 as in page 305, we deduce that the map $E_k \to \mathbb{R} : \psi \mapsto \mathcal{S}_1(\psi)$ is $\mathcal{C}^2$ in a neighborhood of 0. Using (5.33) and (5.34), we deduce that there exists $C > 0$ so that, for all $\psi \in E_k$ small enough, we have

$$|\mathcal{S}_1(\psi)| \leq C\|\psi\|_H^2. \tag{5.35}$$

Now, we remark that the map $\langle\varphi_*\rangle^\perp \to [0, +\infty) : \psi \mapsto \|\psi\|_{L^2(E_0)}$ defines a norm on $\langle\varphi_*\rangle^\perp$. Indeed, the hypotheses ensure that, if $\psi \in \langle\varphi_*\rangle^\perp$ is such that $\|\psi\|_{L^2(E_0)} = 0$, then $\psi = 0$. Since all norms on a finite dimensional vector space are equivalent, we deduce that there exists $D > 0$ so that

$$D\|\psi\|_H^2 \leq \|\psi\|_{L^2(E_0)}^2 \tag{5.36}$$

for every $\psi \in \langle\varphi_*\rangle^\perp$. Finally, if $\psi \in \langle\varphi_*\rangle^\perp$ is small enough, we can ensure that $\|\psi\|_{L^\infty(\mathcal{G})} \leq e^{-C/D}$. In this case, we obtain using (5.35) and (5.36) that

$$\mathcal{S}(\varphi_* + \psi) = \mathcal{S}_1(\psi) - \int_{E_0} \psi^2 \ln|\psi| \, \mathrm{d}x \geq -C\|\psi\|_H^2 + \frac{C}{D}\|\psi\|_{L^2(E_0)}^2 \geq 0,$$

which ends the proof. □

5.2.7 Asymptotic behavior of nodal ground states as $p \to 2$

To conclude this section on "general" results on compact graphs, let us present a theorem describing the behavior of nodal ground states[20] of $(\mathcal{P}_{p,2})$ as $p \approx 2$.

[20]Namely, the minima of the action functional $\mathcal{J}_{p,k}(u) := \frac{1}{2}\|u'\|_{L^2(\mathcal{G})}^2 + \frac{\lambda}{2}\|u\|_{L^2(\mathcal{G})}^2 - \frac{\gamma_2}{p}\|u\|_{L^p(\mathcal{G})}^p$ constrained to its associated nodal Nehari set.

Theorem 5.25. *If* $(p_n)_n \subseteq (2, +\infty)$ *is a sequence converging to* 2 *and* $(u_{p_n})_n$ *is an associated sequence of nodal ground states for* $(\mathcal{P}_{p_n,2})$*, then it is bounded in* $H^1_Z(\mathcal{G})$*. Moreover, if* $(u_{p_n})_n$ *converges weakly to* $u_* \in H^1_Z(\mathcal{G})$*, then* u_* *belongs to* E_2*, is nonzero, is a solution of the reduced problem, minimizes* $\mathcal{J}_{*,2}$ *over* $\mathcal{N}_{*,2}$ *and the convergence of* $(u_{p_n})_n$ *to* u *holds in* H*.*

Proof. All claims can be proved as in [74, Section 4], except the convergence in H which follows using Proposition 5.4. □

Remark 5.26. Of course, up to suitable rescalings, the theorem above also describes the behavior of nodal ground states of $(\mathrm{NLS}_{\mathcal{G},Z})$ as $p \to 2$.

5.3 Compact star graphs

In this section, we will study in detail a family of examples, compact star graphs with Dirichlet conditions. Namely, given an integer $m \geq 2$ and positive numbers $L_1 \geq L_2 \geq L_3 \geq L_4 \geq \cdots \geq L_m$, we consider a metric graph $\mathcal{G}_\mathrm{s}$ made of a node v_0 to which are attached m edges $e_1, \ldots, e_m$ of lengths $L_1, \ldots, L_m$ ending at Dirichlet vertices $v_1, \cdots, v_m$ (see Figure 5.2).

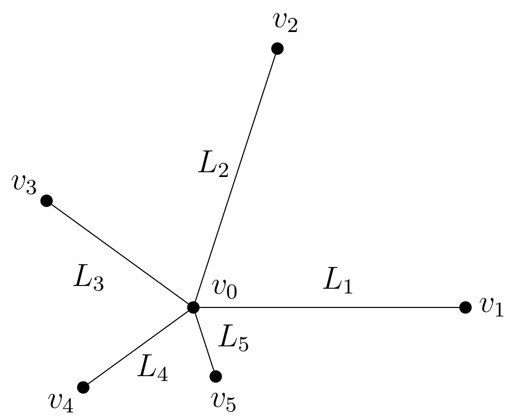


Figure 5.2: Graph $\mathcal{G}_\mathrm{s}$ when $m = 5$

We will study the ground state and the nodal ground state of our problem on this graph. Namely, we consider the problem

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on each edge } e_1, \ldots, e_m, \\ u \text{ is continuous} & \text{at } v_0, \\ \displaystyle\sum_{1\leq i\leq m} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0) = 0, & \\ u(v_i) = 0 & \text{for every } 1 \leq i \leq m. \end{cases} \qquad (\mathrm{NLS}_{\mathcal{G}_\mathrm{s}})$$

As understood in the previous section, the first point consists in the study of the first two eigenvalues of[21]

$$\begin{cases} -u'' = \gamma u & \text{on each edge } e_1, \dots, e_m, \\ u \text{ is continuous} & \text{at } v_0, \\ \displaystyle\sum_{1\le i\le m} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0) = 0, & \\ u(v_i) = 0 & \text{for every } 1 \le i \le m. \end{cases} \tag{$\mathrm{Spec}_{\mathcal{G}_\mathrm{s}}$}$$

Subsequently, we will examine ground states and nodal ground states in this case. We recall that

$$\begin{aligned} \mathcal{J}_{\lambda,\mathcal{G}_\mathrm{s}}(u) &:= \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G}_\mathrm{s})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G}_\mathrm{s})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G}_\mathrm{s})}, \\ \mathcal{N}_\lambda(\mathcal{G}_\mathrm{s}) &:= \Big\{ u \in H^1(\mathcal{G}_\mathrm{s}) \setminus \{0\} \;\Big|\; \|u'\|^2_{L^2(\mathcal{G}_\mathrm{s})} + \lambda\|u\|^2_{L^2(\mathcal{G}_\mathrm{s})} = \|u\|^p_{L^p(\mathcal{G}_\mathrm{s})}, \\ &\qquad\qquad u(v_i) = 0 \text{ for all } i \in \{1, \dots, m\} \Big\}, \\ \mathcal{N}^{nod}_\lambda(\mathcal{G}_\mathrm{s}) &:= \Big\{ u \in H^1(\mathcal{G}_\mathrm{s}) \mid u^\pm \in \mathcal{N}_\lambda(\mathcal{G}_\mathrm{s}) \Big\}, \\ H &:= \Big\{ u \in H^2(\mathcal{G}_\mathrm{s}) \mid \sum_{1\le i\le m} \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0) \text{ and } u(v_i) = 0 \text{ for every } 1 \le i \le m \Big\}. \end{aligned}$$

The space H is a Hilbert space when equipped with the $H^2(\mathcal{G}_\mathrm{s})$-norm.

5.3.1 The two first eigenvalues and eigenspaces of $(\mathrm{Spec}_{\mathcal{G}_\mathrm{s}})$

Proposition 5.27.

- *The first eigenvalue γ_1 of $(\mathrm{Spec}_{\mathcal{G}_\mathrm{s}})$ belongs to $\big(0, (\pi/L_1)^2\big)$.*

  Moreover, the corresponding eigenfunction can be chosen positive on $\mathcal{G}_\mathrm{s}$.
- *If $L_1 > L_2$, the second eigenvalue γ_2 belongs to $\big((\pi/L_1)^2, (\pi/L_2)^2\big)$ and the associated eigenspace has dimension 1.*

  The corresponding eigenfunction has a single point of e_1 as nodal set.
- *If there exists $i > 1$ such that $L_1 = \cdots = L_i > L_{i+1}$, the second eigenvalue of $(\mathrm{Spec}_{\mathcal{G}_\mathrm{s}})$ is given by $\gamma_2 = (\pi/L_1)^2$ and the associated eigenspace has dimension $i - 1$.*

  Moreover, the corresponding eigenfunctions are identically equal to 0 on all the edges e_j for $j > i$.

[21] We may take $\lambda = 0$ in this section since λ simply "shifts" the spectrum without changing the eigenfunctions.

Proof. Considering on each edge the representation of the solution with the origin on the vertex v_i for $1 \le i \le m$ (i.e. the vertex with the Dirichlet condition), we see that the solutions of $(\mathrm{Spec}_{\mathcal{G}_s})$ are given by $u_i(x) = \alpha_i \sin(\sqrt{\gamma}\, x)$ for $1 \le i \le m$ where $\gamma, \alpha_1, \ldots, \alpha_m$ satisfy

$$\begin{cases} \alpha_1 \sin(\sqrt{\gamma}\, L_1) = \cdots = \alpha_m \sin(\sqrt{\gamma}\, L_m), \\ \displaystyle\sum_{i=1}^{m} \alpha_i \cos(\sqrt{\gamma}\, L_i) = 0. \end{cases} \tag{5.37}$$

Step 1. First eigenvalue. Let us observe that the function f defined by

$$f(x) := \sum_{i=1}^{m} \cot(x\, L_i)$$

is continuous, decreasing on $(0, \pi/L_1)$ and such that

$$\lim_{x\to 0^+} f(x) = +\infty \quad \text{and} \quad \lim_{x\to(\pi/L_1)^-} f(x) = -\infty.$$

This implies the existence of $\gamma_1 \in \Big(0, (\pi/L_1)^2\Big)$ such that

$$\sum_{i=1}^{m} \cot\Big(\sqrt{\gamma_1}\, L_i\Big) = 0.$$

Hence, choosing $\alpha_i = \frac{1}{\sin(\sqrt{\gamma_1}\, L_i)}$, for $1 \le i \le m$, we have a nontrivial solution of (5.37). We then conclude that γ_1 is the first eigenvalue of $(\mathrm{Spec}_{\mathcal{G}_s})$ and that the corresponding eigenfunction can be chosen positive on $\mathcal{G}_s$.

Step 2. Second eigenvalue if $L_1 > L_2$.

As f has a unique root in $(0, \pi/L_1)$, the second eigenvalue satisfies

$$\gamma_2 \ge (\pi/L_1)^2.$$

Let us observe that if $L_1 > L_2$, $\gamma = (\pi/L_1)^2$ is not an eigenvalue.

Indeed, if $(\alpha_1, \cdots, \alpha_m)$ is a nontrivial solution of (5.37) with $\gamma = (\pi/L_1)^2$ then, as $\sin(\sqrt{\gamma}\, L_1) = 0$ and, for all $2 \le i \le m$, $\sin(\sqrt{\gamma}\, L_m) > 0$, the first condition in (5.37) implies that $\alpha_i = 0$ for all $2 \le i \le m$. As $(\alpha_1, \cdots, \alpha_m)$ is nontrivial, this contradicts the second condition of (5.37).

Now, let us observe that the function f defined in the first step is continuous, decreasing on $(\pi/L_1, \min(\pi/L_2, 2\pi/L_1))$ and such that

$$\lim_{x\to(\pi/L_1)^+} f(x) = +\infty$$

and

$$\lim_{x\to(\min(\pi/L_2, 2\pi/L_1))^-} f(x) = -\infty.$$

This implies the existence of $\gamma_2 \in \Big((\pi/L_1)^2, (\min(\pi/L_2, 2\pi/L_1)^2\Big)$ such that

$$\sum_{i=1}^{m} \cot\Big(\sqrt{\gamma_2}\, L_i\Big) = 0,$$

$\sin(\sqrt{\gamma_2}\, L_1) < 0$ and, for $2 \le i \le m$, $\sin(\sqrt{\gamma_2}\, L_i) > 0$. Hence, by choosing $\alpha_i = \frac{1}{\sin(\sqrt{\gamma_2}\, L_i)}$, for $1 \le i \le m$, we have a nontrivial solution of (5.37) and the result in that case can be easily deduced.

Step 3. Second eigenvalue if there exists $i > 1$ such that $L_1 = \cdots = L_i > L_{i+1}$.

In that case, $\gamma_2 = (\pi/L_1)^2$ with $\alpha_j = 0$ for $j > i$ and $(\alpha_1, \ldots, \alpha_i)$ solution of $\alpha_1 + \ldots + \alpha_i = 0$ are solutions of (5.37). Hence, the second eigenvalue is given by $\gamma_2 = (\pi/L_1)^2$, the eigenspace is of dimension $i - 1$ with eigenfunctions identically equal to 0 on all the edges e_j for $j > i$. □

5.3.2 Ground states and nodal ground states on intervals

In this section, we give some useful preliminary results on the problem set on a simple interval (see Proposition 5.28). To this end, we consider the functional

$$\mathcal{J}_\lambda : H_0^1(0, L) \to \mathbb{R} : u \mapsto \mathcal{J}_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(0,L)} + \frac{\lambda}{2}\|u\|^2_{L^2(0,L)} - \frac{1}{p}\|u\|^p_{L^p(0,L)}$$

and the two sets

$$\mathcal{N}_\lambda(0, L) := \Big\{u \in H_0^1(0, L) \setminus \{0\} \mid \|u'\|^2_{L^2(0,L)} + \lambda\|u\|^2_{L^2(0,L)} = \|u\|^p_{L^p(0,L)}\Big\},$$

$$\mathcal{N}_\lambda^{nod}(0, L) := \Big\{u \in H_0^1(0, L) \mid u^\pm \in \mathcal{N}_\lambda(0, L)\Big\}.$$

Proposition 5.28. *Let λ, p and L be three real numbers such that $\lambda > 0$, $p > 2$ and $L > 0$. Then, the boundary value problem*

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on } (0, L), \\ u(0) = u(L) = 0 \end{cases} \tag{5.38}$$

has (up to sign) a unique solution u with two nodal zones. Moreover,

1. *u is the nodal ground state of the problem;*
2. *u is such that $u(L/2) = 0$;*
3. *u is the unique positive solution of (5.38) on $[0, L/2]$ and $-u$ is the unique positive solution of (5.38) on $[L/2, L]$;*
4. *one has $\inf_{\mathcal{N}_\lambda^{nod}(0,L)} \mathcal{J}_\lambda = 2 \inf_{\mathcal{N}_\lambda(0,L/2)} \mathcal{J}_\lambda$.*

Proof. Let u be a solution of (5.38) with two nodal zones. Up to replacing u by $-u$, we have that $u'(0) > 0$. Then, u has a unique root $r \in (0, L)$, by assumption on the number of nodal domains. By uniqueness of the solution to the Cauchy problem

$$\begin{cases} -v'' + av = |v|^{p-2}v, \\ v(r) = 0, v'(r) = u'(r), \end{cases}$$

we must have that $r = L/2$ since the first positive root of the solution to this Cauchy problem must be equal to both r and $L - r$ as the ODE is autonomous and as the nonlinearity is odd. Now, the restrictions of $|u|$ to $[0, L/2]$ and $[L/2, L]$ are positive solutions of (5.38) on their respective intervals. As those solutions are unique (see Proposition C.10 in Appendix C), we deduce that solutions with two nodal zones are also unique up to sign. Since nodal ground states exist as we work on a compact interval and since nodal ground states have two nodal domains[22], the proof is complete. □

Another useful property is the following.

Proposition 5.29. *Let $a, p \in \mathbb{R}$ be so that $a > 0$, $p > 2$. Then, the map*

$$\mathcal{J}_\lambda : (0, +\infty) \to (s_\lambda, +\infty) : L \mapsto \mathcal{J}_\lambda(L) := \inf_{\mathcal{N}_\lambda(0,L)} \mathcal{J}_\lambda,$$

is decreasing, continuous, bijective, convex and such that

$$\lim_{L\to 0} \mathcal{J}_\lambda(L) = +\infty \quad \textit{and} \quad \lim_{L\to+\infty} \mathcal{J}_\lambda(L) = s_\lambda.$$

Proof. We show these properties one by one.

Step 1. $\mathcal{J}_\lambda$ is decreasing. Let $0 < L < L'$. Then, one can extend the ground state u in $\mathcal{N}_\lambda(0, L)$ by zero to obtain a function $\bar{u}$ in $\mathcal{N}_\lambda(0, L')$, showing that

$$\mathcal{J}_\lambda(L') = \inf_{\mathcal{N}_\lambda(0,L')} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(\bar{u}) = \mathcal{J}_\lambda(u) = \inf_{\mathcal{N}_\lambda(0,L)} \mathcal{J}_\lambda = \mathcal{J}_\lambda(L).$$

Moreover, the inequality is strict since a ground state in $\mathcal{N}_\lambda(0, L')$ is nonzero inside $(0, L')$, which is not the case of $\bar{u}$. Hence, $\mathcal{J}_\lambda(L') < \mathcal{J}_\lambda(L)$.

Step 2. $\mathcal{J}_\lambda$ is continuous. Let $0 < L < L'$. If u is a ground state in $\mathcal{N}_\lambda(0, L')$, then the function $v(x) := u(L'x/L)$ belongs to $H_0^1(0, L)$ and one has

$$\|v'\|_2^2 = \frac{L'}{L}\|u'\|_2^2, \qquad \|v\|_2^2 = \frac{L}{L'}\|u\|_2^2, \qquad \|v\|_p^p = \frac{L}{L'}\|u\|_p^p.$$

[22]SeeTheorem 2.38.

The projection factor of v on $\mathcal{N}_\lambda(0,L)$ is given by

$$n_\lambda(v) = \left(\frac{\|v'\|_2^2 + \lambda\|v\|_2^2}{\|v\|_p^p}\right)^{\frac{1}{p-2}} = \left(\frac{\frac{(L')^2}{L^2}\|u'\|_2^2 + \lambda\|u\|_2^2}{\|u\|_p^p}\right)^{\frac{1}{p-2}}.$$

Since $\|u'\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p$, we conclude that $n_\lambda(v)$ is close to 1 if L and L' are close enough, so that

$$\mathcal{J}_\lambda(L) \leq \mathcal{J}_\lambda(n_\lambda(v)v) = \mathcal{J}_\lambda(u) + o(1) = \mathcal{J}_\lambda(L') + o(1).$$

Since by Step 1, we have $\mathcal{J}_\lambda(L') < \mathcal{J}_\lambda(L)$, we deduce the continuity of $\mathcal{J}_\lambda$.

Step 3. Limit of $\mathcal{J}_\lambda$ as $L \to +\infty$. If L is long enough, we can truncate solitons to obtain a function $v \in \mathcal{N}_\lambda(0,L)$ with $\mathcal{J}_\lambda(v)$ close to s_λ. This proves that

$$\lim_{L\to+\infty} \mathcal{J}_\lambda(L) = \inf_{\mathcal{N}_\lambda(\mathbb{R})} \mathcal{J}_\lambda = s_\lambda.$$

Step 4. Limit of $\mathcal{J}_\lambda$ as $L \to 0$. For any function $u \in H_0^1(0,L)$ and any $x \in [0,L]$, we have

$$|u(x)| \leq \int_0^x |u'(t)|\,\mathrm{d}t \leq \sqrt{x}\|u'\|_2 \leq \sqrt{L}\|u'\|_2,$$

using the Cauchy-Schwarz inequality. Therefore,

$$\|u\|_\infty \leq \sqrt{L}\|u'\|_2.$$

Now, if $u \in \mathcal{N}_\lambda(0,L)$, we have that

$$\|u'\|_2^2 + \lambda\|u\|_2^2 = \|u\|_p^p \leq \|u\|_2^2\|u\|_\infty^{p-2} \leq \frac{L^{\frac{p}{2}-1}}{a}\Big(\|u'\|_2^2 + \lambda\|u\|_2^2\Big)^p,$$

so that

$$\|u'\|_2^2 + \lambda\|u\|_2^2 \geq \left(aL^{\frac{2-p}{2}}\right)^{\frac{1}{p-2}} = a^{\frac{1}{p-2}}L^{-\frac{1}{2}}.$$

Since this inequality is valid for any $u \in \mathcal{N}_\lambda(0,L)$ and that the right hand side converges to $+\infty$ as $L \to 0^+$, we deduce that

$$\lim_{L\to 0} \mathcal{J}_\lambda(L) = \lim_{L\to 0} \kappa(\|u'\|_2^2 + \lambda\|u\|_2^2) = +\infty.$$

Step 5. $\mathcal{J}_\lambda$ is a bijection from $(0,+\infty)$ to $(s_\lambda,+\infty)$. This follows from all the previous points.

Step 6. Convexity. Let L, $\tilde{L} > 0$ and consider $u \in \mathcal{N}_\lambda(0,L)$ and $\tilde{u} \in \mathcal{N}_\lambda(0,\tilde{L})$ be the positive ground states on the corresponding intervals. Define $v \in H_0^1(0,L+\tilde{L})$ by

$$v(x) := \begin{cases} u(x) & \text{for } x \in [0,L], \\ -\tilde{u}(x-L) & \text{for } x \in [L, L+\tilde{L}]. \end{cases}$$

Then, v belongs to $\mathcal{N}_\lambda^{nod}(0,L+\tilde{L})$. Using Proposition 5.28, we then have

$$2\mathcal{J}_\lambda\Big(\frac{L+\tilde{L}}{2}\Big) = \inf_{\mathcal{N}_\lambda^{nod}(0,L+\tilde{L})} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(v) = \mathcal{J}_\lambda(u) + \mathcal{J}_\lambda(\tilde{u}) = \mathcal{J}_\lambda(L) + \mathcal{J}_\lambda(\tilde{L}).$$

As $\mathcal{J}_\lambda$ is continuous, this implies that it is convex (see e.g. [196]). □

5.3.3 Ground States of $(\mathrm{NLS}_{\mathcal{G}_s})$ – Symmetry breaking

Now, let us turn to the results regarding ground states that can be deduced from the abstract results presented in the first sections.

Proposition 5.30. *Let $\lambda > 0$ and $\gamma_1 \in \left(0, (\pi/L_1)^2\right)$ be such that*

$$\sum_{i=1}^{m} \cot(\sqrt{\gamma_1}\, L_i) = 0$$

Consider the function φ_1 defined on $\mathcal{G}_s$ by $\varphi_1|_{e_i}(x) := \varphi_{1,i}(x) := \frac{\sin(\sqrt{\gamma_1}\, x)}{\sin(\sqrt{\gamma_1}\, L_i)}$, for $i \in \{1, \cdots, m\}$, and the constant

$$k_1 = \exp\left(-\frac{\sum_{i=1}^m \int_0^{L_i} \varphi_{1,i}^2(x) \ln(\varphi_{1,i}(x))\, \mathrm{d}x}{\sum_{i=1}^m \int_0^{L_i} \varphi_{1,i}^2(x)\, \mathrm{d}x}\right).$$

For all $p > 2$, close to 2, let us consider a positive solution u_p of $(\mathrm{NLS}_{\mathcal{G}_s})$.

Then, we have:

1. *u_p is the unique positive solution of $(\mathrm{NLS}_{\mathcal{G}_s})$ and it is the ground state of $(\mathrm{NLS}_{\mathcal{G}_s})$;*
2. *$(\gamma_1 + \lambda)^{-\frac{1}{p-2}} u_p \to k_1 \varphi_1$ in H as $p \to 2$;*
3. *if $L_i = L_j$, for some i, j, then $u_{p,i} = u_{p,j}$ where $u_{p,i}$ denotes the restriction of u_p on e_i.*

Proof. The first item can be deduced from Theorem 5.9.

Let us observe that $k_1\varphi_1$ is a solution of the reduced problem on E_1. The second item can then be deduced from Proposition 5.4, Theorem 5.18 and Lemma 5.21 as in the proof of Theorem 5.9.

To conclude, let us remark that if $L_i = L_j$, by exchanging $u_{p,i}$ and $u_{p,j}$, we define a new positive solution $\tilde{u}_p$ of $(\mathrm{NLS}_{\mathcal{G}_s})$. By uniqueness of the positive solution, we conclude that $\tilde{u}_p = u_p$, which proves the result. □

So we have proved in particular that, for a compact star graph with m edges of the same length L, for L fixed, if p is small enough, the solution is the same on each edge. In the next result, we will prove a symmetry breaking result as we show that, for p fixed, if L is large, the solution is not symmetric anymore.

Proposition 5.31. (Symmetry breaking) *Let us consider the compact star graph $\mathcal{G}_s$ when $m \geq 3$ and $L_1 = \ldots = L_m = L$. For any $p > 2$, if L is large enough, then the ground state on $\mathcal{G}_s$ is **not** symmetric.*

Proof. If L is large enough, we can truncate solitons on an edge of length L to obtain a function $v \in \mathcal{N}_\lambda(0, L)$ with $\mathcal{J}_\lambda(v) < \frac{3}{2} s_\lambda$, where s_λ is the action level of the soliton on the real line (see Appendix C, in particular Proposition C.4). This shows that, when L is large enough, then $\inf_{\mathcal{N}_\lambda(\mathcal{G}_s)} \mathcal{J}_\lambda(u) < \frac{3}{2} s_\lambda$.

Now, if u is a positive symmetric solution of the problem, with the Dirichlet condition at vertices of degree one, then u has m preimages for almost every value in its range. Using Proposition 1.10, we deduce that $\mathcal{J}_\lambda(u) \geq \frac{3}{2} s_\lambda$. This shows that ground states are not symmetric. □

5.3.4 Nodal Ground States of $(\mathrm{NLS}_{\mathcal{G}_s})$.

Let us first prove the results regarding nodal ground states that can be deduced from the abstract results presented in the first sections.

Proposition 5.32. *Let $\lambda > 0$. Let us consider the compact star graph $\mathcal{G}_s$ with $L_1 > L_2$. We define $\gamma_2 \in \big((\pi/L_1)^2, (\pi/L_2)^2\big)$ as the solution of*

$$\sum_{i=1}^{m} \cot(\sqrt{\gamma_2}\, L_i) = 0.$$

Let us consider the function φ_2 defined on $\mathcal{G}_s$ by $\varphi_2|_{e_i}(x) := \varphi_{2,i}(x) = \frac{\sin(\sqrt{\gamma_2}\, x)}{\sin(\sqrt{\gamma_2}\, L_i)}$, for $i \in \{1, \ldots, m\}$. We define the constant

$$k_2 := \exp\left(- \frac{\sum_{i=1}^{m} \int_0^{L_i} \varphi_{2,i}^2(x) \ln |\varphi_{2,i}(x)| \,\mathrm{d}x}{\sum_{i=1}^{m} \int_0^{L_i} \varphi_{2,i}^2(x) \,\mathrm{d}x} \right).$$

For all $p > 2$, close to 2, let us denote by v_p the nodal ground state solution of $(\mathrm{NLS}_{\mathcal{G}_s})$. Then,

1. *there exists a neighborhood U of $(2, k_2\varphi_2)$ in $[2, +\infty) \times H$ and a number $\varepsilon > 0$ such that, for all $p \in [2, 2+\varepsilon]$, there exists a unique $u_p \in H$ so that $\big(p, (\gamma_2 + \lambda)^{-\frac{1}{p-2}} u_p\big)$ belongs to U and u_p is a solution of $(\mathrm{NLS}_{\mathcal{G}_s})$;*
2. *$(\gamma_2 + \lambda)^{-\frac{1}{p-2}} v_p \to \pm k_2 \varphi_2$ in H as $p \to 2$;*
3. *v_p has a single point of e_1 as nodal set.*

Proof. Let us observe that $k_2\varphi_2$ is a solution of the reduced problem on E_2 which is nondegenerate since $\dim(E_2) = 1$. The first item can then be deduced from Theorem 5.8. Moreover, the second idem is a consequence of Theorem 5.25.

For the third idem, by the convergence of second idem, denoting by x_0 the only root of φ_2 in e_1, we know that, for $\epsilon > 0$ small enough and for p close enough to 2, $v_p(x_0-\epsilon) < 0 < v_p(x_0+\epsilon)$. As v_p has two nodal zones[23], this implies that the nodal zone of v_p is localized in the interior of e_1 and that $v_p^{-1}(\{0\})$ is either a point or an interval. By uniqueness of the solution of a Cauchy problem, we conclude that $v_p^{-1}(\{0\})$ is a point since otherwise, we have the contradiction $v_p \equiv 0$ on e_1. □

When $L_1 = L_2$, the situation is more complicated. Indeed, in this case, the second eigenfunctions are identically equal to 0 on all edges with $L_i < L_1$. Then, by Theorem 5.25, $(\gamma_2 + a)^{-\frac{1}{p-2}} u_p \to \tilde{\varphi}_2$ in H as $p \to 2$ with $\tilde{\varphi}_2 \notin S$.

The aim of the rest of this section is to prove that, if $L_1 > L_2$ then the nodal ground state satisfies $u(v_0) \neq 0$ for all $p > 2$ and that, if $L_1 = L_2$ is long enough, the nodal ground state is identically equal to 0 on all edges except the two longest ones.

To this end, let us now consider two important lemmas, both valid on general compact graphs.

Lemma 5.33 (The cutting Lemma)**.** *Let $\mathcal{G}$ be a compact metric graph and u be a solution of* (NLS$_{\mathcal{G},Z}$)*. Let us assume that u is nonzero and is monotone inside $k \geq 1$ edges $e_1, \ldots, e_k$ ending at a Dirichlet vertex. Let us denote $\tilde{\mathcal{G}} = \mathcal{G} \setminus (e_1 \cup \cdots \cup e_k)$. Then,*

$$\mathcal{J}_{\lambda,\mathcal{G}}(u) > \inf_{v \in \mathcal{N}_\lambda(\tilde{\mathcal{G}})} \mathcal{J}_{\lambda,\tilde{\mathcal{G}}}(v).$$

Moreover, if u is a nodal solution, then one has

$$\mathcal{J}_{\lambda,\mathcal{G}}(u) > \inf_{v \in \mathcal{N}_\lambda^{nod}(\tilde{\mathcal{G}})} \mathcal{J}_{\lambda,\tilde{\mathcal{G}}}(v).$$

Proof. Since u belongs to $\mathcal{N}_\lambda(\mathcal{G})$, we have

$$\int_{\mathcal{G}} |u'|^2 \,\mathrm{d}x + \lambda \int_{\mathcal{G}} |u|^2 \,\mathrm{d}x = \int_{\mathcal{G}} |u|^p \,\mathrm{d}x. \tag{5.39}$$

Given an edge e_i with $1 \leq i \leq k$, we parametrize e_i by $[0, L_i]$, with 0 corresponding to the (degree one) Dirichlet vertex and L_i to the other vertex of e_i. Integrating the equality $(-u'' + \lambda u)u = |u|^p$ over $[0, L_i]$, we obtain

$$[-u'u]_0^{L_i} + \int_0^{L_i} |u'|^2 \,\mathrm{d}x + \lambda \int_0^{L_i} |u|^2 \,\mathrm{d}x = \int_0^{L_i} |u|^p \,\mathrm{d}x.$$

[23] See Theorem 2.38.

The boundary term is equal to $-u'(L_i)u(L_i)$, which is nonpositive since u is monotone inside e_i. In other words, one has

$$\int_{e_i} |u'|^2 \,\mathrm{d}x + \lambda \int_{e_i} |u|^2 \,\mathrm{d}x \geq \int_{e_i} |u|^p \,\mathrm{d}x. \tag{5.40}$$

Using (5.39) and adding the inequalities (5.40) for all $1 \leq i \leq k$, we deduce that

$$\int_{\tilde{\mathcal{G}}} |u'|^2 \,\mathrm{d}x + \lambda \int_{\tilde{\mathcal{G}}} |u|^2 \,\mathrm{d}x \leq \int_{\tilde{\mathcal{G}}} |u|^p \,\mathrm{d}x.$$

Denoting by $\tilde{u} : \tilde{\mathcal{G}} \to \mathbb{R}$ the restriction of u to $\tilde{\mathcal{G}}$, we obtain

$$n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}) := \left(\frac{\int_{\tilde{\mathcal{G}}} |u'|^2 \,\mathrm{d}x + \lambda \int_{\tilde{\mathcal{G}}} |u|^2 \,\mathrm{d}x}{\int_{\tilde{\mathcal{G}}} |u|^p \,\mathrm{d}x} \right)^{\frac{1}{p-2}} \leq 1.$$

Now, we remark that

$$\begin{aligned} \mathcal{J}_{\lambda,\mathcal{G}}(u) = \kappa \|u\|^p_{L^p(\mathcal{G})} &> \kappa \|\tilde{u}\|^p_{L^p(\tilde{\mathcal{G}})} \\ &\geq \kappa \|n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u})\tilde{u}\|^p_{L^p(\tilde{\mathcal{G}})} \\ &= \mathcal{J}_{\lambda,\tilde{\mathcal{G}}}(n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u})\tilde{u})(\tilde{u}) \\ &\geq \inf_{v \in \mathcal{N}_\lambda(\tilde{\mathcal{G}})} \mathcal{J}_{\lambda,\tilde{\mathcal{G}}}(v), \end{aligned}$$

which proves the first inequality of Lemma 5.33.

If u is a nodal solution of $(\mathrm{NLS}_{\mathcal{G},Z})$, we may repeat the previous computations with $u^\pm$ and show that $n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}^\pm) \leq 1$ (note that $\tilde{u}^+$ and $\tilde{u}^-$ are nonzero since u does not change sign inside $e_1, \ldots, e_k$). Therefore, in the same way, we deduce,

$$\begin{aligned} \mathcal{J}_{\lambda,\mathcal{G}}(u) &= \kappa \|u^+\|^p_{L^p(\mathcal{G})} + \kappa \|u^-\|^p_{L^p(\mathcal{G})} \\ &> \kappa \|\tilde{u}^+\|^p_{L^p(\tilde{\mathcal{G}})} + \kappa \|\tilde{u}^-\|^p_{L^p(\tilde{\mathcal{G}})} \\ &\geq \kappa \left\| n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}^+)\tilde{u}^+ \right\|^p_{L^p(\tilde{\mathcal{G}})} + \kappa \left\| n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}^-)\tilde{u}^- \right\|^p_{L^p(\tilde{\mathcal{G}})} \\ &\geq \inf_{v \in \mathcal{N}^{nod}_\lambda(\tilde{\mathcal{G}})} \mathcal{J}_{\lambda,\tilde{\mathcal{G}}}(v), \end{aligned}$$

since the function $n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}^+)\tilde{u}^+ + n_{\lambda,\tilde{\mathcal{G}}}(\tilde{u}^-)\tilde{u}^-$ belongs to $\mathcal{N}^{nod}_\lambda(\tilde{\mathcal{G}})$. □

Lemma 5.34. *Let $\mathcal{G}$ be a compact metric graph and u be a one-signed function belonging to $\mathcal{N}_\lambda(\mathcal{G})$. Let us assume that u vanishes at some point of $\mathcal{G}$. Then,*

$$\mathcal{J}_\lambda(u) > \frac{s_\lambda}{2}.$$

Proof. Up to replacing u by $-u$, we may assume that $u \geq 0$. Then, rearranging u in a decreasing way, we obtain a function $\tilde{u}$ that can be extended by 0 to an $H^1(0,+\infty)$ function such that

$$\|\tilde{u}'\|_2^2 \leq \|u'\|_2^2, \qquad \|\tilde{u}\|_2^2 = \|u\|_2^2, \qquad \|\tilde{u}\|_p^p = \|u\|_p^p.$$

Thus,

$$n_\lambda(\tilde{u}) := \left(\frac{\|\tilde{u}'\|_2^2 + \lambda\|\tilde{u}\|_2^2}{\|\tilde{u}\|_p^p}\right)^{\frac{1}{p-2}} \leq 1,$$

so that

$$\mathcal{J}_\lambda(u) = \kappa\|u\|_p^p = \kappa\|\tilde{u}\|_p^p \geq \kappa\|n_\lambda(\tilde{u})\tilde{u}\|_p^p = \mathcal{J}_\lambda(n_\lambda(\tilde{u})\tilde{u}) > \inf_{\mathcal{N}_\lambda(0,+\infty)} \mathcal{J}_\lambda \geq \frac{s_\lambda}{2}$$

where the strict inequality holds since the infimum of $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda(0,+\infty)$ is achieved only by half-solitons[24], and that $n_\lambda(\tilde{u})\tilde{u}$ is not a half-soliton since it vanishes at a point. □

The following lemma describes which are the solutions with two nodal zones vanishing at the central vertex of $\mathcal{G}_\mathrm{s}$.

Lemma 5.35. *Let $\lambda > 0$, $p > 2$ be two real numbers. Let us consider the compact star graph $\mathcal{G}_\mathrm{s}$.*

If a solution u of $(\mathrm{NLS}_{\mathcal{G}_\mathrm{s}})$ *vanishes at v_0 and has two nodal zones, then necessarily it vanishes identically on all edges except two edges e_i and e_j with $1 \leq i < j \leq m$ such that $L_i = L_j$.*

Conversely, given two edges e_i and e_j with $1 \leq i < j \leq m$ such that $L_i = L_j$, there exists a unique solution u of $(\mathrm{NLS}_{\mathcal{G}_\mathrm{s}})$ *with two nodal zones, vanishing at v_0, positive inside e_i and negative inside e_j.*

Moreover, the action of such a solution u is so that

$$\mathcal{J}_\lambda(u) = 2\mathcal{J}_\lambda(L_i)$$

with $\mathcal{J}_\lambda(L)$ defined in Proposition 5.29.

Proof. If u vanishes at v_0 and has two nodal zones, we remark that u cannot vanish identically on all edges but one since otherwise Kirchhoff's condition would imply that u also vanishes identically on the remaining edge.

Since u has two nodal zones, u is identically equal to zero on all edges but two, say e_i and e_j. On those edges, $|u|$ is a positive solution of (5.38). Now, Kirchhoff's condition implies that

$$\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0) + \frac{\mathrm{d}u}{\mathrm{d}x_{e_j}}(v_0) = 0.$$

[24]Which are the only $H^1(0,+\infty)$ solutions of the ODE on the half-line with the Neumann boundary condition at 0, see Proposition C.2.

Without loss of generality, let us assume that $\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0) > 0$. The restrictions of $|u|$ to e_i and to e_j are solutions of the Cauchy problem

$$\begin{cases} -w'' + aw = |w|^{p-2}w, \\ w(0) = 0,\ w'(0) = \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(v_0), \end{cases}$$

so that $L_i = L_j$ by uniqueness of the first positive root of the solution to this Cauchy problem.

Finally, Proposition C.10 implies that[25] $\mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(L_i) + \mathcal{J}_\lambda(L_j) = 2\mathcal{J}_\lambda(L_i)$.

Conversely, if two edges e_i and e_j have the same length, one can join them to form an interval of length $2L_i$ and construct a solution by putting the nodal ground state of (5.38) on their union and 0 on all other edges. The uniqueness of such a solution follows from Proposition 5.28. □

We then have the following result.

Proposition 5.36. *Let $a > 0$, $p > 2$ be two real numbers. Let us consider the compact star graph $\mathcal{G}_\mathrm{s}$. Then,*

$$\inf_{\mathcal{N}_\lambda^{nod}(\mathcal{G}_\mathrm{s})} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(L_1) + \mathcal{J}_\lambda(L_2).$$

Moreover, if $L_1 > L_2$, then no nodal ground states of the problem vanish at v_0.

Proof. Let $u : \mathcal{G}_\mathrm{s} \to \mathbb{R}$ be defined by putting a positive solution of (5.38) over e_1, a negative solution of (5.38) over e_2, and zero over all other edges. Then, one easily checks that u belongs to $\mathcal{N}_\lambda^{nod}(\mathcal{G}_\mathrm{s})$, so that

$$\inf_{\mathcal{N}_\lambda^{nod}(\mathcal{G}_\mathrm{s})} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(L_1) + \mathcal{J}_\lambda(L_2).$$

If we assume, moreover, that $L_1 > L_2$, then

$$\inf_{\mathcal{N}_\lambda^{nod}(\mathcal{G}_\mathrm{s})} \mathcal{J}_\lambda < 2\mathcal{J}_\lambda(L_2) \tag{5.41}$$

since $\mathcal{J}_\lambda$ is decreasing.

Now, let us assume that u is a nodal ground state of ($\mathrm{NLS}_{\mathcal{G}_\mathrm{s}}$) vanishing at v_0. Then, u has two nodal zones and Lemma 5.35 implies that $\mathcal{J}_\lambda(u) = 2\mathcal{J}_\lambda(L)$ where L is the length of the two edges on which u does not vanish identically.

Since $L_1 > L_2$, we have that $L \leq L_2$ so that

$$\inf_{\mathcal{N}_\lambda^{nod}(\mathcal{G}_\mathrm{s})} \mathcal{J}_\lambda = \mathcal{J}_\lambda(u) = 2\mathcal{J}_\lambda(L) \geq 2\mathcal{J}_\lambda(L_2)$$

which contradicts (5.41). □

[25] Remark that the ground state solution of the problem exists on compact intervals $(0, L)$ for all $L > 0$. Its uniqueness for a given L follows from the uniqueness of positive solutions. and since a ground state is (up to a factor of -1) a positive solution.

If the two longest edges have the same length, then one can show that nodal ground states do vanish at the central node assuming that those edges are long enough.

Theorem 5.37. *Let $a > 0$, $p > 2$ and $\bar{L} > 0$ be real numbers so that*

$$\mathcal{J}_\lambda(\bar{L}) = \frac{5}{4} s_\lambda.$$

Let us consider the compact star graph $\mathcal{G}_s$ when $m \geq 3$ and $L_1 = L_2 \geq \bar{L}$.

Then, all nodal ground states of ($\mathrm{NLS}_{\mathcal{G}_s}$) *vanish identically on all edges of the graph but two. Those two edges have length L_1. Moreover, on them, the solution coincides with a one-signed solution of* (5.38) *on* $[0, L_1]$*. In particular, all nodal ground states of* ($\mathrm{NLS}_{\mathcal{G}_s}$) *vanish at the central node v_0.*

Proof. Let u be a nodal ground state of ($\mathrm{NLS}_{\mathcal{G}_s}$). By Proposition 5.36 and, as $\mathcal{J}_\lambda$ is decreasing, we have

$$\mathcal{J}_\lambda(u) \leq 2\mathcal{J}_\lambda(L_1). \tag{5.42}$$

If $u(v_0) = 0$ then, by Lemma 5.35, u must vanish identically on all edges except two which have the same length L. Moreover, equality $\mathcal{J}_\lambda(u) = 2\mathcal{J}_\lambda(L)$ holds. As $\mathcal{J}_\lambda$ is decreasing, (5.42) implies that $L = L_1$, ending the proof when $u(v_0) = 0$.

It thus remains to prove that $u(v_0) = 0$. We assume by contradiction that $u(v_0) \neq 0$. Since $u(v_0) \neq 0$, u is not identically equal to zero inside any edge. Up to replacing u by $-u$, we assume that $u(v_0) > 0$. We now distinguish two cases.

Case 1. The nodal ground state u is monotone inside at least $m - 2$ edges. Since $u(v_0) > 0$, u is increasing inside edges $e_{i_1}, \dots, e_{i_{m-2}}$ where the $m - 2$ indices $i_1, \dots, i_{m-2}$ are all different. In this case, applying Lemma 5.33 to the edges $e_{i_1}, \dots, e_{i_{m-2}}$ gives

$$\mathcal{J}_\lambda(u) > \inf_{v \in \mathcal{N}_\lambda^{nod}\left(\mathcal{G}_s \setminus (e_{i_1} \cup \cdots \cup e_{i_{m-2}})\right)} \mathcal{J}_\lambda(v).$$

Now, we have $\mathcal{G}_s \setminus (e_{i_1} \cup \cdots \cup e_{i_{m-2}}) = e_j \cup e_k$ where j and k are the indices of the two remaining edges of $\mathcal{G}_s$.

Since $e_j \cup e_k$ is an interval of length $L_j + L_k$, applying Proposition 5.28 and recalling that $\mathcal{J}_\lambda$ is decreasing, we obtain the inequalities

$$\mathcal{J}_\lambda(u) > \inf_{v \in \mathcal{N}_\lambda^{nod}(e_j \cup e_k)} \mathcal{J}_\lambda(v) = 2\mathcal{J}_\lambda\Big(\frac{L_j + L_k}{2}\Big) \geq 2\mathcal{J}_\lambda(L_1),$$

which contradicts (5.42).

Case 2. The nodal ground state u is not monotone inside at least three edges. We denote

$$I := \Big\{ i \in \{1, \dots, m\} \mid u \text{ is not monotone inside } e_i \Big\} = \{i_1, \dots, i_k\}.$$

Let us remark that our assumption is that $\#I = k \geq 3$. For any $i \in I$, we call x_i the critical point of u closest to the vertex v_i (see Figure 5.3).

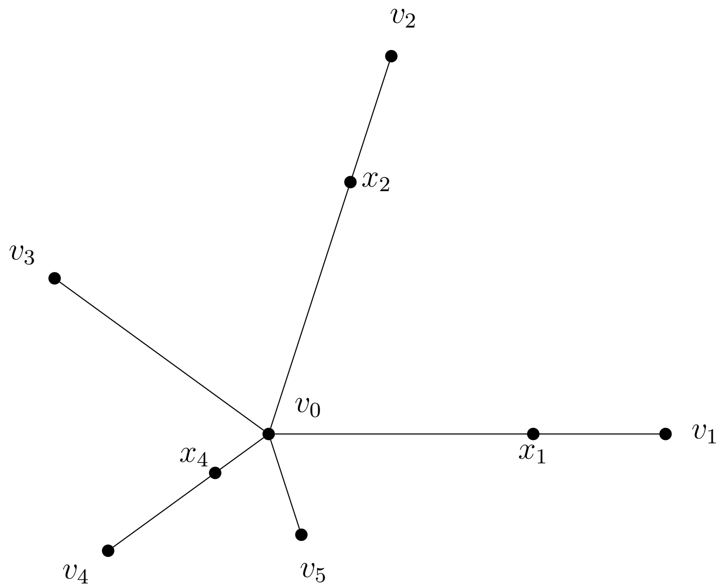


Figure 5.3: The critical points of u: x_1, x_2, x_4 in this example.

For any $i \in I$, we denote by u_i the restriction of u to the interval $[v_i, x_i] \subseteq e_i$. Let us note that for every $i \in I$, u_i belongs to $\mathcal{N}_\lambda([v_i, x_i])$ as it is a portion of a solution to the ODE between a Neumann vertex and a Dirichlet vertex.

Let us also remark that none of the functions u_i change sign inside their domains. Since u changes sign, the graph $\tilde{\mathcal{G}}_{\mathrm{s}} := \mathcal{G}_{\mathrm{s}} \setminus \cup_{i \in I} [v_i, x_i]$ is nonempty. Calling $\tilde{u}$ the restriction of u to $\tilde{\mathcal{G}}_{\mathrm{s}}$, we remark that $\tilde{u}$ belongs to $\mathcal{N}_\lambda^{nod}(\tilde{\mathcal{G}}_{\mathrm{s}})$.

The $k+2$ functions $\tilde{u}^+, \tilde{u}^-, u_{i_1}, \dots, u_{i_k}$ all satisfy the hypotheses of Lemma 5.34 as they are all one-signed, belong to the Nehari manifolds of their respective[26] domains and all vanish at some point. Using Lemma 5.34, we deduce that

$$\mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(\tilde{u}^+) + \mathcal{J}_\lambda(\tilde{u}^-) + \mathcal{J}_\lambda(u_{i_1}) + \dots + \mathcal{J}_\lambda(u_{i_k}) \geq \frac{k+2}{2} s_\lambda > \frac{5}{2} s_\lambda,$$

as $k \geq 3$. Since L_1 is larger than $\bar{L}$ and $\mathcal{J}_\lambda$ is decreasing, we deduce that

$$\mathcal{J}_\lambda(L_1) < \frac{5}{4} s_\lambda.$$

[26]This may be seen by integrating both sides of the equality $(-u'' + \lambda u)u = |u|^p$ and taking into account the suitable Dirichlet, Neumann or Kirchhoff conditions depending on the cases.

Recalling (5.42), this leads to the contradiction

$$\frac{5}{2}s_\lambda \geq 2\mathcal{J}_\lambda(L_1) \geq \mathcal{J}_\lambda(u) > \frac{5}{2}s_\lambda. \qquad \square$$

Example 5.38. **("Symmetry" breaking of nodal ground states (depending on lengths))**

Let $\lambda > 0$ and $p > 2$ be two real numbers. Let $\bar{L} > 0$ be such that

$$\mathcal{J}_\lambda(\bar{L}) = \frac{5}{4}s_\lambda.$$

Let $\mathcal{G}_\mathrm{s}$ be a compact star graph consisting of three edges e_1, e_2 and e_3 of lengths L_1, L_2 and L_3 respectively (see Figure 5.4) ending at Dirichlet vertices. We assume that $L_1 = L_2 = L$.

Then, Proposition 5.36 and Theorem 5.37 imply that:

- if $L_3 > L$, then nodal ground states do not vanish at v_0;
- if $L_3 \leq L$ and if $L \geq \bar{L}$, then all nodal ground states vanish at v_0.

In particular, if $L \geq \bar{L}$ is fixed, then:

- all nodal ground states vanish at v_0 if $L_3 \leq L$;
- none of them vanish at v_0 if $L_3 > L_1$.

Therefore, there is a threshold on L_3 which determines whether the nodal ground states vanish at v_0 or not.

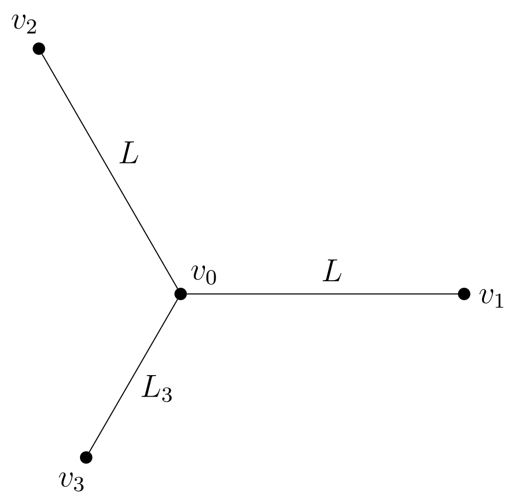


Figure 5.4: Symmetry breaking of nodal ground states

Let us now turn to the study of another compact metric graph: the *tetrahedron*.

5.4 The tetrahedron

In this final section, we are interested in the behavior of nodal ground states[27] of $(\mathrm{NLS}_{\mathcal{G},Z})$ as $p \to 2$ on the tetrahedron graph $\mathcal{G}_\mathrm{t}$ depicted in Figure 5.5. It is made of four vertices v_0, v_1, v_2, v_3 joined by six edges v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3, all of unit length.

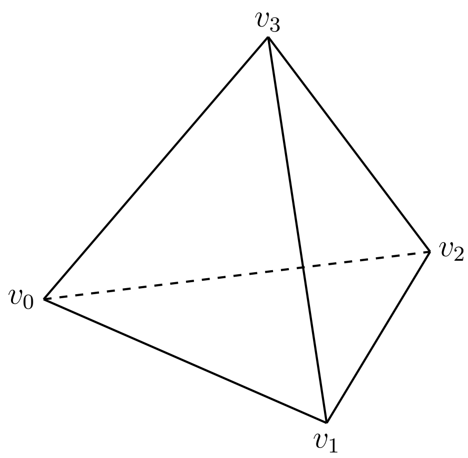


Figure 5.5: The tetrahedron graph $\mathcal{G}_\mathrm{t}$, an equilateral graph made of four vertices v_0, v_1, v_2, v_3 and six edges of length one.

We will parametrize the edges by $[0,1]$, going from the vertex with a lower index to the one with a bigger index. In this way, functions on the tetrahedron correspond to sextuplets[28] of functions from $[0,1]$ to $\mathbb{R}$, indexed by ij with $0 \le i < j \le 3$. There are no vertices having degree one, thus no Dirichlet vertices (i.e. $Z = \varnothing$).

The first step of our approach is to study the spectral problem on the graph in order to determine the second eigenspace E_2.

5.4.1 Determination of the second eigenspace

The spectral problem on the graph is given by[29]

$$\begin{cases} -u'' = k^2 u & \text{on every edge } e \text{ of } \mathcal{G}_\mathrm{t}, \\ u \text{ is continuous} & \text{at every vertex } v \text{ of } \mathcal{G}_\mathrm{t}, \\ \displaystyle\sum_{e \succ v} \frac{\mathrm{d}u}{\mathrm{d}x_e}(v) = 0 & \text{at every vertex } v \text{ of } \mathcal{G}_\mathrm{t}, \end{cases} \tag{$\mathrm{Spec}_{\mathcal{G}_\mathrm{t}}$}$$

where $\gamma = k^2$ is the eigenvalue (taking $k \ge 0$).

[27] And more generally to solutions converging to eigenfunctions of E_2 as $p \to 2$.

[28] Namely, vectors having 6 components.

[29] We take $\lambda = 0$ in this section since λ simply “shifts” the spectrum without changing the eigenfunctions.

Since the graph is compact and has no Dirichlet vertices, the first eigenvalue is equal to zero and the first eigenspace is the space of constant functions.

To determine the second eigenvalue, we need to find the smallest $k > 0$ for which there exist nonzero solutions to ($\mathrm{Spec}_{\mathcal{G}_t}$). A method to compute the spectra of *equilateral* efficiently graphs was developed independently by S. Nicaise [251, 253] and J. von Below [57] (see also [264] and [68, Section 3.6]).

Let us first assume that $k \notin \pi\mathbb{Z}$. In this case, the key observation is to remark that for any $a, \tilde{a} \in \mathbb{R}$, the boundary value problem

$$\begin{cases} -u'' = k^2 u & \text{on } (0,1), \\ u(0) = a, \ u(1) = \tilde{a} \end{cases}$$

has a unique solution given by the function

$$x \mapsto \frac{a\sin(k(1-x)) + \tilde{a}\sin(kx)}{\sin(k)}.$$

Thus, assuming that $k \notin \pi\mathbb{Z}$, the spectral problem amounts to find all quadruplets $a = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ of possible values that the eigenfunctions can take at the vertices. Given $a \in \mathbb{R}^4$, and $k \in \mathbb{R} \setminus \pi\mathbb{Z}$, we define $\varphi_{a,k}$ by

$$\varphi_{a,k}(x) = \frac{a_i \sin(k(1-x)) + a_j \sin(kx)}{\sin(k)} \tag{5.43}$$

if x belongs to the edge[30] $v_i v_j$ $(0 \le i < j \le 3)$. Its derivative is given by

$$\varphi'_{a,k}(x) = k\frac{-a_i \cos(k(1-x)) + a_j \cos(kx)}{\sin(k)}.$$

Kirchhoff's condition at vertex v_i leads to four equations, given by

$$-3a_i \cos(k) + \sum_{\substack{0 \le j \le 3 \\ i \ne j}} a_j = 0, \quad 0 \le i \le 3. \tag{5.44}$$

Adding these four equalities together leads to

$$(a_0 + a_1 + a_2 + a_3)(1 - \cos(k)) = 0, \quad 0 \le i \le 3.$$

Since $\sin(k) \ne 0$, $1 - \cos(k)$ is nonzero. We deduce that $a_0 + a_1 + a_2 + a_3 = 0$. Plugging this last equality into (5.44) leads to

$$a_i(3\cos(k) + 1) = 0.$$

If $3\cos(k) + 1 \ne 0$, all a_i are equal to zero and we do not find any eigenfunctions. Thus necessarily $\cos(k) = -1/3$. Conversely, if $\cos(k) = -1/3$, then all conditions (5.44) rewrite as $a_0 + a_1 + a_2 + a_3 = 0$.

[30]We recall that the edge $v_i v_j$ is parametrized by $[0, 1]$, going from v_i to v_j.

In conclusion, we have found eigenvalues $\Big(\arccos(-1/3)+2\pi\ell\Big)^2$ with $\ell \in \mathbb{Z}^{\geq 0}$ and $\Big(2\pi\ell - \arccos(-1/3)\Big)^2$ with $\ell \in \mathbb{Z}^{\geq 1}$. All those eigenvalues $\gamma > 0$ correspond to eigenspaces of dimension three. All other possible eigenvalues of the problem are necessarily of the form $\pi^2 l^2$ with $\ell \in \mathbb{Z}^{\geq 1}$, and are in particular larger than $\Big(\arccos(-1/3)\Big)^2$.

To summarize, the second eigenvalue of the problem is $\gamma_2 = \Big(\arccos(-1/3)\Big)^2$ and the second eigenspace is given by (recall (5.43))

$$E_2 = \Big\{\varphi_a \;\Big|\; a = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4, a_0 + a_1 + a_2 + a_3 = 0\Big\},$$

where $\varphi_a := \varphi_{a,\,\arccos(-1/3)}$ is such that

$$\varphi_{a,k_2}(x) = \frac{a_i \sin(k_2(1-x)) + a_j \sin(k_2 x)}{\sin(k_2)}, \quad k_2 := \arccos(-1/3), \tag{5.45}$$

if x belongs to the edge $v_i v_j$. In particular, E_2 has dimension three.

5.4.2 Symmetries

Symmetries of the tetrahedron

Let us present the symmetry group of the tetrahedron. We will do so in two different manners, one more geometrical, the other more combinatorial.

- Consider the tetrahedron depicted in Figure 5.5 as a solid polyhedron in $\mathbb{R}^3$. Determining its symmetries amounts to determining how many rotations[31] in $\mathbb{R}^3$ send the tetrahedron to itself. It can be seen[32] that there are twelve such rotations (there are four choices for the vertex chosen as the “top” and three choices of orientation for the lower face of the tetrahedron). If we also allow reflections, then the number of symmetries is doubled, so that there are 24 symmetries in total (see [32, Exercise 1.6]). Even though those symmetries came from our geometric intuition in $\mathbb{R}^3$, they all correspond to symmetries of the metric graph $\mathcal{G}_\mathrm{t}$, whose vertices and edges are those of the tetrahedron.
- More combinatorially, the graph $\mathcal{G}_\mathrm{t}$ is the complete graph with four vertices[33]. It is clear that in such a graph, “all vertices are the same”. Here, the natural symmetry group is the *permutation group* S_4, corresponding to all bijections of a set of four elements. Namely,

$$S_4 := \Big\{\sigma : \{0,1,2,3\} \to \{0,1,2,3\} \mid \sigma \text{ is a bijection}\Big\}.$$

[31]Where the origin of the rotation in $\mathbb{R}^3$ is let free.
[32]A pedagogical explanation is given in [32, Chapter 1].
[33]I.e. the graph with four vertices all linked to one another by an edge.

Given a permutation $\sigma \in S_4$, it sends the vertex v_i to the vertex $v_{\sigma(i)}$ $(0 \le i \le 3)$ and the edge v_iv_j to the edge $v_{\sigma(i)}v_{\sigma(j)}$ $(0 \le i < j \le 3)$. Let us remark that one might have $\sigma(i) > \sigma(j)$ here but, by convention, we will identify the edge v_jv_i with v_iv_j for all $0 \le i < j \le 3$.

In what follows, it will be more convenient for us to use the combinatorial language. Nevertheless, both explanations describe the same group of symmetries[34].

Action of the symmetry group on functions

We identify functions $f : \mathcal{G}_{\mathrm{t}} \to \mathbb{R}$ with sextuplets

$$(f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}) \tag{5.46}$$

of functions giving the restrictions of f to the edges. Namely, f_{ij} is the restriction of f to v_iv_j, parametrized by $[0,1]$ so that 0 corresponds to v_i and 1 to v_j.

For convenience in what follows, let us define

$$f_{ji}(x) := f_{ij}(1-x) \tag{5.47}$$

for all $0 \le i < j \le 3$, so that f_{ji} parametrizes the edge v_iv_j going from v_j to v_i.

We introduce some definitions[35,36] (see e.g. [335, Section 1.6]).

Definition 5.39. The *action* of a group G on a vector space E is a map $G \times E \to E : (g,u) \mapsto g \cdot u$ such that the identities $1_G \cdot u = u$, $(gh)\cdot u = g\cdot(h\cdot u)$ hold and such that $u \mapsto g \cdot u$ is linear.

If E is a normed space, we say that the action is *isometric* if $\|g \cdot u\|_E = \|u\|_E$ for all $g \in G$ and all $u \in E$.

A function $J : E \to \mathbb{R}$ is *invariant* if $J(g \cdot u) = J(u)$ for every $g \in G$ and every $u \in E$.

We denote by $\mathcal{F}(\mathcal{G}_{\mathrm{t}};\mathbb{R})$ the vector space of functions from $\mathcal{G}_{\mathrm{t}}$ to $\mathbb{R}$. Two group actions are naturally defined on $\mathcal{F}(\mathcal{G}_{\mathrm{t}};\mathbb{R})$.

- The one associated with the group of symmetries of the tetrahedron, given by

$$\begin{cases} S_4 \times \mathcal{F}(\mathcal{G}_{\mathrm{t}};\mathbb{R}) & \to \mathcal{F}(\mathcal{G}_{\mathrm{t}};\mathbb{R}) \\ (\sigma, f) & \mapsto \sigma \cdot f, \end{cases}$$

  where (recalling conventions (5.46) and (5.47))

$$\sigma \cdot f := \Big(f_{\sigma(0)\sigma(1)}, f_{\sigma(0)\sigma(2)}, f_{\sigma(0)\sigma(3)}, f_{\sigma(1)\sigma(2)}, f_{\sigma(1)\sigma(3)}, f_{\sigma(2)\sigma(3)}\Big).$$

[34] See e.g. [32, Exercise 8.11].

[35] In [335, Section 1.6], one finds the definition of a *topological* group action on a *normed* vector space, requiring continuity assumptions. Here, we will consider *finite* groups with the discrete topology, and those continuity assumptions hold automatically.

[36] See also [32, Chapter 17] for a general introduction to group actions.

- One independent of the domain of functions and corresponding to the "changes of sign". It is the action of the two-element group $\{\pm 1\}$ (with the multiplication operation) given by

$$\begin{cases} \{\pm 1\} \times \mathcal{F}(\mathcal{G}_{\mathrm{t}}; \mathbb{R}) & \to \mathcal{F}(\mathcal{G}_{\mathrm{t}}; \mathbb{R}) \\ (s, f) & \mapsto s \cdot f, \end{cases}$$

where $s \cdot f := (sf_{01}, sf_{02}, sf_{03}, sf_{12}, sf_{13}, sf_{23})$.

We can combine both actions. To do so, we define the set $G_t := S_4 \times \{\pm 1\}$. It is a group if the multiplication is defined by

$$(\sigma_1, s_1) \cdot (\sigma_2, s_2) := (\sigma_1 \circ \sigma_2, s_1 s_2),$$

where $\sigma_1 \circ \sigma_2$ is the permutation given by $(\sigma_1 \circ \sigma_2)(i) := \sigma_1(\sigma_2(i))$ for every $0 \leq i \leq 3$ and where $s_1 s_2$ is the product of s_1 and s_2 as real numbers. The identity element of the group is $(\mathrm{Id}, 1)$ where Id is the identity permutation such that $\mathrm{Id}(i) = i$ for all $0 \leq i \leq 3$. Given an element $(\sigma, s) \in G_t$, its inverse is given by[37] (σ^{-1}, s), where σ^{-1} is the inverse of the permutation σ in S_4, so that $\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = \mathrm{Id}$.

The group G_t acts on $\mathcal{F}(\mathcal{G}_{\mathrm{t}}; \mathbb{R})$ in the following way:

$$\begin{cases} G_t \times \mathcal{F}(\mathcal{G}_{\mathrm{t}}; \mathbb{R}) & \to \mathcal{F}(\mathcal{G}_{\mathrm{t}}; \mathbb{R}) \\ \big((\sigma, s), f\big) & \mapsto (\sigma, s) \cdot f \end{cases} \tag{5.48}$$

where (recalling again (5.46) and (5.47))

$$(\sigma, s) \cdot f := \Big(sf_{\sigma(0)\sigma(1)}, sf_{\sigma(0)\sigma(2)}, sf_{\sigma(0)\sigma(3)}, sf_{\sigma(1)\sigma(2)}, sf_{\sigma(1)\sigma(3)}, sf_{\sigma(2)\sigma(3)}\Big).$$

In other words, one may switch the roles of some vertices and one may multiply the whole function by -1.

One easily checks that, given $(\sigma, s) \in G_t$, if f belongs to $H^1(\mathcal{G}_{\mathrm{t}})$ or to H, then so does $(\sigma, s) \cdot f$. Restricting the function space to $H^1(\mathcal{G}_{\mathrm{t}})$ or to H, we obtain an *isometric* group action (recall Definition 5.39).

Now, let us show how G_t acts on the second eigenspace. We have seen that $E_2 = \{\varphi_a \mid a \in A_0\}$ where φ_a is defined by (5.45) and

$$A_0 := \Big\{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \mid a_0 + a_1 + a_2 + a_3 = 0\Big\}.$$

We may define a group action from G_t on A_0 by

$$\begin{cases} G_t \times A_0 & \to A_0 \\ \big((\sigma, s), a\big) & \mapsto (\sigma, s) \cdot a \end{cases} \tag{5.49}$$

where $(\sigma, s) \cdot a := (sa_{\sigma(0)}, sa_{\sigma(1)}, sa_{\sigma(2)}, sa_{\sigma(3)})$.

[37] As $s^2 = 1$ since $s \in \{\pm 1\}$.

In other words, one may permute the coefficients and one may change all of their signs. One easily checks from the respective definitions that the two group actions (the one on functions defined by (5.48) and the one on A_0 defined by (5.49)) are compatible for eigenfunctions in the sense that $(\sigma, s) \cdot \varphi_a = \varphi_{(\sigma,s)\cdot a}$. Further restricting the action to E_2 provides a group action on E_2 under which the reduced functional $\mathcal{J}_{*,2}$ (see Definition 5.3) is invariant (see Definition 5.39).

Let us now introduce two further definitions (see [335, Section 1.6]).

Definition 5.40. The *space of invariant points (or fixed points)* of E under the action of a subgroup $H < G$ is given by

$$\mathrm{Fix}(H) := \Big\{u \in E \mid \forall g \in H, g \cdot u = u\Big\}.$$

Moreover, the *stabilizer* of an element $u \in E$ is given by

$$\mathrm{Stab}_u(G) := \Big\{g \in G \mid g \cdot u = u\Big\}.$$

We then have the following theorem (see [335, Theorem 1.28] and [261]).

Theorem 5.41 (Principle of symmetric criticality, Palais, 1979)**.** *Assume that the action of the topological group G on the Hilbert space E is isometric. If $J \in \mathcal{C}^1(E, \mathbb{R})$ is invariant and if u is a critical point of J restricted to* $\mathrm{Fix}(G)$*, then u is a critical point of J.*

Let us now apply this result to the action of the group[38] G_t and to the reduced functional $\mathcal{J}_* := \mathcal{J}_{*,2}$.

To better understand the role played by symmetries, let us present hereunder the possible symmetries eigenfunctions may have.

5.4.3 Symmetries of eigenfunctions

Using the isomorphism $A_0 \to E_2 : a \mapsto \varphi_a$, we deduce that

$$\mathrm{Stab}_{\varphi_a} = \Big\{(\sigma, s) \in G_t \mid (\sigma, s) \cdot a = a\Big\}.$$

Classifying the possible symmetries that an eigenfunction may have thus amounts to determining the symmetries that a quadruplet $a \in A_0$ may have. Let us perform this (slightly tedious) task. We reason combinatorially.

[38]Technically, we endow G_t with the discrete topology to obtain a continuous group action.

We recall that the permutations of S_4 have one of the following forms (see e.g. [32, Page 30, below (6.5)]):

- the identity Id (corresponding to a single element of S_4);
- a transposition (for instance $(0\,1)$, which swaps elements 0 and 1 and fixes 2 and 3). There are 6 transpositions in total;
- a 3-cycle (for instance $(0\,1\,2)$, which maps 0 to 1, 1 to 2, 2 to 0 and 4 to itself). There are 8 such 3-cycles in total;
- a 4-cycle (for instance $(0\,1\,2\,3)$, which maps 0 to 1, 1 to 2, 2 to 3 and 3 to 0). There are 6 such 4-cycles in total;
- a product of two transpositions (for instance $(0\,1)(2\,3)$, which swaps 0 and 1 and 2 and 3). There are 3 such permutations in total;

Let us now translate symmetries on the quadruplet a as equations. This will be useful in the next section.

- $\Big(\mathrm{Id}, 1\Big)$ is the identity element of G_t and is in the stabilizer of all $a \in A_0$;
- $\Big(\mathrm{Id}, -1\Big) \in \mathrm{Stab}_a$ if and only if $a = 0$;
- $\Big((0\,1), 1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = a_1$;
- $\Big((0\,1), -1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = -a_1$ and $a_2 = a_3 = 0$;
- $\Big((0\,1\,2), 1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = a_1 = a_2$, in which case $a_3 = -3a_0$ (recall that a belongs to A_0);
- $\Big((0\,1\,2), -1\Big) \in \mathrm{Stab}_a$ if and only if $(a_0, a_1, a_2, a_3) = -(a_1, a_2, a_0, a_3)$, which rewrites as $a_0 = -a_1 = a_2 = -a_0$ so that $a_0 = a_1 = a_2 = a_3 = 0$;
- $\Big((0\,1\,2\,3), 1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = a_1 = a_2 = a_3 = 0$;
- $\Big((0\,1\,2\,3), -1\Big) \in \mathrm{Stab}_a$ if and only if $(a_0, a_1, a_2, a_3) = -(a_1, a_2, a_3, a_0)$ which rewrites as $a_0 = -a_1 = a_2 = -a_3$;
- $\Big((0\,1)(2\,3), 1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = a_1$ and $a_2 = a_3$, in which case necessarily $a_0 = a_1 = -a_2 = -a_3$ since a belongs to A_0;
- $\Big((0\,1)(2\,3), -1\Big) \in \mathrm{Stab}_a$ if and only if $a_0 = -a_1$ and $a_2 = -a_3$.

Now, we will see how some of those symmetries give rise to critical points via the principle of symmetric criticality.

5.4.4 Critical points created by the symmetries

In what follows, we note simply $\mathcal{J}_* := \mathcal{J}_{*,2}$, $\mathcal{N}_* := \mathcal{N}_{*,2}$, $\pi_{\mathcal{N}_*} := \pi_{\mathcal{N}_{*,2}}$, recalling the expressions (5.24), (5.26) and (5.31).

Proposition 5.42. *The following eigenfunctions are critical points of* $\mathcal{J}_*$*:*

- $f_0 := \pi_{\mathcal{N}_*}(\varphi_{(1,-1,0,0)})$*;*
- $f_1 := \pi_{\mathcal{N}_*}(\varphi_{(1,-1/3,-1/3,-1/3)})$*;*
- $f_2 := \pi_{\mathcal{N}_*}(\varphi_{(1,1,-1,-1)})$*;*
- $f_3 := \pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})$ *where* $c \in (0,1)$ *maximizes the function*

$$[0,1] \to \mathbb{R} : c \mapsto \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})\Big).$$

Proof. We will use similar arguments for all four critical points. Each of them corresponds to the action of a specific subgroup of G_t having order two.

- Taking $H_0 = \Big\{\big(\mathrm{Id}, 1\big), \big((0\,1), -1\big)\Big\}$, we have that $\mathrm{Fix}(H_0) = \mathrm{span}\{f_0\}$. Since f_0 is a critical point of $\mathcal{J}_*$ in the one-dimensional vector space $\mathrm{Fix}(H_0)$ (as f_0 belongs to $\mathcal{N}_*$), the principle of symmetric criticality (see Theorem 5.41) implies that $\mathcal{J}_*'(f_0) = 0$.
- Taking $H_1 = \Big\{\big(\mathrm{Id}, 1\big), \big((1\,2\,3), 1\big)\Big\}$, we have that $\mathrm{Fix}(H_1) = \mathrm{span}\{f_1\}$. Since f_1 is a critical point of $\mathcal{J}_*$ in $\mathrm{Fix}(H_1)$, we have $\mathcal{J}_*'(f_1) = 0$.
- Taking $H_2 = \Big\{\big(\mathrm{Id}, 1\big), \big((0\,1)(2\,3), 1\big)\Big\}$, we have that $\mathrm{Fix}(H_2) = \mathrm{span}\{f_2\}$. Since f_2 is a critical point of $\mathcal{J}_*$ in $\mathrm{Fix}(H_2)$, we have $\mathcal{J}_*'(f_2) = 0$.
- Taking $H_3 = \Big\{\big(\mathrm{Id}, 1\big), \big((0\,1)(2\,3), -1\big)\Big\}$, we have that

$$\mathrm{Fix}(H_3) = \Big\{(a, -a, b, -b) \mid a, b \in \mathbb{R}\Big\}.$$

  This time, the vector-space $\mathrm{Fix}(H_3)$ is two dimensional. Up to the symmetries of the problem and up to multiplying vectors by a constant (which we can do since we will project vectors on $\mathcal{N}_*$ anyway), we can study vectors of the form $\varphi_{(1,-1,c,-c)}$ with $c \in [0,1]$. Let us note that both f_0 and $\tilde{f}_2 := \pi_{\mathcal{N}_*}(\varphi_{(1,-1,1,-1)})$ belong to $\mathrm{Fix}(H_3)$ and are critical points of $\mathcal{J}_*$ (for $\tilde{f}_2$, we remark that there exists $g \in \mathcal{G}_t$ such that $\tilde{f}_2 = g \cdot f_2$). This implies that the map

$$\mathbb{R} \to \mathbb{R} : c \mapsto \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})\Big) \tag{5.50}$$

  has critical points at $c = 0$ and at $c = 1$.

  It turns out that this map (see Figure 5.6 for a picture and the next section for a rigorous computer-assisted proof of this fact) is such that

$$\sup_{c\in[0,1]} \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})\Big) > \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,0,0)})\Big) > \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,1,-1)})\Big).$$

Thus, the map (5.50) has a maximum for some $c_M \in (0,1)$, that we will prove to be unique (in the next section). We define $f_3 := \pi_{\mathcal{N}_*}(\varphi_{(1,-1,c_M,-c_M)})$. Due to our reasoning by symmetry, we have

$$\mathcal{J}_*(f_3) = \max_{\psi \in \mathrm{Fix}(H_3)\setminus\{0\}} \mathcal{J}_*(\pi_{\mathcal{N}_*}(\psi)).$$

Thus, f_3 is a critical point of $\mathcal{J}_*$ in $\mathrm{Fix}(H_3)$ (since it maximizes $\mathcal{J}_*$ on the corresponding Nehari manifold), so that the principle of symmetric criticality implies that we have $\mathcal{J}_*'(f_3) = 0$. □

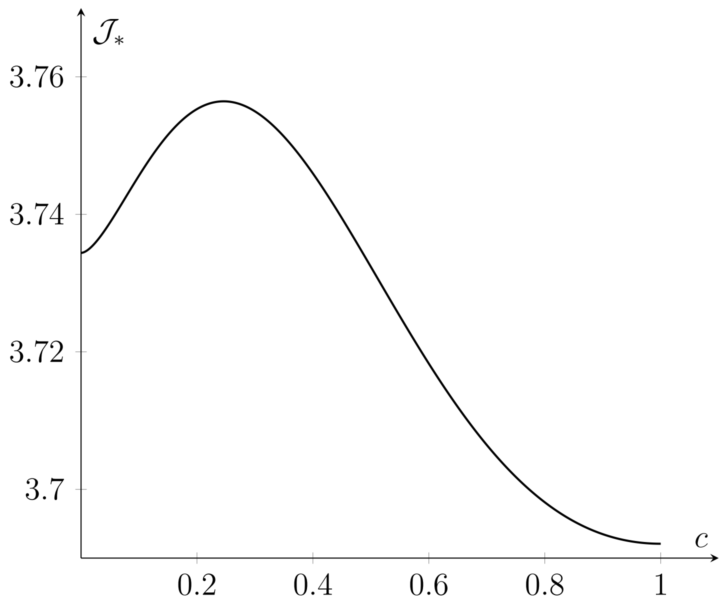


Figure 5.6: The map $c \mapsto \mathcal{J}_*(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)}))$

Remark 5.43. Of course, there are many others critical points in E_2, obtained by symmetries. Namely, the eigenfunction $g \cdot f_i$ is a critical point of $\mathcal{J}_*$ for every $g \in G_t$ and every index i such that $0 \le i \le 3$. However, those critical points are "the same as" f_0, f_1, f_2 and f_3, up to the symmetries of the problem.

Several questions naturally arise.

- *Are there other critical points, possibly ones without any symmetry?*
- *How are the critical points we obtained characterized variationally in $\mathcal{N}_*$?*
- *Are f_1, f_2 and f_3 non-degenerate?*[39]
- *Which of the critical points minimizes the action, and corresponds to limits of nodal ground states?*

[39]Our notion of nondegeneracy (see Definition 5.5) is not well-defined for f_0 since it does not belong to S.

None of those questions seem to be simple to tackle "by hand". Even if it was possible to provide answers to the questions using tedious computations, it is unclear that this would shed any insight on the understanding of the problem.

For those reasons, we decided to resort to a *computer-assisted proof* to study the functional $\mathcal{J}_*$ on E_2. How it proceeds will be detailed in the following section 5.5. First, let us state (without proof, for now) in the next proposition the results we obtain thanks to the computer-assisted proof.

Proposition 5.44 (Conclusions of the computer-assisted proof). *The action levels of the critical points we found are so that*

$$\mathcal{J}_*(f_2) < \mathcal{J}_*(f_0) < \mathcal{J}_*(f_3) < \mathcal{J}_*(f_1).$$

Moreover, f_0, f_1, f_2 and f_3 are the only nonzero critical points of $\mathcal{J}_*$ up to symmetries, in the sense that

$$\forall \varphi \in E_k \setminus \{0\}, \Big[\big(\mathcal{J}_*'(\varphi) = 0\big) \implies \big(\exists i \in \{0,1,2,3\}, \exists g \in G_t, \varphi = g \cdot f_i\big)\Big].$$

The variational characterizations of the critical points are the following:

- *f_0 is a* ***strict local minimum*** *of $\mathcal{J}_*$ on $\mathcal{N}_*$;*
- *f_1 is a strict* ***global maximum*** *of $\mathcal{J}_*$ on $\mathcal{N}_*$;*
- *f_2 is a strict* ***global minimum*** *of $\mathcal{J}_*$ on $\mathcal{N}_*$;*
- *f_3 is a* ***saddle point*** *of $\mathcal{J}_*$ on $\mathcal{N}_*$.*

Finally, f_1, f_2 and f_3 are nondegenerate.

Remark 5.45. The fact that the eigenfunction f_0 is a strict local minimum of $\mathcal{J}_*$ on $\mathcal{N}_*$ is a consequence of Theorem 5.24, taking the set E_0 equal to the edge on which f_0 vanishes identically. For the other critical points, we will resort to the computer-assisted proof in order to determine their variational characterization.

Since $\mathcal{N}_*$ is locally $\mathcal{C}^2$ around f_1, f_2 and f_3, we can consider Hessian matrices to study the types of these critical points. This makes precise the notion of saddle point: it is a point whose Hessian has one positive and one negative eigenvalue.

Using the previous proposition, we deduce the following theorem.

Theorem 5.46. *There exists $\delta > 0$ such that, for every $p \in (2, 2+\delta]$, there exists $\tilde{u}_p$, a nodal ground state of $(\mathcal{P}_{p,k})$, such that*

$$\tilde{u}_p(v_0) = \tilde{u}_p(v_1) = -\tilde{u}_p(v_2) = -\tilde{u}_p(v_3) > 0.$$

More precisely, $\tilde{u}_p$ is invariant under the action of $\Big\{\big(\mathrm{Id}, 1\big), \big((1\,2)(3\,4), 1\big)\Big\}$.

Moreover, $\tilde{u}_p$ is unique up to symmetries, in the sense that

$$\forall u_p \in H^1(\mathcal{G}_{\mathrm{t}}), \Big[u_p \text{ is a nodal ground state of } (\mathcal{P}_{p,2}) \implies \big(\exists g \in G_t, u_p = g\cdot\tilde{u}_p\big)\Big].$$

Proof. Theorem 5.25 claims that nodal ground states converge weakly and up to subsequences to global minima of $\mathcal{J}_*$ on $\mathcal{N}_*$ as $p \to 2$. Proposition 5.44 claims that there is a unique such minimizer up to symmetries, given by f_2. Since f_2 is nondegenerate, the claim follows using Theorem 5.8. □

Remark 5.47. Morse theory tells us that the *Euler-characteristic* $\chi(M)$ of a two-dimensional compact smooth manifold M is given by

$$\chi(M) = C_0(F) - C_1(F) + C_2(F),$$

where $F : M \to \mathbb{R}$ is a Morse function[40] and where for every $0 \le i \le 2$, $C_i(F)$ denotes the number of critical points of index i of F (see e.g. [242, Chapter §6] and [178, Section 5.2]). Since $\mathcal{N}_*$ is $\mathcal{C}^1$-diffeomorphic to a two-dimensional sphere, $\chi(\mathcal{N}_*) = 2$ (see e.g. [242, Example, page 39]). Now, $\mathcal{N}_*$ and $\mathcal{J}_*$ are not locally $\mathcal{C}^2$ around f_2 so that the notion of nondegeneracy is not well-defined. Let us nevertheless reason heuristically.

- Even if $\mathcal{J}_*$ is not $\mathcal{C}^2$ around f_0, it is natural to consider that f_0 has index 0 since it is a strict local minimum of $\mathcal{J}_*$.

  Now, let us count all critical points obtained from f_0 thanks to the symmetries. Namely, we have to determine the cardinality of the set $\{g \cdot f_0 \mid g \in G_t\}$. Using the correspondence between eigenfunctions and quadruplets in A_0, this amounts to[41] determine the cardinality of the set $\{g \cdot (1, -1, 0, 0) \mid g \in G_t\}$. This is the set of quadruplets containing a "1", a "−1" and two zeroes. So, there are $4 \cdot 3 = 12$ critical points corresponding to f_0.

- f_1 has index 2 since it is a global maximum of $\mathcal{J}_*$ on $\mathcal{N}_*$.

  There are $\#\{g \cdot (1, -1/3, -1/3, -1/3) \mid g \in G_t\}$ corresponding critical points, namely 8 in total (4 positions to choose where is the coordinate having absolute value 1 and 2 choices of sign).

[40] Namely, a smooth function such that all its critical points are nondegenerate.

[41] Technically, we consider positive multiples of $(1, -1, 0, 0)$ due to the projection factor on $\mathcal{N}_*$, but this does not affect the reasoning.

- f_2 has index 0 since it is a global minimum of $\mathcal{J}_*$ on $\mathcal{N}_*$.

  There are $\#\{g \cdot (1,1,-1,-1) \mid g \in G_t\}$ corresponding critical points, namely 6 in total (6 positions where the two "1s" are).
- f_3 has index 1 since it is a saddle point of $\mathcal{J}_*$ on $\mathcal{N}_*$. There are 24 corresponding critical points (there are $4! = 24$ ways to order "1", "-1", "c" and "$-c$").

Thus, $\chi(\mathcal{N}_*) = 2$, $C_0 = 12 + 6 = 18$, $C_1 = 24$ and $C_2 = 8$, so that $2 = 18 - 24 + 8$.

We believe that understanding better what happens for eigenfunctions which vanish on edges, for instance those satisfying the hypotheses of Theorem 5.24, would be very interesting. This might require to replace the "classical" Lyapunov-Schmidt method by another method better suited to our lower regularity setting.

5.5 Computer-assisted proofs[42]

Let us cite René Thom[43] (in French).

Pour moi, la mathématique, c'est la conquête du continu par le discret.

One could translate this sentence as follows.

For me, mathematics is the conquest of the continuous by the discrete.

We believe that this quote describes really well the philosophy of the computer-assisted proofs based on interval arithmetic that we will describe hereunder.

5.5.1 A few words on interval arithmetic

Floating-point computations and numerical errors

As we mentioned in the introduction (see section II.11.5), doing *rigorous proofs* based on *numerical computations* is not a trivial task.

Indeed, those computations are subject to *two* sources of numerical errors.

1. Computations have to be discretized so that the computer only performs a *finite number of steps*. Working with finer discretizations allows to improve the precision, but never to obtain exact results.

[42]Many thanks to Prof. Christophe Troestler for his course on interval arithmetic given at UPHF (on which most of section 5.5.1 is based) and for his invaluable help in the computer-assisted proof to study the tetrahedron graph that we will present in this section 5.5.

[43]Fields medalist in 1958 for his works on differential topology. The citation comes from the article "La planète de l'oncle Thom – René Thom géometrise et philosophe avec Jean-François Fogel et Jean-Louis Hue." in the (mainstream) journal "Le Sauvage", in January 1977, Paris, pp. 74-80. We found the reference of the quoted sentence thanks to the documents "Bibliographie des œuvres de René Thom" `https://www.maths.ed.ac.uk/~v1ranick/papers/thom/data/biblio.pdf` along with "Citations de René Thom" `https://www.maths.ed.ac.uk/~v1ranick/papers/thom/data/citations.pdf`, by Michèle Porte.

2. Even very basic operations such as additions, multiplications and divisions lead to small errors in general. This is due to the fact that computers may only deal with *floating-point numbers*[44], containing only finitely many digits in their representation. We thus only have access to *finitely many values* which implies that we cannot represent arbitrarily big numbers, or numbers arbitrarily close to 0.

Despite all this, we want to use numerical computations and yet being *one hundred percent sure* that their conclusions are (mathematically) correct. A way to do so consists in using *certified computations* based on *interval arithmetic*.

Brief presentation of interval arithmetic

The basic idea of interval arithmetic is rather simple.

Replace a real number $x \in \mathbb{R}$ by an interval $[\underline{x}, \overline{x}]$ *that accounts for the numerical errors,*   ***in such a way that the mathematical answer***   ***lies in the returned interval.***

First, let us present this concept "mathematically", letting aside floating point numbers for a moment.

The intervals we will consider are the topologically closed and connected subsets of $\mathbb{R}$ (as specified in the standard IEEE-1788 devoted to interval arithmetic[45]). Namely, they belong to the class $\mathcal{I}_{\mathbb{R}}$ of subsets of $\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}_{\mathbb{R}} := \Big\{\varnothing\Big\} &\cup \Big\{[a,b] \mid a,b \in \mathbb{R}, a \leq b\Big\} \\ &\cup \Big\{[a,+\infty) \mid a \in \mathbb{R}\Big\} \\ &\cup \Big\{(-\infty,b] \mid b \in \mathbb{R}\Big\} \\ &\cup \Big\{(-\infty,+\infty) := \mathbb{R}\Big\}.\end{aligned}$$

Now, let us see a few operations one may perform with intervals.

- Given two intervals $\mathbf{x}$ and $\mathbf{y}$, their *sum* is given by

  $$\mathbf{x} + \mathbf{y} = \Big\{x + y \mid x \in \mathbf{x}, y \in \mathbf{y}\Big\},$$

  For instance,

  $$[-1,3] + [4,7] = \Big\{x + y \mid -1 \leq x \leq 3, 4 \leq y \leq 7\Big\} = [3,10].$$

  To compute the sum of two intervals, it thus suffices to add the endpoints together. Easy!

[44] We refer to [248] to know more about floating-point arithmetic.
[45] See `https://standards.ieee.org/ieee/1788/4431/`.

- Nevertheless, one should be careful not to use "reflexes" used when computing with real numbers. For instance, the *difference* between two intervals $\mathbf{x}$ and $\mathbf{y}$ is given by
$$\mathbf{x}-\mathbf{y} := \Big\{x-y \mid x\in\mathbf{x}, y\in\mathbf{y}\Big\}.$$
But then, one sees that
$$[-1,3]-[-1,3] = \Big\{x_1-x_2 \mid -1\leq x_1\leq 3, -1\leq x_2\leq 3,\Big\} = [-4,4],$$
so that we *do not have* $\mathbf{x}-\mathbf{x}=[0,0]$, only $\mathbf{x}-\mathbf{x}\supseteq[0,0]$.
This shows that computing $[-1,3]-[-1,3]$ is *not* the same as determining $\{x-x \mid -1\leq x\leq 3\}=[0,0]$, since the difference between the two intervals does not assume that the values coming from both intervals are correlated.

Interval extensions

We have the following mathematical definition.

Definition 5.48. Let $D\subseteq\mathbb{R}$ be a set and let $F: D\to\mathbb{R}$ be a map.

An *interval extension* of F is an application $\mathbf{F}:\mathcal{I}_\mathbb{R}\to\mathcal{I}_\mathbb{R}$ which satisfies the *containment property*, namely so that for all $\mathbf{x}\in\mathcal{I}_\mathbb{R}$, the set
$$F(\mathbf{x}) := \Big\{F(x) \mid x\in\mathbf{x}\cap D\Big\}$$
is included in $\mathbf{F}(\mathbf{x})$.

This notion is useful when using computer-assisted methods in analysis.

For instance, given a real function F, one may want to locate all possible roots of F in $[0,1]$. Now, if $\mathbf{F}$ is an interval extension of F and if $\mathbf{x}\in\mathcal{I}_\mathbb{R}$ is included in $[0,1]$, then the implication
$$\Big(0\notin\mathbf{F}(\mathbf{x})\Big) \implies \Big(\mathbf{x}\text{ does not contain any roots of }F\Big)$$
holds. We may thus divide $[0,1]$ into many "small" intervals and discard all those for which we are sure that F has no roots, this being determined by evaluating the interval extension $\mathbf{F}$. We end up with (possibly many) small intervals such that all potential roots of F belong to one of those.

Let us now make an easy but crucial observation.

Theorem 5.49 ("Fundamental theorem of interval arithmetic")**.** *If interval extensions of real functions $f_1,\ldots,f_k$ are composed, the result is an interval extension of the composition $f_1\circ\cdots\circ f_k$.*

This observation allows to obtain interval extensions of complicated functions by composing suitable interval extensions of its subparts.

Computer implementation of interval arithmetic

The concepts of intervals and of interval extensions we just presented are still "mathematical". On a computer, we will deal with the set[46]

$$\mathcal{I}_{\text{representable}} := \Big\{\mathbf{x} = [\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x} \text{ are two floating-point numbers}\Big\} \cup \Big\{\varnothing\Big\}.$$

A few comments are in order.

- The previous set is "large", but *finite*. Indeed, there are only finitely many floating-point numbers. For instance, one typically uses "`f64` numbers", the floating point numbers represented using 64 binary digits. Thus, there are less than[47] 2^{64} of them.
- There are many practical details one should pay attention to here: some floating-point numbers encode "$+\infty$" and "$-\infty$", some include "Not-a-number", there are two ways[48] to represent 0, etc.

  We will not go into any of those implementation details and refer to [248] (see in particular [248, Section 2.1.4]) for details about special floating-point data.
- When manipulating intervals from $\mathcal{I}_{\text{representable}}$, one is faced again with the errors inherent to floating-point computations. One thus needs to make sure that elementary operations are *correctly rounded*.

  Luckily, modern processors possess several *rounding modes*:
  - round to nearest: the default mode, well suited for simulations but often unsuitable for interval arithmetic;
  - round towards $-\infty$;
  - round towards $+\infty$;
  - round towards 0.

  Using those rounding modes, one obtains enclosures of elementary operations. For instance, given two nonempty intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$, their "sum on the computer" is given by $[\nabla(\underline{x} + \underline{y}), \Delta(\overline{x} + \overline{y})]$, where ∇ means that the result was rounded down and Δ means that it was rounded up. This is indeed an interval containing all possible sums $x + y$ with $x \in \mathbf{x}$ and $y \in \mathbf{y}$.
- Using the suitable rounding modes, processors can round correctly the four basic operations of sums, differences, products, quotients and square roots for floating-point numbers such as `f64`. This is sadly not the case for built-in transcendental functions.

[46] In this definition, by "floating-point number" we mean values which are not "special" such as $+\infty$, $-\infty$, "Not-a-number", etc (see the discussion below the definition).

[47] This is not an equality due to the existence of special floating-point numbers, mentioned in the next point.

[48] Namely, 0^+ and 0^- have different representations in order to allow computations such as $1/(+\infty) = 0^+$, $1/(-\infty) = 0^-$, which is useful in some numerical simulations.

Rounding elementary functions such as sin or exp to the nearest floating-point number is an highly nontrivial task. It was performed by the authors of the CR-LIBM library, we refer to their paper [95] for more details.

A few words on integration

Given a function $F : \mathbb{R} \to [a, b]$, we want to estimate the integral $\int_a^b F(x)\,\mathrm{d}x$ using interval arithmetic, using interval enclosures of F as well as its derivatives.

First, it is clear that one may split the domain of integration into smaller subdomains. Namely, if we consider $a = x_0 \le x_1 \le \cdots \le x_N = b$ and enclose all integrals on intervals $[x_i, x_{i+1}]$, $0 \le i \le N-1$, then an interval enclosure of the integral over $[a, b]$ is given by the sum of those enclosures.

The easiest method one may think of to estimate an integral over a "small interval" $[a, b]$ with $a \le b$ is simply

$$(b-a) \inf_{x\in[a,b]} F(x) \le \int_a^b F(x)\,\mathrm{d}x \le (b-a) \sup_{x\in[a,b]} F(x).$$

Therefore, if $\mathbf{F}$ is an interval extension of F, $\mathbf{x} \in \mathcal{I}_{\text{representable}}$ is an interval which contains $[a, b]$ and if the interval $\mathbf{l}$ contains the value of the length of the interval (namely, $b - a$), then an enclosure of the result is given by $\mathbf{l} \cdot \mathbf{F}$.

In practice, the previous method is both simple and robust but usually too slow "because it does not use any information on the derivatives". When possible, we prefer to use higher order methods, which make use of interval extensions of some derivatives of F.

A typical tool used for integration is *Simpson's method*. It relies on the fact that, if F is $\mathcal{C}^4$, then there exists $\xi \in [a, b]$ such that

$$\int_a^b F(x)\,\mathrm{d}x = \frac{b-a}{6}\Big(F(a) + 4F\big(\tfrac{a+b}{2}\big) + F(b)\Big) - \frac{(b-a)^5}{2880}\partial^4 F(\xi).$$

Basically, we will thus estimate the integral by both evaluating

$$\frac{b-a}{6}\Big(F(a) + 4F\big(\tfrac{a+b}{2}\big) + F(b)\Big),$$

using an interval extension of F, as well as the fourth order term $\frac{(b-a)^5}{2880}\partial^4 F(\xi)$ which is contained in an interval computed[49] by evaluating an interval extension of $\partial^4 F$ on $[a, b]$.

We refer to [326, Section 5.3] for more details about numerical integration, in particular about Simpson's method.

[49] In practice, this requires to compute expressions for the fourth derivatives, which is an elementary but sometimes tedious task.

An important practical remark

In "classical numerical analysis" (without controls using interval arithmetic), the *termination criteria* of the numerical methods that are used are very important, both in controlling the precision of the result and its speed.

Somehow, they are less crucial when dealing with interval methods. Indeed, our first focus will always be to *ensure that we only deal with interval enclosures of the quantities we are interested in*. Making sure that the intervals we obtain are "small enough" will depend on several factors and sometimes we will only determine whether we obtain "good enough" intervals *after running the code*. In a way, this is more of an art than a science.

Let us nevertheless emphasize once again that we obtain interval enclosures *by design* of our methods. We thus succeeded in our goal of obtaining numerical methods producing *one hundred percent sure* results.

Further reading on interval arithmetic

We refer the interested reader to the introductory article [327] and the book [326], both written by W. Tucker, for more information about interval arithmetic.

Let us also mention [235, 294], two papers which prove uniqueness results for semilinear elliptic partial differential equations on squares using computer-assisted proofs relying on interval arithmetic.

A computer-assisted proof to study $\mathcal{J}_*$ on the tetrahedron graph

Now, let us present the computer-assisted proof of Proposition 5.44. It will be split in several successive steps. For each of them, we will state *precise mathematical statements* of what is proved and we will also give some details about our computer-assisted methods.

We put our focus on explaining our proof strategies and make connections between the numerical conclusions and the corresponding mathematics. For the implementation details, we refer the reader to the code, available on request by contacting the author.

The code is written in the language *Rust*[50], using the "crate[51]" `inari`[52] to perform interval arithmetic computations. We also use the CRlibm library[53] to compute some functions with correct rounding.

[50] See its official webpage `https://www.rust-lang.org/`.

[51] The Rust term used to refer to libraries.

[52] See `https://crates.io/crates/inari/`.

[53] See [95] and `https://crates.io/crates/crlibm` for the Rust binding.

5.5.2 Study of the function $c \mapsto \pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})$

To get started in our presentation of computer-assisted proofs, let us focus on the one-dimensional study of the map $c \mapsto \mathcal{J}_*(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)}))$.

The first result that we prove is the following.

Proposition 5.50. *Global maxima of the continuous map*

$$\mathcal{J}_c : [0,1] \to \mathbb{R} : c \mapsto \mathcal{J}_*\Big(\pi_{\mathcal{N}_*}(\varphi_{(1,-1,c,-c)})\Big)$$

belong to the interval $[0.246054, 0.246064]$. *In particular, there exists at least one value of* c *in this interval which is a maximum of the map.*

A few comments are in order.

- The previous claim corresponds to what can be observed in Figure 5.6.
- We *do not claim the uniqueness of the maximum point* in the statement of Proposition 5.50. This property is true but it will be proved later.
- The preceding statement is *mathematically precise* but it does not look like a claim one wants to prove "by hand", due to the presence of very precise values of constants. Such statements are typical cases where one is interested in using computer-assisted tools in order to prove the claim while avoiding very tedious computations.

The strategy to prove the previous result is rather direct. First, let us note that $\mathcal{J}_c$ is a continuous function over $[0,1]$, so that the maximum exists. Hence, we just need to locate where maxima may lie. To do so, we proceed as follows.

1. A simple look at the graph in Figure 5.6 shows that we expect $\mathcal{J}_c$ to have a maximum for $c \approx 0.25$. More precisely, we make an *initial guess* by defining

   $$c_{\text{guess}} := 0.2460592 \quad \text{and} \quad \mathcal{J}_{\text{guess}} := \mathcal{J}_c(c_{\text{guess}}).$$

2. We split the interval $[0,1]$ in which we want to locate the maximum into many little intervals. For each such small interval I, we know that I *does not* contain any maximum point if either:
   - *all* values of $\mathcal{J}_c(c)$ with $c \in I$ are smaller than $\mathcal{J}_{\text{guess}}$;
   - $0 \notin \mathcal{J}_c'(I) \neq 0$.

   We thus discard all the intervals I for which one of those two possibilities occurs.

3. After the previous step, we regroup all the remaining small intervals into a single one, which necessarily contains all potential maximum points.

   Executing the code shows that this interval is included in $[0.246054, 0.246064]$, which proves the claim.

Of course, all the steps presented above have to be performed using interval arithmetic, as we presented in the previous section. For instance, even our initial guess c_{guess} and its level $\mathcal{J}_{\text{guess}}$ have to be represented by small intervals which contain the "true values". To keep our explanation light, we will not always mention such facts, but we warn the reader that caution is required.

Now, let us locate where critical points of $\mathcal{J}_*$ may lie.

5.5.3 Locating critical points: exploring the tetrahedron

Proposition 5.51. *If $\varphi \in E_2$ is a nonzero critical point of $\mathcal{J}_*$, then there exists $g \in \mathcal{G}_{\text{t}}$ such that $g \cdot \varphi$ belongs to $\Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3$ where*

$$\Phi_i := \left\{ \pi_{\mathcal{N}_*}(\varphi_a) \mid a \in \mathcal{A}_i \right\}$$

for every $0 \leq i \leq 3$ and where[a]

$$\begin{aligned}
\mathcal{A}_0 &:= A_0 \cap \Big(\{1\} \times [-1, -0.999877] \times [-0.000275, 0.000145] \times [-0.000268, 0.000275] \Big), \\
\mathcal{A}_1 &:= A_0 \cap \Big(\{1\} \times [-0.333962, -0.333274] \\
&\qquad\qquad \times [-0.333840, -0.332672] \times [-0.333710, -0.332542] \Big), \\
\mathcal{A}_2 &:= A_0 \cap \Big(\{1\} \times [0.998237, 1] \times [-1, -0.997837] \times [-1, -0.998237] \Big), \\
\mathcal{A}_3 &:= A_0 \cap \Big(\{1\} \times [-1, -0.999694] \times [0.242614, 0.249299] \times [-0.249451, -0.242767] \Big).
\end{aligned}$$

Moreover, the sets Φ_0, Φ_1, Φ_2 and Φ_3 all contain at least one critical point.

[a] Recalling that $A_0 := \left\{ (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \;\middle|\; a_0 + a_1 + a_2 + a_3 = 0 \right\}$.

To prove Proposition 5.51, we proceed in the following way.

1. Using the symmetries of the problem (described thanks to the action of the group G_t in section 5.4.2), we can sort the components a_0, a_1, a_2, a_3 of vectors in A_0 by nonincreasing order of absolute values and we can make sure that the first one is positive. Up to a multiplicative factor, we may normalize the first component to one. More precisely, defining

$$A_{\text{sorted}} := \left\{ (a_0, a_1, a_2, a_3) \in A_0 \mid a_0 = 1 \geq |a_1| \geq |a_2| \geq |a_3| \right\},$$

then for every $\varphi \in \mathcal{N}_*$, there exists $g \in G_t$ such that $g \cdot \varphi$ belongs to

$$\mathcal{N}_{\text{sorted}} := \left\{ \pi_{\mathcal{N}_*}(\varphi_a) \mid a \in A_{\text{sorted}} \right\}.$$

Thus, if we prove that for every $\varphi \in \mathcal{N}_{\text{sorted}}$, there exists $g \in G_t$ such that $g \cdot \varphi$ belongs to $\mathcal{A}_i$ for some $0 \leq i \leq 3$, then the claim will follow.

2. All elements of A_{sorted} are of the form $(1, a_1, a_2, 1 - a_1 - a_2)$. A simple way to consider all elements of the set A_{sorted} consists in considering all couples $(a_1, a_2) \in [-1, 1]^2$ and then only keeping the couples (a_1, a_2) for which $(1, a_1, a_2, 1 - a_1 - a_2)$ belongs to A_{sorted}.

3. More precisely, we divide the square $[-1, 1]^2$ into small rectangular boxes $I \times J$. Each box corresponds to a set of couples $(a_1, a_2) \in I \times J$. Then:
   - if none of the associated quadruplets $(1, a_1, a_2, 1 - a_1 - a_2)$ for $(a_1, a_2) \in I \times J$ belong to A_{sorted}, we discard the box since we just want to consider all elements of A_{sorted};
   - if the box is such that the set

$$\left\{\pi_{\mathcal{N}_*}(\varphi_a) \mid a := (1, a_1, a_2, 1 - a_1 - a_2), (a_1, a_2) \in I \times J\right\}$$

   does not contain any critical point[54], we also discard it.

4. After the previous step, we know that all critical points in $\mathcal{N}_{\text{sorted}}$ are of the form $\pi_{\mathcal{N}_*}(\varphi_{(1,a_1,a_2,1-a_1-a_2)})$ where (a_1, a_2) belongs to a box from an explicit list of small boxes in $\mathbb{R}^2$. Then, we group the many small boxes into "hulls", which are bigger rectangular boxes containing the connected components in $\mathbb{R}^2$ of the union of all the boxes from the list. After this step, we end up with *seven* disjoint hulls. All critical points of $\mathcal{N}_{\text{sorted}}$ belong to one of those hulls.

5. We remark that some of the hulls are "the same", up to the symmetries. We identify those hulls and merge them, so that we end up with *four* hulls.

6. In order to continue the reasoning with small boxes, we perform again step 3 on the four boxes we obtained with higher precision. Then, we collect again the very small boxes in hulls so that we end up with four hulls again in the end. *Executing the code* shows that those hulls are respectively included in $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$.

7. To conclude the proof of the proposition, it remains to prove that every set Φ_i, $0 \leq i \leq 3$, contains a critical point of $\mathcal{J}_*$. For this, it suffices to note that f_i belongs to[55] $\mathcal{A}_i$ for every $0 \leq i \leq 3$ and to apply Proposition 5.42.

5.5.4 Uniqueness of critical points corresponding to $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and their nondegeneracy

Our strategy to prove uniqueness of critical points is based on the next proposition, classical in convex analysis (see [140, Chapter 6, in particular Proposition 1.5]). Since its proof is informative and elementary, we include it here for completeness.

[54] This is checked by estimating the Jacobian matrix of $\mathcal{J}_*$ using interval arithmetic.

[55] Remark that, even though we do not know if c is unique, we know that it belongs to $[0.246054, 0.246064]$ so that we know that $(1, -1, c, -c)$ belongs to $\mathcal{A}_3$.

Proposition 5.52. *Let $\Omega_1 \subseteq E_1$ and $\Omega_2 \subseteq E_2$ be open subsets of two finite dimensional real vector spaces E_1 and E_2. Let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Let $K_1 \subset \Omega_1$ and $K_2 \subset \Omega_2$ be two nonempty convex compact sets. Assume that:*

1. *for every $x_1 \in K_1$, the map $K_2 \to \mathbb{R} : x_2 \mapsto F(x_1, x_2)$ is strictly convex;*
2. *for every $x_2 \in K_2$, the map $K_1 \to \mathbb{R} : x_1 \mapsto F(x_1, x_2)$ is strictly concave.*

Then, the function f has at most one critical point inside $K_1 \times K_2$. Moreover, if such a critical point $(y_1, y_2) \in K_1 \times K_2$ exists, it is the only element of $K_1 \times K_2$ satisfying the equality

$$F(y_1, y_2) = \min_{x_1 \in K_1} \max_{x_2 \in K_2} F(x_1, x_2).$$

Proof. Let us assume that (y_1, y_2) and (z_1, z_2) are two critical points of F inside $K_1 \times K_2$. Then, the convexity hypotheses imply that:

1. y_1 is a strict minimum of $K_1 \to \mathbb{R} : x_1 \mapsto F(x_1, y_2)$;
2. z_1 is a strict minimum of $K_1 \to \mathbb{R} : x_1 \mapsto F(x_1, z_2)$;
3. y_2 is a strict maximum of $K_2 \to \mathbb{R} : x_2 \mapsto F(y_1, x_2)$;
4. z_2 is a strict maximum of $K_2 \to \mathbb{R} : x_2 \mapsto F(z_1, x_2)$.

Using successively points 1, 4, 2 and 3, we deduce that

$$F(y_1, y_2) \leq F(z_1, y_2) \leq F(z_1, z_2) \leq F(y_1, z_2) \leq F(y_1, y_2).$$

Remarking that one of those inequalities is strict unless $(y_1, y_2) = (z_1, z_2)$ proves the uniqueness claim. The min-max characterization follows from the inequalities

$$\begin{aligned} \min_{x_1 \in K_1} \max_{x_2 \in K_2} F(x_1, x_2) &\leq \max_{x_2 \in K_2} F(y_1, x_2) \\ &= F(y_1, y_2) \\ &= \min_{x_1 \in K_1} F(x_1, y_2) \\ &\leq \min_{x_1 \in K_1} \max_{x_2 \in K_2} F(x_1, x_2). \end{aligned}$$

□

Remark 5.53. If we further assume that f is $\mathcal{C}^2$ on $\Omega_1 \times \Omega_2$, the convexity conditions will follow if we show, for every $(x_1, x_2) \in K_1 \times K_2$, that the second differential $\partial_{11} F(x_1, x_2)$ is positive definite and that $\partial_{22} F(x_1, x_2)$ is negative definite.

The previous proposition is very convenient for our purposes. Indeed, the functional $\mathcal{J}_*$ is $\mathcal{C}^1$ and is $\mathcal{C}^2$ on open sets containing Φ_1, Φ_2, Φ_3 (since those sets are included in S).

In the space E_2, we expect the functional[56] $\mathcal{J}_*$ to possess a "min-max" type structure around f_2 and f_3. For vectors in the tangent plane $T_{f_i}\mathcal{N}_*$, the min-max structure depends on the type of critical point. We thus need to choose directions suitably to identify in which directions $\mathcal{J}_*$ is convex and in which it is concave[57]. The functional will always be concave in the direction of functions themselves, due to the Nehari-type geometry.

A small inconvenience occurs. Indeed, Proposition 5.52 requires to consider functions defined on direct products. Mathematically, it seems that this does not matter much (we just need to apply a change of basis, after all). However, *all our methods in the code label eigenfunctions using their associated quadruplet in* A_0. Moreover, *we deal with "boxes" of parameters.*

Therefore, we use a method to wrap our boxes of quadruplet into a larger box *aligned with given basis vectors*, which we choose to be $b_0 := (1, -1, 0, 0)$, $b_1 := (0, 0, 1, -1)$ and $b_2 := (1, 1, -1, -1)$ in our application. Let us illustrate this in two dimensions in Figure 5.7. Starting with a box $\mathbf{B_1}$ in axis (a_1, a_2) (in which we are locating critical points), we include it in a box $\mathbf{B_2}$ in axis $(\widetilde{a_1}, \widetilde{a_2})$. Then, we wrap the obtained box in another yet larger box, *parametrized again by* a. This is illustrated by boxes $\mathbf{B_2}$ and $\mathbf{B_3}$. We thus create a "box in a box in a box", so that we keep using quadruplets a as parameters while making sure to enclose a box parallel to the axis given by the vectors b, as required by Proposition 5.52.

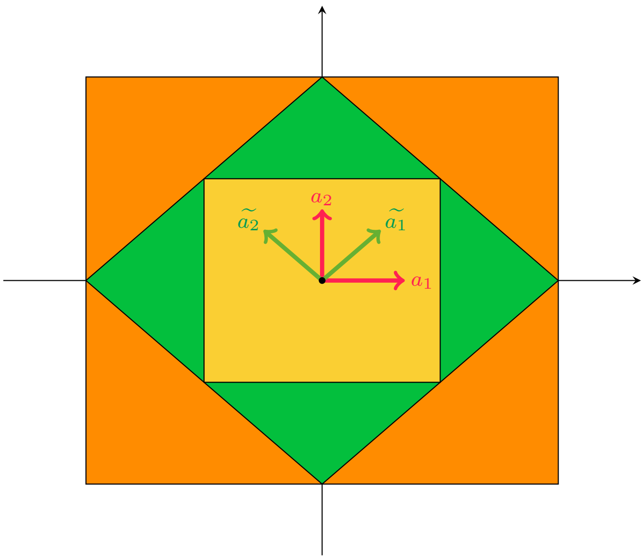


Figure 5.7: A box $\mathbf{B_1}$ in a box $\mathbf{B_2}$ in a box $\mathbf{B_3}$

Now that all those concepts have been introduced, let us describe our method.

[56] That we consider here as a "free" functional on E_2, not constrained to $\mathcal{N}_*$.

[57] In elementary terms, we are just looking for a convenient basis in which one may compute the Hessian matrix of $\mathcal{J}_*$ at critical points.

1. We make the boxes $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ larger, in order to be sure to contain large enough boxes aligned with the axis b_0, b_1, b_2, using the "box in a box in a box" method.
2. We compute second derivatives by evaluating the relevant integrals explicitly. Some care is required due to the presence of terms of the type $\int_0^1 \ln|\varphi(x)|\psi(x)\,\mathrm{d}x$, where φ has a root inside the interval on which we integrate its logarithm. Let us explain how we deal with this singular integral.
   - The expressions of eigenfunctions are explicit, so that we can compute the position of the root (though, it will only be located into a small interval, as usual).
   - In the parts of the domain where there are no roots, we use our "standard" methods of integration, based on Simpson's method.
   - In the small region close to the root, we use a bound based on information on the derivative of φ, which is basically the same as the bound (5.6) we used to prove Lemma 5.14.
3. In the case of $\mathcal{A}_1$, *executing the code* shows that the matrix containing the second derivatives is negative definite, so that there is at most one critical point corresponding to $\mathcal{A}_1$, which is necessarily f_1.
4. In the case of $\mathcal{A}_2$, *executing the code* shows that the matrix is positive definite in $\operatorname{span}\{b_0, b_1\}$ and negative definite in $\operatorname{span}\{b_2\}$.
5. In the case of $\mathcal{A}_3$, *executing the code* shows that the matrix is negative definite in $\operatorname{span}\{b_0, b_1\}$ and positive definite in $\operatorname{span}\{b_2\}$.
6. In both cases, using Proposition 5.52, we deduce that $\mathcal{A}_2$ and $\mathcal{A}_3$ each contain at most one critical point. In fact, those two critical points exist and are given by f_2 and f_3 respectively.

Let us summarize what we obtained so far.

Proposition 5.54. *The sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ each contain exactly one critical point of $\mathcal{J}_*$, namely f_1, f_2 and f_3. In particular, the constant "c" corresponding to f_3 is unique.*

Since we explicitly studied Hessian-type matrices and checked their invertibility (by determining subspaces for which they are positive definite or negative definite), we deduce that f_1, f_2 and f_3 are nondegenerate.

We can be a bit more precise.

5.5.5 Variational characterization of f_1, f_2 and f_3

Since f_1, f_2 and f_3 belong to S, the reduced Nehari manifold $\mathcal{N}_*$ is $\mathcal{C}^2$ in the neighborhood of those critical points, as we saw in section 5.2.5.

We may thus compute the signature of the Hessian of $\mathcal{J}_*$, constrained to the tangent space $T_{f_i}\mathcal{N}_*$, for every $1 \leq i \leq 3$. *Executing the code* shows that f_1 has index 2, f_2 has index 0 and f_3 has index 1. This corresponds to their description as a minimum, a maximum, and a saddle point of $\mathcal{J}_*$ on the two-dimensional manifold $\mathcal{N}_*$, using Morse theoretical arguments.

5.5.6 (Local) uniqueness around f_0

To end the proof of Proposition 5.44, it remains to prove that f_0 is the only critical point in Φ_0, where we recall that, as in Proposition 5.51,

$$\Phi_0 := \left\{\pi_{\mathcal{N}_*}(\varphi_a) \mid a \in \mathcal{A}_0\right\}.$$

As in the general theory, we are faced with a challenge: *f_0 vanishes identically on an edge, so that $\mathcal{J}_*$ is not $\mathcal{C}^2$ in the neighborhood of f_0.* Thus, we cannot "simply" conclude by using Hessian matrices, as we did for f_1, f_2 and f_3.

We recall that, since f_0 is a critical point of $\mathcal{J}_*$, the tangent space to $\mathcal{N}_*$ at f_0 is given by

$$T_{f_0}\mathcal{N}_* = \langle f_0\rangle^\perp := \left\{\psi \in E_2 \,\middle|\, (f_0 \mid \psi)_{L^2(\mathcal{G}_t)} = 0\right\},$$

using Proposition 5.23. Let us prove the following result.

Proposition 5.55. *There exists $\varepsilon > 0$ such that for all $s > 0$ and every $\psi \in \langle f_0\rangle^\perp$ such that $sf_0 + \psi$ belongs to Φ_0, the inequality*

$$\mathcal{J}_*'(sf_0 + \psi)[\psi] \geq \varepsilon \|\psi\|_{E_2}^2 \tag{5.51}$$

holds.

Proof. Let us denote by e_0 the edge on which f_0 vanishes identically. Given $s > 0$ and $\psi \in \langle f_0\rangle^\perp$ such that $sf_0 + \psi$ belongs to Φ_0, we have (recalling (5.25))

$$\begin{aligned}\mathcal{J}_*'(sf_0 + \psi)[\psi] &= -\int_{\mathcal{G}_t} (sf_0 + \psi)\ln|sf_0 + \psi|\psi \,\mathrm{d}x \\ &= -\int_{\mathcal{G}_t\setminus e_0} (sf_0 + \psi)\ln|sf_0 + \psi|\psi \,\mathrm{d}x + \int_e \psi^2(-\ln|\psi|)\,\mathrm{d}x \end{aligned} \tag{5.52}$$

Now, let us state a claim that is checked by the computer-assisted proof. We will later detail how it is obtained .

- **Claim 1.** Taking $D := 3.125$, the inequality $\|\psi\|_{L^\infty(e_0)} \leq \mathrm{e}^{-D}$ is satisfied[58].

[58]Recall that $sf_0 + \psi$ belongs to the "small box" Φ_0 and that f_0 vanishes identically on e_0, so the fact that ψ is small on e_0 is not surprising. Nevertheless, it has to be verified.

Using this claim and (5.52), we deduce that

$$\mathcal{J}_*'(sf_0+\psi)[\psi] \geq -\int_{\mathcal{G}_t\setminus e_0}\left[\int_0^1\Big(1+\ln|sf_0+t\psi|\Big)\,\mathrm{d}t\right]\psi^2\,\mathrm{d}x + D\int_{e_0}\psi^2\,\mathrm{d}x. \tag{5.53}$$

We now state a second claim, obtained thanks to the computer-assisted proof.

- **Claim 2.** There exists $\varepsilon>0$ such that for every $v\in\Phi_0$ and every $\psi\in\langle f_0\rangle^\perp$, the inequality

$$Q_v(\psi,\psi)\geq\varepsilon\|\psi\|_{E_2}^2$$

holds, where the bilinear form Q_v is defined by

$$Q_v(\psi_1,\psi_2) := -\int_{\mathcal{G}_t\setminus e_0}\left[\int_0^1\Big(1+\ln|v|\Big)\,\mathrm{d}t\right]\psi_1\psi 2\,\mathrm{d}x + D\int_{e_0}\psi_1\psi 2\,\mathrm{d}x.$$

Using this claim and (5.53) (remarking that all functions $sf_0+t\psi$ in (5.53) belong to the convex set Φ_0), we deduce that

$$\mathcal{J}_*'(sf_0+\psi)[\psi]\geq Q_v(\psi,\psi)\geq\varepsilon\|\psi\|_{E_2}^2,$$

which proves (5.51).

Thus, it remains to check that claim 1 and claim 2 hold. This may be done using a rather direct method.

1. For claim 1, we may bound the L^∞ norm of eigenfunctions using the explicit expression of φ_a for a in the suitable box of parameters. We then make sure that the bound is always less or equal than e^{-D}.
2. For claim 2, we compute the matrix of the associated bilinear form in a basis of the tangent plane. More precisely, we compute a matrix of intervals where each entry encloses the corresponding entries of the Hessian matrices of Q_v where v lies in Φ_0.

   Executing the code, we check that those Hessian matrices are positive definite, which is the case. Here, it was important to choose D "large enough" so that the term $\|\psi\|_{L^2(e_0)}^2$ "brings positive definiteness", as we saw in the proof of Theorem 5.24. □

It remains to end the proof of Proposition 5.44, by showing that f_0 is the only critical point in Φ_0. But this follows easily from Proposition 5.55.

Let us assume that $sf_0+\psi$ belongs to Φ_0 and is such that $\mathcal{J}'(sf_0+\psi)=0$. Then, $\mathcal{J}'(sf_0+\psi)[\psi]=0$, and Proposition 5.55 implies that $\psi=0$. Thus, $\mathcal{J}'(sf_0)=0$. Since $s>0$ and f_0 belongs to $\mathcal{N}_*$, we deduce[59] that $s=1$, which ends the argument.

Thus, Proposition 5.44 is proved. As we saw, it follows from a sequence of more elementary statements, but several of them (propositions 5.50, 5.51, 5.54, 5.55) crucially require the computer-assisted tools we presented.

[59]Recalling the geometry of $\mathcal{N}_*$, see e.g. (5.30).

Appendix A
What are metric graphs?

In this appendix, we define metric graphs and describe their metric space and their measured space structures. We also define the Sobolev space H^1 on metric graphs and prove Sobolev embedding theorems. Finally, we state and prove a coarea formula for H^1 functions on metric graphs. Some care is required since we are working in a rather low regularity setting.

The first sections hereunder are based on [247], where one can find a formal definition of metric graphs. A reference book on analysis problems set on metric graphs is [68].

A.1 What are metric graphs?

A metric graph is made of a finite or countable number of vertices (or nodes) joined by finitely many or countably many edges that join the vertices or go to infinity. The edges going to infinity are called *half-lines*.

We say that two different vertices are *adjacent* if they share an edge.

Some vertices may be joined to themselves by an edge. In this case, we say that this edge is a *loop*.

Let us consider the example of metric graph depicted in the following figure.

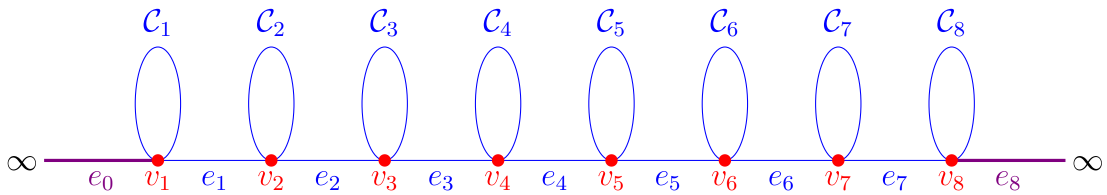


Figure A.1: An example of metric graph $\mathcal{G}$ with 8 vertices $v_1, \dots, v_8$; 7 edges $e_1, \dots, e_7$ with finite length joining distinct vertices; 8 loops $\mathcal{C}_1, \dots, \mathcal{C}_8$ joining a vertex to itself; two half-lines e_0 and e_8.

In this example:

- the vertices v_1 and v_2 are adjacent since they are joined by the edge e_1. The vertices v_3 and v_7 are not adjacent;
- the edge $\mathcal{C}_3$ is a loop joining the vertex v_3 to itself;
- the edges e_0 and e_8 are half-lines;
- there are 8 vertices and $2 + 7 + 8 = 17$ edges in total.

The *degree of a vertex* v is the number of edges incident at v, counting twice any self-loop at v. We denote it by $\deg(v)$.

In the previous example, all vertices have degree 4. For instance, v_6 is joined to edges e_5, e_6 and $\mathcal{C}_6$ (which counts twice since it is "attached twice" at v_6).

The same vertices may be joined several times by different edges, as is the case for the triple-bridge graph depicted hereunder.

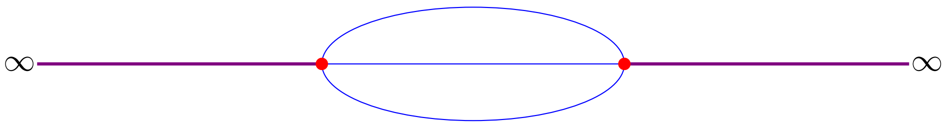


Figure A.2: The triple-bridge

Given a metric graph $\mathcal{G}$, we denote by $\mathbb{V}$ its set of vertices and by $\mathbb{E}$ its set of edges. We consider $\mathcal{G}$ as a *set of points* (the vertices and the points in the interior of the edges), endowed with the metric space and measured space structures defined in the following sections.

Nevertheless, we will often simply write $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ where $\mathbb{V}$ is the set of vertices of $\mathcal{G}$ and $\mathbb{E}$ is the set of edges of $\mathcal{G}$.

So far, we only described the structure of a *graph*. Our graphs are *metric*, which means that all edges have a given *length*. We denote the length of an edge e by $|e|$. If e joins two vertices, then $|e|$ belongs to $(0, +\infty)$. If e is a half-line, it has an infinite length and we set $|e| := +\infty$.

In all the thesis, we only consider *combinatorially locally finite*[1] metric graphs, i.e. so that the degree of every vertex is finite: $\deg(v) < \infty$ for all $v \in \mathbb{V}$.

We assume that all our metric graphs are *connected*, i.e. that one may join any pair of points of the graph by a continuous path. We will make this notion more precise in the following section, where we will use the *length of shortest paths* between points to turn metric graphs into metric spaces.

A.2 Metric graphs are metric spaces

Connected metric graphs are metric spaces when equipped with the shortest path metric $d_{\mathcal{G}}$ (see e.g. [247]), defined for all $x, y \in \mathcal{G}$ by

$$d_{\mathcal{G}}(x, y) := \inf_{p \in P_{x,y}} \ell(p),$$

where $P_{x,y}$ is the set of paths joining x and y, and where $\ell(p)$ denotes the length of such a path p. By definition, $\mathcal{G}$ is connected if $P_{x,y}$ is non-empty for all $x, y \in \mathcal{G}$.

[1] Using the same terminology as in [247].

More precisely, $P_{x,y}$ is given by

$$P_{x,y} := \Big\{(w_1, \dots, w_k) \in \mathcal{G}^k \;\Big|\; k \in \mathbb{Z}^{\geq 1}, x = w_1, y = w_k, w_2, \dots, w_{k-1} \text{ are vertices,}$$
$$w_i \text{ and } w_{i+1} \text{ belong to the same edge for all } 1 \leq i \leq k-1\Big\}.$$

Moreover, given a path $p = (w_1, \dots, w_k) \in P_{x,y}$, its length is given by

$$\ell(p) := \sum_{1 \leq i \leq k-1} |w_i w_{i+1}|$$

where $w_i w_{i+1}$ denotes the edge[2] joining w_i and w_{i+1}.

Let us now characterize compact metric graphs.

Proposition A.1. *The metric space $(\mathcal{G}, d_{\mathcal{G}})$ associated to a (combinatorially finite) metric graph $\mathcal{G}$ is compact if and only if $\mathcal{G}$ is made of a finite number of edges of finite length.*

Proof. **Step 1. If $\mathcal{G}$ is made of a finite number of edges of finite length, then $\mathcal{G}$ is compact**. Indeed, any sequence of points $(x_n)_{n\geq 1} \subseteq \mathcal{G}$, contains a subsequence of points all belonging to the same edge of $\mathcal{G}$, which admits a convergent subsequence in $\mathcal{G}$ since edges of finite length are compact as an edge is closed by definition.

Step 2. If $\mathcal{G}$ is compact, then $\mathcal{G}$ is made of a finite number of edges of finite length. Otherwise, assume that either $\mathcal{G}$ contains at least one half-line or $\mathcal{G}$ contains infinitely many edges of finite length.

- If $\mathcal{G}$ contains at least one half-line, then a sequence of points going to infinity along the half-line does not admit any converging sequence to a point of $\mathcal{G}$ and $\mathcal{G}$ is not compact.
- If $\mathcal{G}$ contains infinitely many edges of finite length, the sequence $(m_n)_{n\geq 1} \subseteq \mathcal{G}$ of midpoints of those edges does not admit any converging subsequence. Indeed, if a subsequence $(m'_n)_{n\geq 1}$ of $(m_n)_{n\geq 1}$ converges to some $x \in \mathcal{G}$, then:
  - if x belongs to the interior of some edge $e \in \mathbb{E}$, then m_n also belongs to e for every n large enough, which contradicts the construction;
  - if x is a vertex of $\mathcal{G}$ (i.e. $x \in \mathbb{V}$), then m_n would need to belong to one of the finitely many edges adjacent to x for n large enough, which contradicts the construction.

We thus found a sequence of points of $\mathcal{G}$ without a converging subsequence, showing that $\mathcal{G}$ is not compact. □

[2]Note that $w_1 w_2$ and $w_{k-1} w_k$ are in general only portions of the original edges since x and y are not necessarily vertices.

Let us now discuss the hypothesis $\inf_{e\in\mathbb{E}}|e|>0$, that holds in all chapters, see in particular the definition of classes $\mathbf{G_1}$ (on page 156), $\mathbf{G_2}$ (on page 181), and $\mathbf{G_4}$ (on page 253).

It comes naturally into play while studying a metric graph as a metric space, as can be seen in the following proposition.

Proposition A.2. *Let $\mathcal{G}$ be a metric graph such that $\inf_{e\in\mathbb{E}}|e|>0$. Then $(\mathcal{G},d_{\mathcal{G}})$ is a complete metric space.*

Proof. Let $(x_n)_{n\geq 1}\subseteq\mathcal{G}$ be a Cauchy sequence in $(\mathcal{G},d_{\mathcal{G}})$, so that

$$\forall\varepsilon>0,\exists n_0\geq 1,\forall n,m\geq n_0,\qquad d_{\mathcal{G}}(x_n,x_m)\leq\varepsilon.$$

We define

$$D_{\mathbb{V}}:=\liminf_{n\to\infty}\inf_{v\in\mathbb{V}}d_{\mathcal{G}}(x_n,v)\in[0,+\infty].$$

Let us distinguish two cases.

Case 1. If $D_{\mathbb{V}}=0$.

Then, there exists a subsequence $(x'_n)_{n\geq 1}\subseteq(x_n)_{n\geq 1}$ and a sequence of nodes $(v_n)_{n\geq 1}\subseteq\mathbb{V}$ such that

$$\lim_{n\to\infty}d_{\mathcal{G}}(x'_n,v_n)=0. \tag{A.1}$$

Given $n\geq 1$, we have

$$d_{\mathcal{G}}(v_n,v_{n+1})\leq d_{\mathcal{G}}(v_n,x'_n)+d_{\mathcal{G}}(x'_n,x'_{n+1})+d_{\mathcal{G}}(x'_{n+1},v_{n+1})\xrightarrow[n\to\infty]{}0.$$

Since $\inf_{e\in\mathbb{E}}|e|>0$, we deduce that there exists $v\in\mathbb{V}$ such that one has $v_n=v$ for all n large enough.

Then, (A.1) shows that $(x'_n)_n$ converges to v. Since $(x_n)_n$ is a Cauchy sequence, we deduce that it also converges to v.

Case 2. If $D_{\mathbb{V}}>0$.

In this case, the sequence eventually stays far away from the vertices. Thus, there exists an edge e such that x_n belongs to e for all n large enough. Moreover, identifying this edge with an interval of real numbers, the distance $d_{\mathcal{G}}(x_n,x_m)$ is equal to $|x_n-x_m|$ for all indices n,m large enough. We conclude using the completeness of closed real intervals for the usual Euclidean distance. □

Remark A.3. The hypothesis $\inf_{e\in\mathbb{E}}|e|>0$ cannot in general be removed in the previous proposition.

Indeed, let us consider the binary tree with "shorter and shorter edges" shown in Figure A.3. Then, the sequence of vertices "on the leftmost side" (see the sequence $(x_n)_{n\geq 1}$ depicted on the figure) is a Cauchy sequence but does not converge to a point of the graph since there is no corresponding limit point.

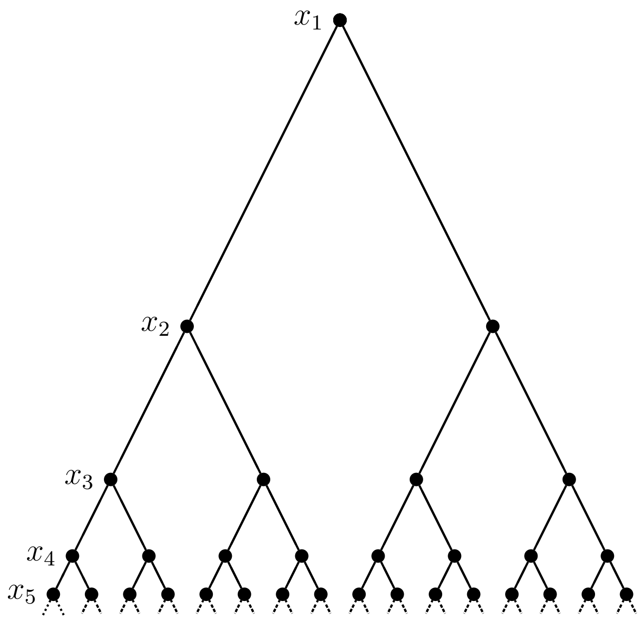


Figure A.3: A binary tree with shorter and shorter edges

A.3 Metric graphs are measured spaces

We will not go into details on integration in metric graphs, referring the reader e.g. to [247] for more information.

The point is that we may reduce ourselves to functions defined on each edge, naturally identified with intervals of $\mathbb{R}$ (see Figure A.4). We can thus integrate using the rule

$$\int_{\mathcal{G}} f(x)\,\mathrm{d}x = \sum_{e\in\mathbb{E}} \int_e f_{|e}(x)\,\mathrm{d}x, \tag{A.2}$$

where $f_{|e}$ is the restriction of f to e, so that the integrals on the right hand side are usual Lebesgue integrals of real-valued functions defined on intervals.

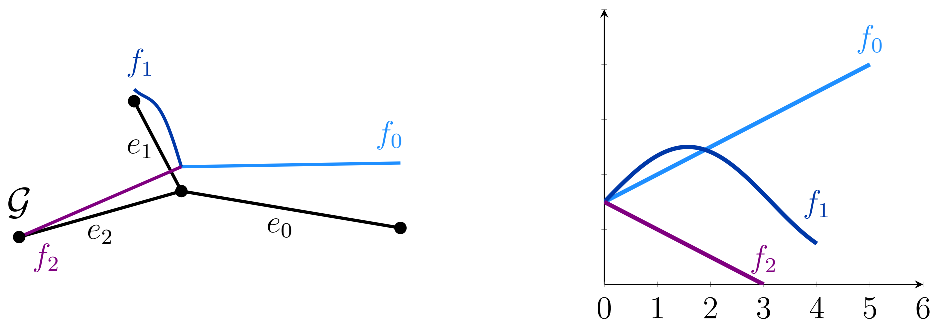


Figure A.4: A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3) and three real functions f_0, f_1 and f_2, associated to a function $f : \mathcal{G} \to \mathbb{R}$.

Metric graphs can be naturally endowed with a measured space structure, using their Borel σ-algebra associated to their metric $d_{\mathcal{G}}$ and a measure $\lambda_{\mathcal{G}}$ which simply amounts to sum the lengths over all edges. Indeed, if $A \subseteq \mathcal{G}$ is measurable, then

$$|A| := \lambda_{\mathcal{G}}(A) := \sum_{e \in \mathbb{E}} |A \cap e|,$$

where we use the notation $|\cdot|$ to denote the Lebesgue measure on subsets of $\mathbb{R}$ as well as its natural generalization to metric graphs. In particular, we have

$$|\mathcal{G}| = \sum_{e \in \mathbb{E}} |e|.$$

As we already mentioned, we can interpret the integral on the left hand side of (A.2) as a Lebesgue integral with respect to the measure $\lambda_{\mathcal{G}}$. We will simply note $\mathrm{d}x$ in integrals instead of $\mathrm{d}\lambda_{\mathcal{G}}$. In particular, we may define L^p spaces in the usual sense. When p belongs to $[1, +\infty)$, we have

$$L^p(\mathcal{G}) := \left\{ u : \mathcal{G} \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_{\mathcal{G}} |u(x)|^p \, \mathrm{d}x < +\infty \right\}.$$

Remark A.4. **A word of caution:** *if a metric graph $\mathcal{G}$ has finitely many edges*, then a function $u : \mathcal{G} \to \mathbb{R}$ is integrable if and only if all its restrictions to edges are integrable as well. This is however wrong if $\mathcal{G}$ has infinitely many edges.

A.4 Weak derivatives and the space $H^1(\mathcal{G})$

One may naturally define a notion of weak derivatives for functions defined on metric graphs by *differentiating them edge by edge.*

Definition A.5 (Weak derivatives on metric graphs)**.** Let $\mathcal{G}$ be a metric graph and $f : \mathcal{G} \to \mathbb{R}$ be a locally integrable mapping. Then, a function f' is a *weak derivative* of f if and only if its restrictions to all edges of $\mathcal{G}$ are weak derivatives[a] of the restrictions of f to the edges.

[a] See e.g. [334, Section 6.1] for a definition of the weak derivative. We will denote by $W^{1,1}_{\mathrm{loc}}(D)$ the space of functions which possess weak derivatives on a domain D.

As for function define on real intervals, one may show that weak derivatives are unique up to almost everywhere equality. Note that in the previous definition, continuity of f at the vertices is not required. However, we will require continuity in the following definition.

Definition A.6. The *Sobolev space* $H^1(\mathcal{G})$ is defined as

$$H^1(\mathcal{G}) := \Big\{ u : \mathcal{G} \to \mathbb{R} \;\Big|\; u \text{ is continuous, has a weak derivative } u' \text{ such that } u \text{ and } u' \text{ belong to } L^2(\mathcal{G}) \Big\}.$$

It is a Hilbert space[a] when endowed with the scalar product

$$(u \mid v)_{H^1(\mathcal{G})} := \int_{\mathcal{G}} u(x)v(x)\,\mathrm{d}x + \int_{\mathcal{G}} u'(x)v'(x)\,\mathrm{d}x.$$

The norm on H^1 is given by $\|u\|_{H^1(\mathcal{G})} := \sqrt{(u \mid u)_{H^1(\mathcal{G})}}$.

[a] Since strong H^1 convergence implies L^∞_{loc} convergence (see the next proposition), the continuity is preserved when taking strong H^1 limits.

We now state and prove Sobolev embedding theorems.

Proposition A.7. *Let $\mathcal{G}$ be a metric graph. Then, the Sobolev space $H^1(\mathcal{G})$ is continuously embedded in the space of continuous bounded functions on $\mathcal{G}$. For all $u \in H^1(\mathcal{G})$, the Sobolev inequality*

$$\|u\|_{L^\infty(\mathcal{G})} \le \max\Big\{2, 1/|\mathcal{G}|\Big\}^{1/2} \|u\|_{H^1(\mathcal{G})}$$

holds, where $|\mathcal{G}|$ is the total length of $\mathcal{G}$. Moreover, if $|\mathcal{G}| = +\infty$, the more precise Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty(\mathcal{G})} \le \sqrt{2}\|u\|^{1/2}_{L^2(\mathcal{G})}\|u'\|^{1/2}_{L^2(\mathcal{G})}$$

holds.

Proof. All $H^1(\mathcal{G})$ functions are continuous by definition (this definition made sense since H^1 functions on intervals of $\mathbb{R}$ are necessarily continuous). We want to be more precise about Sobolev embeddings on metric graphs. Let us fix two points $x, y \in \mathcal{G}$. Since the graph is assumed to be connected, and writing $L := d_{\mathcal{G}}(x, y)$, we know there is a path $\gamma : [0, L] \to \mathcal{G}$, so that $\gamma(0) = x$ and $\gamma(L) = y$, going through finitely many edges, and such that $|\gamma'| = 1$ almost everywhere. The composition $u_\gamma := u \circ \gamma$ is an $H^1([0, L])$ function. We have

$$u(y)^2 - u(x)^2 = u_\gamma(L)^2 - u_\gamma(0)^2 = \int_0^L 2u_\gamma(t)u'_\gamma(t)\,\mathrm{d}t \le 2\|u_\gamma\|_{L^2([0,L])}\|u'_\gamma\|_{L^2([0,L])},$$

using the Cauchy-Schwarz inequality. Therefore, for every $x, y \in \mathcal{G}$, we have

$$u(y)^2 \le u(x)^2 + 2\|u\|_{L^2(\mathcal{G})}\|u'\|_{L^2(\mathcal{G})}. \tag{A.3}$$

If $|\mathcal{G}| = +\infty$, then $\inf_{x\in\mathcal{G}} |u(x)| = 0$ (since u belongs to $L^2(\mathcal{G})$), hence by taking the infimum on x in (A.3), we obtain

$$u(y)^2 \leq 2\|u\|_{L^2(\mathcal{G})}\|u'\|_{L^2(\mathcal{G})}$$

for all $y \in \mathcal{G}$, which proves the Gagliardo-Nirenberg inequality.

If $|\mathcal{G}| < +\infty$, we remark that

$$\|u\|^2_{L^2(\mathcal{G})} = \int_{\mathcal{G}} |u|^2 \,\mathrm{d}x \geq \left(\inf_{x\in\mathcal{G}} |u(x)|\right)^2 |\mathcal{G}|,$$

hence

$$\left(\inf_{x\in\mathcal{G}} |u(x)|\right)^2 \leq \frac{\|u\|^2_{L^2(\mathcal{G})}}{|\mathcal{G}|}.$$

Using the previous inequality in (A.3) gives

$$u(y)^2 \leq \frac{\|u\|^2_{L^2(\mathcal{G})}}{|\mathcal{G}|} + 2\|u\|_{L^2(\mathcal{G})}\|u'\|_{L^2(\mathcal{G})}$$

for all $y \in \mathcal{G}$. Taking the supremum for $y \in \mathcal{G}$ gives

$$\|u\|_{L^\infty(\mathcal{G})} \leq \max\left\{2, 1/|\mathcal{G}|\right\}^{1/2} \|u\|_{H^1(\mathcal{G})},$$

which proves the announced Sobolev inequality. □

Corollary A.8. *Let $\mathcal{G}$ be a metric graph and let $p \in [2, +\infty]$. Then, there exists a constant $C(p, |\mathcal{G}|)$ such that, for all $u \in H^1(\mathcal{G})$, the Sobolev inequality*

$$\|u\|_{L^p(\mathcal{G})} \leq \max\left\{2, 1/|\mathcal{G}|\right\}^{1/2-1/p} \|u\|_{H^1(\mathcal{G})}$$

holds. Moreover, if $|\mathcal{G}| = +\infty$, the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(\mathcal{G})} \leq 2^{1/2-1/p} \|u\|^{1/2+1/p}_{L^2(\mathcal{G})} \|u'\|^{1/2-1/p}_{L^2(\mathcal{G})}$$

holds.

Proof. We remark that

$$\|u\|^p_{L^p(\mathcal{G})} = \int_{\mathcal{G}} |u(x)|^p \leq \|u\|^{p-2}_{L^\infty(\mathcal{G})} \|u\|^2_{L^2(\mathcal{G})}. \tag{A.4}$$

If $|\mathcal{G}| < +\infty$, we have

$$\|u\|_{L^\infty(\mathcal{G})} \leq \max\left\{2, 1/|\mathcal{G}|\right\}^{1/2} \|u\|_{H^1(\mathcal{G})},$$

by the $L^\infty(\mathcal{G})$ Sobolev inequality. Since $\|u\|_{L^2(\mathcal{G})} \leq \|u\|_{H^1(\mathcal{G})}$, we obtain

$$\|u\|^p_{L^p(\mathcal{G})} \leq \max\left\{2, 1/|\mathcal{G}|\right\}^{p/2-1} \|u\|^p_{H^1(\mathcal{G})},$$

which proves the Sobolev inequality.

If $|\mathcal{G}| = +\infty$, we have

$$\|u\|_{L^\infty(\mathcal{G})} \leq \sqrt{2}\|u\|_{L^2(\mathcal{G})}^{1/2}\|u'\|_{L^2(\mathcal{G})}^{1/2}$$

by the $L^\infty(\mathcal{G})$ Gagliardo-Nirenberg inequality. Inequality (A.4) gives

$$\|u\|_{L^p(\mathcal{G})}^p \leq 2^{p/2-1}\|u\|_{L^2(\mathcal{G})}^{p/2+1}\|u'\|_{L^2(\mathcal{G})}^{p/2-1},$$

which proves the $L^p(\mathcal{G})$ Gagliardo-Nirenberg inequality. □

Finally, let us show that the injection of $H^1(\mathcal{G})$ in $L^p(\mathcal{G})$ is *compact* when $\mathcal{G}$ is compact and $p \in [2, +\infty]$.

Proposition A.9. *If $\mathcal{G}$ is a compact metric graph then, for every $p \in [2, +\infty]$, the Sobolev embedding $H^1(\mathcal{G}) \hookrightarrow L^p(\mathcal{G})$ is compact.*

Proof. Using Proposition A.1 we know that $\mathcal{G}$ is made of m edges $e_1, \cdots, e_m$ of finite length, so that $\mathbb{E} = \{e_1, \cdots, e_m\}$.

Let $(u_n)_{n\geq 1} \subseteq H^1(\mathcal{G})$ be a bounded sequence in $H^1(\mathcal{G})$. We have to show that there exists a subsequence of $(u_n)_{n\geq 1}$ converging strongly in $L^p(\mathcal{G})$ to some function $v \in L^p(\mathcal{G})$.

Since $(u_n)_{n\geq 1}$ is bounded in $H^1(\mathcal{G})$, its restriction to any edge e_i $(1 \leq i \leq m)$ is bounded in $H^1(e_i)$. Since the embedding of $H^1(I)$ into $L^p(I)$ is compact for any bounded interval $I \subseteq \mathbb{R}$ (see [85, Theorem 8.8]), we deduce, up to passing to a subsequence m times, that $(u_n)_{n\geq 1}$ has a subsequence $(u_{n_k})_{k\geq 1}$ so that for all $1 \leq i \leq m$, the sequence of restrictions $(u_{n_k|e_i})_{k\geq 1}$ converges strongly in $L^p(e_i)$ to some $v_i \in L^p(e_i)$.

Let us define a function $v : \mathcal{G} \to \mathbb{R}$ by[3]

$$v(x) := \begin{cases} v_i(x) & \text{if } x \in \text{int}(e_i), \\ 0 & \text{if } x \in \mathbb{V}. \end{cases}$$

Then, v belongs to $L^p(\mathcal{G})$ and the sequence $(u_{n_k})_{k\geq 1}$ converges strongly to v in $L^p(\mathcal{G})$ as it converges strongly in $L^p(e_i)$ for every $1 \leq i \leq m$. Therefore, the Sobolev embedding $H^1(\mathcal{G}) \hookrightarrow L^p(\mathcal{G})$ is compact. □

A.5 Coarea formulas

A.5.1 A simple case: $\mathcal{C}^1$-diffeomorphisms on intervals

Let $I = (a, b)$ be a bounded nonempty interval of $\mathbb{R}$. Let $f \in \mathcal{C}^1(\bar{I})$ and assume that $f' > 0$ on $[a, b]$. Let $J := (f(a), f(b))$.

[3] In fact, the value given for $x \in \mathbb{V}$ does not matter since $\mathbb{V}$ has measure zero in $\mathcal{G}$.

Then, f^{-1} is a well-defined $\mathcal{C}^1(\bar{J})$ function so that f is a $\mathcal{C}^1$-diffeomorphism between the open sets I and J.

Moreover, if $g \in \mathcal{C}(\bar{I})$, then the change of variables formula with $t = f(x)$ gives

$$\int_a^b g(x) f'(x)\,\mathrm{d}x = \int_{f(a)}^{f(b)} g(f^{-1}(t))\,\mathrm{d}t.$$

Rewriting this equality in a way that will be useful later, we also have

$$\int_I g(x) f'(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left(\sum_{x \in f^{-1}(\{t\})} g(x) \right) \mathrm{d}t, \tag{A.5}$$

where the set $f^{-1}(\{t\})$ is a singleton if y belongs to $\bar{J}$ and is empty otherwise.

A.5.2 Change of variables and coarea formulas for $W^{1,1}_{\mathrm{loc}}(I)$ functions, I interval of $\mathbb{R}$

In all this section, let I be an open, possibly unbounded, interval of $\mathbb{R}$.

The following theorem generalizes the change of variables to $W^{1,1}_{\mathrm{loc}}$ functions. We refer to [176, Theorem 2] or [230, Theorem 1.1] for a proof.

Theorem A.10 (Coarea formula for $W^{1,1}_{\mathrm{loc}}(I)$ functions)**.** *Let $f : I \to \mathbb{R}$ be a $W^{1,1}_{\mathrm{loc}}(I)$ mapping, which we identify with its unique continuous representative. Let $g : I \to [0, +\infty]$ be a nonnegative measurable function. Then,*

1. *For almost every $t \in \mathbb{R}$, the set $f^{-1}(\{t\})$ is countable or finite.*
2. *The map*
$$\mathbb{R} \to [0, +\infty] : t \mapsto \sum_{x \in f^{-1}(\{t\})} g(x)$$
is measurable (note that the sum is taken over an at most countable set for a.e. $t \in \mathbb{R}$ according to the previous claim).
3. *The equality*
$$\int_I g(x) |f'(x)|\,\mathrm{d}x = \int_{\mathbb{R}} \sum_{x \in f^{-1}(\{t\})} g(x)\ \mathrm{d}t \tag{A.6}$$
holds, where both integrals are given by Lebesgue integrals of nonnegative functions, taking values in $[0, +\infty]$.

Remark A.11. There are two important differences between equality (A.6) and the change of variables for $\mathcal{C}^1$-diffeomorphisms (equality (A.5)): the regularity was lowered from $\mathcal{C}^1$ to $W^{1,1}_{\mathrm{loc}}$ and f is not assumed to be injective[4] anymore.

[4]Which was formulated by requiring that $f' > 0$.

A.5.3 The coarea formula on metric graphs

Theorem A.12 (Coarea formula on metric graphs)**.** *Let $\mathcal{G}$ be a metric graph. Let $f : \mathcal{G} \to \mathbb{R}$ be a measurable mapping, whose restrictions to the interior of all edges of $\mathcal{G}$ are assumed to belong to $W^{1,1}_{\text{loc}}$. Let $g : \mathcal{G} \to [0, +\infty]$ be a nonnegative measurable function. Then,*

1. *For almost every $t \in \mathbb{R}$, the set $f^{-1}(\{t\})$ is countable or finite.*
2. *The map*
$$\mathbb{R} \to [0, +\infty] : t \mapsto \sum_{x \in f^{-1}(\{t\})} g(x)$$
is measurable.
3. *The equality*
$$\int_{\mathcal{G}} g(x)|f'(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \sum_{x \in f^{-1}(\{t\})} g(x) \, \mathrm{d}t$$
holds.

Remark A.13. We do not assume that f is continuous at the vertices of $\mathcal{G}$, as this is not required in the proof.

Proof of Theorem A.12. We write $\mathcal{G} = (\mathbb{V}, \mathbb{E})$. For every $e \in \mathbb{E}$, Theorem A.10 applied to the restrictions of f and g to the interior of e implies that $e \cap f^{-1}(\{t\})$ is countable or finite for almost all $e \in \mathbb{E}$ and that

$$\mathbb{R} \to [0, +\infty] : t \mapsto \sum_{x \in e \cap f^{-1}(\{t\})} g(x)$$

is measurable. Since $\mathbb{E}$ and $\mathbb{V}$ are countable or finite, we deduce, again from Theorem A.10, that $f^{-1}(\{t\})$ is countable or finite for almost every e and that

$$t \mapsto \sum_{x \in f^{-1}(\{t\})} g(x) = \sum_{e \in \mathbb{E}} \sum_{x \in e \cap f^{-1}(\{t\})} g(x)$$

is measurable, since a series[5] of measurable functions is measurable. Moreover,

$$\int_{e} g(x)|f'(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \sum_{x \in e \cap f^{-1}(\{t\})} g(x) \, \mathrm{d}t$$

holds for every $e \in \mathbb{E}$.

[5] Or a finite sum.

Applying the Fubini-Tonelli theorem for the Lebesgue integral of nonnegative functions, we obtain

$$\begin{aligned}\int_{\mathcal{G}} g(x)|f'(x)|\,\mathrm{d}x &= \sum_{e\in\mathbb{E}} \int_e g(x)|f'(x)|\,\mathrm{d}x \\ &= \sum_{e\in\mathbb{E}} \int_{\mathbb{R}} \sum_{x\in e\cap f^{-1}(\{t\})} g(x)\ \mathrm{d}t \\ &= \int_{\mathbb{R}} \sum_{e\in\mathbb{E}} \sum_{x\in e\cap f^{-1}(\{t\})} g(x)\ \mathrm{d}t \\ &= \int_{\mathbb{R}} \sum_{x\in f^{-1}(\{t\})} g(x)\ \mathrm{d}t. \qquad \square\end{aligned}$$

From Theorem A.10, we can deduce the following "Sard-type[6]" result.

Proposition A.14. *Let $\mathcal{G}$ be a metric graph. Let $f : \mathcal{G} \to \mathbb{R}$ be a measurable mapping, whose restrictions to the interior of all edges of $\mathcal{G}$ are assumed to belong to $W^{1,1}_{\mathrm{loc}}$. We consider a fixed representative $f' : \mathcal{G} \to \mathbb{R}$ of the weak derivative of f. Then, for almost every $t \in \mathbb{R}$, the set $f^{-1}(\{t\})$ does not contain any critical value, namely*

$$\forall x \in f^{-1}(\{t\}), f'(x) \neq 0.$$

Proof. We define

$$g(x) := \begin{cases} 0 & \text{if } f'(x) \neq 0, \\ 1 & \text{if } f'(x) = 0. \end{cases}$$

Then, g is a nonnegative measurable function such that, for all $x \in \mathcal{G}$, the equality $g(x)|f'(x)| = 0$ holds. Using the coarea formula (Theorem A.10), we obtain

$$0 = \int_{\mathcal{G}} g(x)|f'(x)|\,\mathrm{d}x = \int_{\mathbb{R}} \left(\sum_{x\in f^{-1}(\{t\})} g(x) \right) \mathrm{d}t.$$

Therefore, for almost every $t \in \mathbb{R}$, one has

$$\sum_{x\in f^{-1}(\{t\})} g(x) = 0,$$

which ends the proof by definition of g. $\square$

[6]For a proof of Sard's theorem for smooth functions, see e.g. [242, Chapter 3].

Appendix B
Decreasing rearrangement on metric graphs

In the following sections, we present the notion of *decreasing rearrangement* of functions from a measured space to the half-line. Then, we focus on the case of H^1 functions defined on a metric graph. Using the coarea formula presented in Appendix A (Theorem A.12), we prove precise versions of the Pólya-Szegő inequality, refined by taking into account the "number of preimages" of functions. We show how the symmetric rearrangement of functions is obtained as a variant of the decreasing rearrangement.

The first sections that follow are heavily based on the first chapter of the author's master's thesis [159], itself based on lecture notes by A. Burchard, see [87]. They both contain further references about the rearrangement techniques, their history and their use to study variational problems.

We will use several notions from measure theory. A reference on the topic is the book of J.F. Le Gall [215].

B.1 Decreasing rearrangement on the half-line

B.1.1 Rearrangement of measurable sets on the half-line

Let $(\Omega, \mathcal{A}, \mu)$ be a measured space (e.g. a metric graph equipped with its Borel σ-algebra and the Lebesgue measure of the graph $\lambda_{\mathcal{G}}$, see Appendix A). We denote by λ the Lebesgue measure on $\mathbb{R}$ and do not distinguish it with its restrictions to measurable subsets of $\mathbb{R}$.

Definition B.1. The *rearrangement* of a set $A \in \mathcal{A}$ is the interval A^* given by

$$A^* := (0, \mu(A)),$$

with the convention that $(0,0) := \varnothing$.

Remark B.2. If $A \in \mathcal{A}$, then $\mu(A) = \lambda(A^*)$.

Remark B.3. If $A, B \in \mathcal{A}$, then $A \subseteq B \Rightarrow A^* \subseteq B^*$.

Notation B.4. We denote by $\mathcal{F}^*$ the family of subsets of $(0,+\infty)$ defined by

$$\mathcal{F}^* := \{\varnothing\} \cup \{(0,r) \mid r > 0\} \cup \{(0,+\infty)\}.$$

We remark that $\mathcal{F}^*$ is stable by unions and by finite intersections.

Proposition B.5. *If $A, B \in \mathcal{A}$, then*

$$\mu(A \cap B) \leq \lambda(A^* \cap B^*).$$

Proof. Since A^* and B^* belong to $\mathcal{F}^*$, we have

$$\lambda(A^* \cap B^*) = \min\Big(\lambda(A^*), \lambda(B^*)\Big) = \min\Big(\mu(A), \mu(B)\Big).$$

Moreover,

$$\mu(A \cap B) \leq \mu(A) \quad \text{and} \quad \mu(A \cap B) \leq \mu(B).$$

Therefore,

$$\mu(A \cap B) \leq \min\Big(\mu(A), \mu(B)\Big) = \lambda(A^* \cap B^*).$$

□

Proposition B.6. *If $(A_n)_{n\geq 1} \subseteq \mathcal{A}$ is an increasing sequence, then*

$$\bigcup_{n\geq 1} A_n^* = \left(\bigcup_{n\geq 1} A_n\right)^*.$$

Proof. We first recall that if $(E_n)_{n\geq 1} \subseteq \mathcal{A}$ is an increasing sequence of sets, then

$$\mu\left(\bigcup_{n\geq 1} E_n\right) = \sup_{n\geq 1} \mu(E_n). \tag{B.1}$$

In our case, $(A_n)_{n\geq 1}$ and $(A_n^*)_{n\geq 1}$ are two increasing sequences of sets. Thus, for all $n \geq 1$, we obtain

$$\lambda\left(\bigcup_{n\geq 1} A_n^*\right) = \sup_{n\geq 1} \lambda(A_n^*) = \sup_{n\geq 1} \mu(A_n) = \mu\left(\bigcup_{n\geq 1} A_n\right) = \lambda\left(\left(\bigcup_{n\geq 1} A_n\right)^*\right)$$

where we used equality (B.1) with $(A_n)_{n\geq 1}$ and $(A_n^*)_{n\geq 1}$ as well as the fact that the rearrangement of sets preserves the measure.

The sets $\left(\bigcup_{n\geq 1} A_n^*\right)$ and $\left(\bigcup_{n\geq 1} A_n\right)^*$ belong to $\mathcal{F}^*$ because all sets A_n^* belong to $\mathcal{F}^*$ and this family is stable by union. This ends the proof since two distinct elements of $\mathcal{F}^*$ have distinct measures. □

B.1.2 Decreasing rearrangement of admissible functions on the half-line

Definitions and first properties

Definition B.7. Let $u : \Omega \to [0, +\infty]$ be a nonnegative function. Its *superlevel sets* are defined by

$$\{u > t\} := \Big\{x \in \Omega \mid u(x) > t\Big\}$$

for a parameter $t \in \mathbb{R}$. We say that u is *admissible* if it is measurable and if its superlevel sets have finite measure for all $t > 0$.

Theorem B.8. *Let $u : \Omega \to [0, +\infty]$ be an admissible function. There exists a unique nonnegative function u^* on the open interval $(0, \mu(\Omega))$ (equal to the whole open half-line $(0, +\infty)$ if $\mu(\Omega) = +\infty$) whose superlevel sets are given for all $t \in \mathbb{R}$ by $\{u^* > t\} = \{u > t\}^*$.*

Proof. If such a function u^* exists, it is unique since a nonnegative function is determined by its superlevel sets. To prove the existence, we define

$$u^*(x) := \sup\Big\{t \in \mathbb{R} \,\Big|\, x \in \{u > t\}^*\Big\},$$

for all $x \in (0, \mu(\Omega))$. Given $t \in \mathbb{R}$, let us show that equality $\{u^* > t\} = \{u > t\}^*$ holds. We will prove both inclusions separately.

- If $x \in \{u^* > t\}$, i.e. if $u^*(x) > t$, then there exists $s > t$ such that x belongs to $\{u > s\}^*$, by definition of u^* as a supremum. Therefore, x belongs to $\{u > t\}^*$ since the inclusion $\{u > s\} \subseteq \{u > t\}$ implies that $\{u > s\}^* \subseteq \{u > t\}^*$.
- Using Proposition B.6, we have

$$\{u > t\}^* = \left(\bigcup_{s>t} \{u > s\}\right)^* = \bigcup_{s>t} \{u > s\}^*.$$

  If $x \in \{u > s\}^*$, then $u^*(x) \geq s$ by definition of u^* as a supremum, so $u^*(x) > t$ since $s > t$. Hence, we have $\{u > t\}^* \subseteq \{u^* > t\}$. $\square$

Definition B.9. The function u^* defined in the previous Theorem B.8 is called the *decreasing rearrangement* of u.

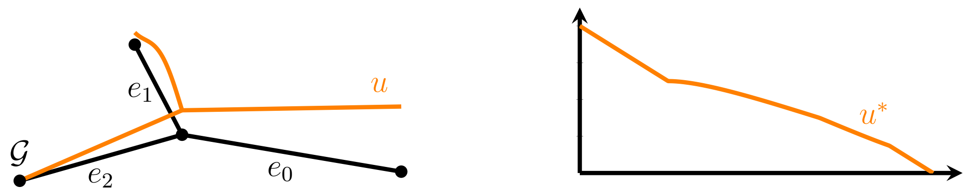


Figure B.1: A function on a metric graph and its decreasing rearrangement

Remark B.10. The following properties hold.

- Rearrangement of functions is compatible with rearrangement of sets, since for every measurable set $A \in \mathcal{A}$, we have $\mathbb{1}_{A^*} = (\mathbb{1}_A)^*$.
- The rearrangement u^* is a decreasing function on $(0, \mu(\Omega))$.
- Two admissible functions equal almost everywhere have the same decreasing rearrangement.

The following useful proposition can be proved using standard techniques (see e.g. [159, Sections I.2 and I.3] for more details).

Proposition B.11. *Let $u : \Omega \to [0, +\infty]$ and $f : [0, +\infty] \to [0, +\infty]$ be two measurable functions. If we assume that $f(0) = 0$ or that $u > 0$, then*

$$\int_\Omega f(u(x))\,\mathrm{d}\mu = \int_0^{\mu(\Omega)} f(u^*(x))\,\mathrm{d}x.$$

Remark B.12. We cannot assume that $f(0) = 0$ without loss of generality, since the fact that two functions $f_1, f_2 : [0, +\infty] \to [0, +\infty]$ are equal almost everywhere does not imply that $\Omega \to \mathbb{R} : x \mapsto f_1(u(x))$ and $\Omega \to \mathbb{R} : x \mapsto f_2(u(x))$ are equal almost everywhere, as can be seen by considering $u \equiv 0$, $f_1 \equiv 0$ and f_2 given by

$$f_2(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Decreasing rearrangement of nonnegative L^2 functions

Notation B.13 (Nonnegative $L^2(\Omega, \mathcal{A}, \mu)$ functions)**.** We denote

$$L^2_+(\Omega, \mathcal{A}, \mu) := \{u \in L^2(\Omega, \mathcal{A}, \mu) \mid u \geq 0\}.$$

Proposition B.14 (Conservation of L^2 norms)**.** *Let $u \in L^2_+(\Omega, \mathcal{A}, \mu)$. The function u^* belongs to $L^2_+(0, \mu(\Omega))$ (where the open half-line is equipped with its Borel σ-algebra and the Lebesgue measure) and we have*

$$\|u\|_{L^2(\Omega,\mathcal{A},\mu)} = \|u^*\|_{L^2(0,\mu(\Omega))}.$$

Remark B.15. This operation is well defined on $L^2_+(\Omega, \mathcal{A}, \mu)$ since two functions equal almost everywhere have equal decreasing rearrangements, see Remark B.10.

Proof. It is clear that the rearranged functions are nonnegative. To prove equality of the L^2 norms, it suffices to apply Proposition B.11 to the function $f(t) := t^2$ (extended by $f(+\infty) = +\infty$). □

The following theorem claims that the decreasing rearrangement increases the L^2 scalar product between functions.

Theorem B.16 (Hardy-Littlewood)**.** *Let $u, v \in L^2_+(\Omega, \mathcal{A}, \mu)$. Then,*

$$\int_\Omega u(x)v(x)\,\mathrm{d}\mu \le \int_0^{\mu(\Omega)} u^*(x)v^*(x)\,\mathrm{d}x.$$

Proof. For every $t \ge 0$ and every $x \in \Omega$, we have

$$\mathbb{1}_{\{u>t\}}(x) = \mathbb{1}_{[0,u(x))}(t). \tag{B.2}$$

The map $t \mapsto \mathbb{1}_{\{u>t\}}(x)$ is measurable since it is the indicator function of the interval $[0, u(x))$. Integrating equality (B.2) on $[0, +\infty]$, we obtain

$$u(x) = \int_{[0,+\infty]} \mathbb{1}_{[0,u(x))}(t)\,\mathrm{d}t = \int_{[0,+\infty]} \mathbb{1}_{\{u>t\}}(x)\,\mathrm{d}t.$$

Applying the same reasoning to v, we deduce that

$$u(x)v(x) = \left(\int_0^{+\infty} \mathbb{1}_{\{u>s\}}(x)\,\mathrm{d}s\right)\left(\int_0^{+\infty} \mathbb{1}_{\{v>t\}}(x)\,\mathrm{d}t\right).$$

Therefore, we obtain

$$\begin{aligned}
\int_\Omega u(x)v(x)\,\mathrm{d}\mu &= \int_\Omega \left(\int_0^{+\infty} \mathbb{1}_{\{u>s\}}(x)\,\mathrm{d}s\right)\left(\int_0^{+\infty} \mathbb{1}_{\{v>t\}}(x)\,\mathrm{d}t\right)\mathrm{d}\mu \\
&= \int_0^{+\infty}\int_0^{+\infty} \left(\int_\Omega \mathbb{1}_{\{u>s\}\cap\{v>t\}}(x)\,\mathrm{d}\mu\right)\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{+\infty}\int_0^{+\infty} \mu(\{u>s\}\cap\{v>t\})\,\mathrm{d}s\,\mathrm{d}t \qquad \text{(B.3)}
\end{aligned}$$

using Fubini's theorem for positive functions.

Similarly, we have

$$\int_0^{+\infty} u^*(x)v^*(x)\,\mathrm{d}x = \int_0^{+\infty}\int_0^{+\infty} \lambda(\{u^* > s\} \cap \{v^* > t\})\,\mathrm{d}s\,\mathrm{d}t. \tag{B.4}$$

By definition of the rearrangement operation, equalities

$$\{u^* > s\} = \{u > s\}^* \text{ and } \{v^* > t\} = \{v > t\}^*$$

are satisfied for every $s, t \geq 0$. Using Proposition B.5, we obtain

$$\mu(\{u > s\} \cap \{v > t\}) \leq \lambda(\{u^* > s\} \cap \{v^* > t\}). \tag{B.5}$$

Hardy-Littlewood's inequality is deduced from (B.3), (B.4) and (B.5). □

The preceding propositions imply that the decreasing rearrangement decreases the L^2 norms between functions, as shown in the following result.

Proposition B.17. *If $u, v \in L^2_+(\Omega, \mathcal{A}, \mu)$, then*

$$\|u^* - v^*\|_{L^2(0,\mu(\Omega))} \leq \|u - v\|_{L^2(\Omega,\mathcal{A},\mu)}\,.$$

In particular, the decreasing rearrangement defines a continuous map

$$L^2_+(\Omega, \mathcal{A}, \mu) \to L^2(0, \mu(\Omega)) : u \mapsto u^*.$$

Proof. Let $u, v \in L^2_+(\Omega, \mathcal{A}, \mu)$. Using the conservation of L^2 norms and Hardy-Littlewood's inequality (Theorem B.16), we obtain

$$\begin{aligned}
\|u^* - v^*\|^2_{L^2(0,\mu(\Omega))} &= \|u^*\|^2_{L^2(0,\mu(\Omega))} + \|v^*\|^2_{L^2(0,\mu(\Omega))} - 2\,(u^* \,|\, v^*)_{L^2(0,\mu(\Omega))} \\
&= \|u\|^2_{L^2(\Omega,\mathcal{A},\mu)} + \|v\|^2_{L^2(\Omega,\mathcal{A},\mu)} - 2\,(u^* \,|\, v^*)_{L^2(0,\mu(\Omega))} \\
&\leq \|u\|^2_{L^2(\Omega,\mathcal{A},\mu)} + \|v\|^2_{L^2(\Omega,\mathcal{A},\mu)} - 2\,(u \,|\, v)_{L^2(\Omega,\mathcal{A},\mu)} \\
&= \|u - v\|^2_{L^2(\Omega,\mathcal{A},\mu)}.
\end{aligned}$$

□

B.2 Decreasing rearrangement of functions in H^1 and the Pólya-Szegő inequality

B.2.1 Piecewise affine functions on intervals

First, we will introduce a precise form of the main inequality under study in a simple setting. This will allow to have some intuition of what is behind the result without needing to use technical tools.

Proposition B.18 (Pólya-Szegő inequality for piecewise affine functions). *Let I be a possibly unbounded open interval of $\mathbb{R}$ and let $u : I \to \mathbb{R}$ be a nonnegative piecewise affine function with bounded support. Let $A \subseteq [0, +\infty)$ be a measurable set and let $N \geq 1$ be an integer so that*

$$\# u^{-1}(\{t\}) \geq N \qquad \text{for a.e. } t \in A.$$

Then, the decreasing rearrangement $u^* : (0, |I|) \to \mathbb{R}$ belongs to $H^1(0, |I|)$ and is such that

$$\|(u^*)'\|_{L^2((u^*)^{-1}(A))} \leq \tfrac{1}{N} \|u'\|_{L^2(u^{-1}(A))}.$$

Proof. We write $u(I) \subseteq \overline{J_1} \cup \cdots \cup \overline{J_K}$, where $J_1, \cdots, J_K$ are disjoint non-empty intervals so that for all $1 \leq i \leq K$, $u^{-1}(J_i)$ consists of a disjoint union of open intervals on which u is affine. Let J be one of those open intervals (see Figure B.2).

Let $I_1, \cdots, I_N$ be so that $u^{-1}(J) = I_1 \cup \cdots \cup I_N$. We define $\ell_i := |I_i|$. A direct computation shows that

$$\|u'\|^2_{L^2(u^{-1}(J))} = \sum_{1 \leq i \leq N} \|u'\|^2_{L^2(I_i)} = \sum_{1 \leq i \leq N} \ell_i \frac{|J|^2}{\ell_i^2} = \sum_{1 \leq i \leq N} \frac{|J|^2}{\ell_i}.$$

Since u^* is decreasing and since $|\{x \in (0, |I|) \mid u^*(x) > t\}| = |\{x \in I \mid u(x) > t\}|$ for all $t \in \mathbb{R}$, the set $(u^*)^{-1}(J)$ is an interval of length $\sum_{1 \leq i \leq N} l_i$, thus

$$\|(u^*)'\|^2_{L^2((u^*)^{-1}(J))} = \frac{|J|^2}{\sum_{1 \leq i \leq N} l_i}.$$

The inequality between arithmetic and harmonic means implies that

$$\frac{\sum_{1 \leq i \leq N} \ell_i}{N} \geq \frac{N}{\sum_{1 \leq i \leq N} \frac{1}{\ell_i}}$$

so that

$$\|u'\|^2_{L^2(u^{-1}(J))} \geq N^2 \|(u^*)'\|^2_{L^2((u^*)^{-1}(J))} \geq \|(u^*)'\|^2_{L^2((u^*)^{-1}(J))}.$$

Adding the previous inequality over all intervals $J_1, \cdots, J_K$ ends the proof. □

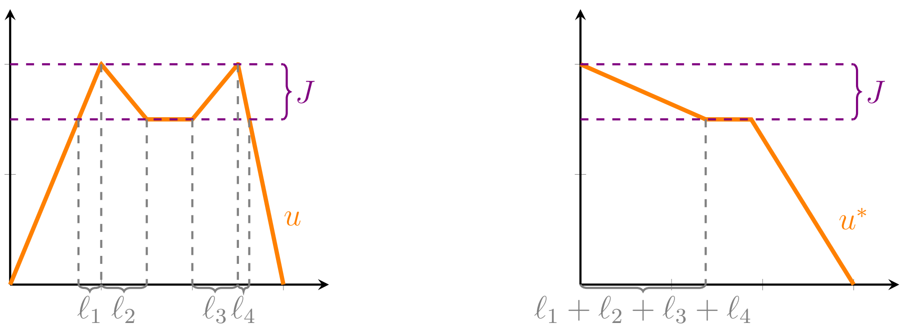


Figure B.2: A piecewise affine function u and its decreasing rearrangement u^*

In what follows, we will prove a similar result on metric graphs and for $H^1(\mathcal{G})$ functions instead of piecewise affine functions. Since the proofs rely on "slicing the range of functions", we will need to use the coarea formulas presented in Section A.5 of Appendix A.

First, let us lower the regularity required in the Pólya-Szegő inequality on intervals of $\mathbb{R}$. This goal is achieved in Proposition B.21.

B.2.2 H^1 functions on intervals

Let us begin by proving the following Lemma.

Lemma B.19. *Let I be a possibly unbounded open interval of $\mathbb{R}$. If there exists $\mathcal{R} \subseteq H^1(I)$, a dense subset of the set of nonnegative $H^1(I)$ functions such that the Pólya-Szegő inequality*

$$\|(u^*)'\|_{L^2(0,|I|)} \leq \|u'\|_{L^2(I)}$$

holds for all $u \in \mathcal{R}$, then it holds for all nonnegative $H^1(I)$ functions.

Remark B.20. One may wonder why we first prove the "classical" Pólya-Szegő inequality for H^1 functions before its refined version as stated in Theorem B.23 below. The main reason for this is that it is important to know *a priori* that u^* belongs to H^1 before applying the reasoning of Section B.2.3.

Moreover, even though we proved a refined Pólya-Szegő inequality for piecewise affine functions in Proposition B.18, it seems delicate to control the number of preimages when approximating H^1 functions by piecewise affine functions, so we "lose the refinement when passing to the limit".

Proof of Lemma B.19. Let u be a nonnegative $H^1(I)$ function. By density, there exists a sequence $(u_n)_{n\geq 1} \subseteq \mathcal{R}$ converging strongly to u in H^1 and such that

$$\|(u_n^*)'\|_{L^2(0,|I|)} \leq \|u_n'\|_{L^2(I)} \tag{B.6}$$

for all $n \geq 1$. Moreover, one has

$$\|u_n^*\|_{L^2(0,|I|)} = \|u_n\|_{L^2(I)}. \tag{B.7}$$

Using (B.6) and (B.7), we deduce that the sequence $(u_n^*)_n$ is bounded in $H^1(0,|I|)$. We can thus assume that it converges weakly to some $v \in H^1(0,|I|)$.

Proposition B.17 implies that $(u_n^*)_n$ converges strongly to u^* in $L^2(0,|I|)$. Since both limits also take place in the sense of distributions, one has $v = u^*$. So $(u_n^*)_n$ converges weakly to u^* in $H^1(0,|I|)$, thus

$$\|(u^*)'\|_{L^2(0,|I|)} \leq \liminf_{n\to\infty} \|(u_n^*)'\|_{L^2(0,|I|)} \leq \liminf_{n\to\infty} \|u_n'\|_{L^2(I)} = \|u'\|_{L^2(I)}$$

by lower semi-continuity, (B.6) and strong H^1 convergence of $(u_n)_n$ to u. □

We now have at our disposal all the required tools to prove the Pólya-Szegő inequality for H^1 functions on intervals.

Proposition B.21 (Pólya-Szegő inequality for H^1 functions on intervals)**.** *Let I be a possibly unbounded open interval of $\mathbb{R}$. Let $u \in H^1(I)$ with $u \geq 0$. Then, its decreasing rearrangement u^* belongs to $H^1(0,|I|)$ and the Pólya-Szegő inequality*

$$\|(u^*)'\|_{L^2(0,|I|)} \leq \|u'\|_{L^2(I)}$$

holds.

Proof. Let $\mathcal{R}$ be the set of piecewise affine functions with bounded support in I. The set $\mathcal{R}$ is dense in H^1. Indeed, one can first approximate H^1 functions by H^1 functions with bounded support, then approximate them by piecewise affine functions[1]. It then suffices to use Proposition B.18 (with A equal to the image of u and $N = 1$) and Lemma B.19. □

Remark B.22. Other choices could have been performed for the dense set $\mathcal{R}$, for instance $\mathcal{C}^1$ functions (see e.g. [135, Theorem 1]). In [135], G.F.D. Duff refines the Pólya-Szegő inequality for $\mathcal{C}^1$ functions on intervals by considering the number of preimages of functions, as we saw for piecewise affine functions in Proposition B.18.

B.2.3 The number of preimages and a refined Pólya-Szegő inequality on graphs

Our goal in this section is to prove the following theorem[2].

Theorem B.23 (Refined Pólya-Szegő inequality)**.** *Let $\mathcal{G}$ be a metric graph and $u \in H^1(\mathcal{G})$ be a nonnegative function. Let $A \subseteq [0,+\infty)$ be a measurable set and let $N \geq 1$ be an integer so that*

$$\#u^{-1}(\{t\}) \geq N \qquad \text{for a.e. } t \in A.$$

Then, the decreasing rearrangement $u^* : (0,|\mathcal{G}|) \to \mathbb{R}$ belongs to $H^1(0,|\mathcal{G}|)$ and is such that

$$\|(u^*)'\|_{L^2((u^*)^{-1}(A))} \leq \tfrac{1}{N}\|u'\|_{L^2(u^{-1}(A))}.$$

Moreover, if the equality

$$\|(u^*)'\|_{L^2((u^*)^{-1}(A))} = \tfrac{1}{N}\|u'\|_{L^2(u^{-1}(A))}$$

holds, then $\#u^{-1}(\{t\}) = N$ for a.e. $t \in A$.

[1] A well-known technique when using finite element methods, see e.g. [84, Section (0.3)].

[2] In what follows, we will consider metric graphs as in Appendix A. In particular, they may have infinitely many edges but are combinatorially locally finite and satisfy $\inf_{e\in\mathbb{E}} |e| > 0$.

Remark B.24. Since $\#u^{-1}(\{t\}) \geq N$ for a.e. $t \in A$, the set $(\mathbb{R} \setminus u(\mathcal{G})) \cap A$ is negligible, i.e. A is "essentially included" in $u(\mathcal{G})$.

Before proving Theorem B.23, we will need the classical Pólya-Szegő inequality. Its proof will turn out to be quite delicate.

Lemma B.25. *Let $u \in H^1(\mathcal{G})$, with $u \geq 0$. Then u^* belongs to $H^1(0, |\mathcal{G}|)$ and*

$$\|(u^*)'\|_{L^2((0,|\mathcal{G}|))} \leq \|u'\|_{L^2(\mathcal{G})}.$$

In the proof of Lemma B.25, we will need the following result.

Lemma B.26 (The "cut-and-paste" lemma)**.** *Let $a, b \in (0, +\infty]$. Consider $u \in H^1(0,a)$, $v \in H^1(0,b)$. We assume that $u \geq 0, v \geq 0$ and that*

$$\overline{\mathrm{Range}(u)} \cap \overline{\mathrm{Range}(v)} \neq \varnothing,$$

where $\mathrm{Range}(u) := \{u(x) \mid x \in (0,a)\}$, $\mathrm{Range}(v) := \{v(x) \mid x \in (0,b)\}$. *Then, there exists a nonnegative decreasing function $\psi \in H^1(0, a+b)$ such that for all $t > 0$, the equality*

$$|\{\psi > t\}| = |\{u > t\}| + |\{v > t\}| \tag{B.8}$$

and the inequality

$$\|\psi'\|^2_{L^2(0,a+b)} \leq \|u'\|^2_{L^2(0,a)} + \|v'\|^2_{L^2(0,b)} \tag{B.9}$$

hold.

Proof of Lemma B.26. According to Proposition B.21, we can assume that u and v are decreasing, up to replacing u by u^* and v by v^*, as this operation does not change the measure of the superlevel sets and decreases the L^2-norm of the derivatives. We extend u and v to decreasing continuous functions on $[0,a]$ and $[0,b]$ respectively, still denoted by u and v.

By assumption, the two intervals $\{u(x) \mid x \in [0,a]\}$ and $\{v(x) \mid x \in [0,b]\}$ have a non-empty intersection, which is a closed interval J.

As u is decreasing, the sets $u^{-1}(J) = [a_1, a_2]$ and $v^{-1}(J) = [b_1, b_2]$ are nonempty closed intervals (possibly reduced to a singleton if $a_1 = a_2$ or $b_1 = b_2$). We split the intervals $[0,a]$ and $[0,b]$ as

$$[0,a] = [0,a_1] \cup [a_1,a_2] \cup [a_2,a], \qquad [0,b] = [0,b_1] \cup [b_1,b_2] \cup [b_2,b].$$

One has[3]

$$u(a_1) = v(b_1) = \sup J = \min(u(0), v(0))$$

[3] If $a = +\infty$ or $b = +\infty$, then $\inf J = 0$ so that $a_2 = a$, $b_2 = b$, $u(+\infty) = v(+\infty) = 0$ and we should just split the intervals in two parts.

and

$$u(a_2) = v(b_2) = \inf J = \max(u(a), v(b)).$$

Moreover, we have either $a_1 = 0$ (if $u(0) \leq v(0)$) or $b_1 = 0$ (if $u(0) \geq v(0)$). Similarly, we have either $a_2 = a$ (if $u(a) \geq v(b)$) or $b_2 = b$ (if $u(a) \leq v(b)$).

We define a function $w : (-(a_2 - a_1), b_2 - b_1) \to \mathbb{R}$ by

$$w(x) := \begin{cases} u(a_1 - x) & \text{if } x \in (-(a_2 - a_1), 0), \\ v(b_1 + x) & \text{if } x \in (0, b_2 - b_1). \end{cases}$$

Since w is continuous at 0 (as $u(a_1) = v(b_1)$), it belongs to $H^1(-(a_2 - a_1), b_2 - b_1)$. The function w^* belongs to $H^1(0, a_2 - a_1 + b_2 - b_1)$ and is such that

$$w^*(0) = u(a_1) = v(b_1), \quad w^*(a_2 - a_1 + b_2 - b_1) = u(a_2) = v(b_2), \tag{B.10}$$

$$\|(w^*)'\|^2_{L^2(0,a_2-a_1+b_2-b_1)} \leq \|u'\|^2_{L^2(a_1,a_2)} + \|v'\|^2_{L^2(b_1,b_2)} \tag{B.11}$$

and, for all $t > 0$, one has

$$|\{w^* > t\}| = |\{w > t\}| = |\{u > t\} \cap [a_1, a_2]| + |\{v > t\} \cap [a_1, a_2]|. \tag{B.12}$$

We are finally ready to define ψ. Since $a_1 = 0$ or $b_1 = 0$, we assume without loss of generality (up to switching the roles of u and v) that $b_1 = 0$. There remain two cases, depending on whether $a_2 = a$ or $b_2 = b$.

We first assume that $b_2 = b$. In this case, we define[4]

$$\psi(x) := \begin{cases} u(x) & \text{if } x \in [0, a_1], \\ w^*(x - a_1) & \text{if } x \in (a_1, a_2 + b], \\ u(x - b) & \text{if } x \in (a_2 + b, a + b]. \end{cases}$$

Using (B.10), we deduce that ψ belongs to $H^1(0, a + b)$. One then checks that it satisfies (B.8) and (B.9) using (B.12) and (B.11). Moreover, ψ is decreasing by definition.

If we do not have $b_2 = b$ but $a_2 = a$, then we define

$$\psi(x) := \begin{cases} u(x) & \text{if } x \in [0, a_1], \\ w^*(x - a_1) & \text{if } x \in (a_1, a + b_2], \\ v(x - a) & \text{if } x \in (a + b_2, a + b], \end{cases}$$

and we end the proof similarly. □

We will also need the following fact about the existence of a suitable covering of $\mathcal{G}$ by graphs with finitely many edges.

---

[4]If $a_2 + b_2 = +\infty$, we define $\psi(x) := \begin{cases} u(x) & \text{if } x \in [0, a_1], \\ w^*(x - a_1) & \text{if } x \in (a_1, +\infty) \end{cases}$ and conclude similarly.

Lemma B.27. *Given a metric graph $\mathcal{G}$, there exists a (possibly finite) sequence $(\mathcal{G}_n)_{n\geq 1}$ of connected metric graphs made of some vertices and edges from $\mathcal{G}$ such that:*

1. *$\mathcal{G}_1$ is made of only one edge e_1;*
2. *for all $n \geq 2$, $\mathcal{G}_n$ is connected and made by adding one edge $e_n \not\subseteq \mathcal{G}_{n-1}$ to $\mathcal{G}_{n-1}$;*
3. *for every edge e of $\mathcal{G}$, there exists $n_0 \geq 1$ such that e belongs to $\mathcal{G}_n$ for all $n \geq n_0$.*

Proof. Let us perform *breadth-first search* on $\mathcal{G}$ (see e.g. [109, Section 22.2]).

More precisely, we pick a vertex $p \in \mathbb{V}$ and choose it as a "start vertex". For every vertex $v \in \mathbb{V}$, there exists a path from p to v consisting of a finite number of edges, by the connectedness assumption. We call $d_{\mathbb{V}}(v)$ the distance between p and v, measured by the least number of edges needed to go from p to v (not taking lengths of edges into account). Since $\deg(v) < \infty$ for all $v \in \mathbb{V}$, we deduce that for every positive integer k, there are only finitely many paths of length k starting from p. So for every integer $k \geq 1$, one has $d_{\mathbb{V}}(v) = k$ only for finitely many $v \in \mathbb{V}$. We can order edges $e \in \mathbb{E}$ by distance too, defining $d_{\mathbb{E}}(vw) := \min(d_{\mathbb{V}}(v), d_{\mathbb{V}}(w))$ for every edge $vw \in \mathbb{E}$ joining vertices v and w. Note that $d_{\mathbb{E}}(e) < \infty$ for every edge $e \in \mathbb{E}$ and that for every $k \geq 1$, there are only finitely many edges with $d_{\mathbb{E}}(e) = k$ (again the assumption that $\deg(v) < \infty$ for all $v \in \mathbb{V}$ is essential).

To construct the sequence of graphs $(\mathcal{G}_n)_{n\geq 1}$, we consider any edge e_1 with $d_{\mathbb{E}}(e_1) = 1$ (there must be at least one since $\deg(p) \geq 1$) and take $\mathcal{G}_1$ consisting of just e_1. Then, we construct the sequence iteratively, where $\mathcal{G}_n$ is obtained by adding exactly one edge e_n to $\mathcal{G}_{n-1}$. We ensure that edges are added by increasing order of $d_{\mathbb{E}}$, breaking ties arbitrarily.

Property (1) is then satisfied by construction, as well as property (3) since there are only finitely many $e \in \mathbb{E}$ with $d_{\mathbb{E}}(e) = k$ for every integer $k \geq 1$ so that every edge will eventually be added.

It remains to check the connectedness property of $\mathcal{G}_n$ claimed in point (2). If an edge $e_n = vw$ is added to $\mathcal{G}_{n-1}$ to obtain $\mathcal{G}_n$, then assuming without loss of generality that $d_{\mathbb{E}}(e_n) = d_{\mathbb{V}}(v) \leq d_{\mathbb{V}}(w)$, we remark that all edges $f_1, \dots, f_{d_{\mathbb{V}}(v)}$ from any shortest path between p and v satisfy $d_{\mathbb{E}}(f_i) < d_{\mathbb{V}}(v) = d_{\mathbb{E}}(e_n)$ and are thus present in $\mathcal{G}_{n-1}$ since edges are added by increasing order of $d_{\mathbb{E}}$. Therefore, there exists a path from p to v using edges of $\mathcal{G}_{n-1}$. Using e_n, one can join p and w using edges of $\mathcal{G}_n$. Since by assumption $\mathcal{G}_{n-1}$ is connected, there exists a path between p and every vertex of $\mathcal{G}_{n-1}$ using edges of $\mathcal{G}_{n-1} \subseteq \mathcal{G}_n$. Thus $\mathcal{G}_n$ is connected, as all vertices of $\mathcal{G}_n$ can be joined by finite paths going through p, which ends the proof. □

We can now prove the classical Pólya-Szegő inequality.

Proof of Lemma B.25. We consider the sequence $(\mathcal{G}_n)_{n\geq 1}$ of metric graphs given by Lemma B.27 and we define a sequence $(u_n)_{n\geq 1} \subseteq L^2(\mathcal{G})$ by

$$u_n(x) := \begin{cases} u(x) & \text{if } x \in \mathcal{G}_n, \\ 0 & \text{otherwise.} \end{cases}$$

By assumption on the sequence $(\mathcal{G}_n)_{n\geq 1}$ and monotone convergence, one has

$$u_n \xrightarrow[n\to\infty]{L^2(\mathcal{G})} u.$$

Therefore, using Proposition B.17, we have

$$u_n^* \xrightarrow[n\to\infty]{L^2(0,|\mathcal{G}|)} u^*. \tag{B.13}$$

Let us prove that for all $n \geq 1$, u_n^* belongs to $H^1(0, |\mathcal{G}_n|)$ and that

$$\|(u_n^*)'\|_{L^2(0,|\mathcal{G}_n|)} \leq \|u'\|_{L^2(\mathcal{G}_n)}. \tag{B.14}$$

We proceed by (possibly finite) induction on $n \geq 1$.

- Base case: if $n = 1$, the Pólya-Szegő inequality for intervals (Proposition B.21) implies that
$$\|(u_1^*)'\|_{L^2(0,|e_1|)} \leq \|u'\|_{L^2(e_1)},$$
which is what we wanted to prove since $\mathcal{G}_1$ is only made of e_1.
- Induction step: by induction hypothesis, we know that
$$\|(u_{n-1}^*)'\|_{L^2(0,|\mathcal{G}_{n-1}|)} \leq \|u'\|_{L^2(\mathcal{G}_{n-1})}. \tag{B.15}$$
We consider the function v_n defined on $\mathcal{G}$ by
$$v_n(x) := \begin{cases} u(x) & \text{if } x \in e_n, \\ 0 & \text{otherwise.} \end{cases}$$
The Pólya-Szegő inequality for intervals (Proposition B.21) implies that
$$\|(v_n^*)'\|_{L^2(0,|e_n|)} \leq \|u'\|_{L^2(e_n)}. \tag{B.16}$$
We use the "cut-and-paste" Lemma B.26 with functions $u_{n-1}^* \in H^1(0, |\mathcal{G}_{n-1}|)$ and $v_n^* \in H^1(0, |e_n|)$. Note that the hypothesis $\overline{\text{Range}(u_{n-1}^*)} \cap \overline{\text{Range}(v_n^*)} \neq \varnothing$ is satisfied since $\overline{\text{Range}(u_{n-1}^*)} = \overline{\text{Range}(u_{n-1})}$, $\overline{\text{Range}(v_n^*)} = \overline{\text{Range}(v_n)}$, and u_{n-1} as well as v_n are restrictions of the continuous function u on $\mathcal{G}_{n-1}$ and e_n, which share at least one vertex by the construction of the graphs $\mathcal{G}_n$.

This gives a function $\psi_n \in H^1(0,|\mathcal{G}_n|)$ such that for all $t>0$,

$$|\{\psi_n > t\}| = |\{u_{n-1}^* > t\}| + |\{v_n^* > t\}| \tag{B.17}$$

and such that

$$\|\psi_n'\|^2_{L^2(0,|\mathcal{G}_n|)} \leq \|(u_{n-1}^*)'\|^2_{L^2(0,|\mathcal{G}_{n-1}|)} + \|(v_n^*)'\|^2_{L^2(0,|e_n|)}. \tag{B.18}$$

Using equality (B.17), we obtain for all $t>0$,

$$|\{\psi_n > t\}| = |\{u_{n-1}^* > t\}| + |\{v_n^* > t\}| = |\{u_{n-1} > t\}| + |\{v_n > t\}| = |\{u_n > t\}|,$$

hence $\psi_n = u_n^*$ since ψ_n is continuous and decreasing. Using inequalities (B.15), (B.16) and (B.18), we obtain

$$\begin{aligned}\|(u_n^*)'\|^2_{L^2(0,|\mathcal{G}_n|)} &= \|\psi_n'\|^2_{L^2(0,|\mathcal{G}_n|)}\\ &\leq \|(u_{n-1}^*)'\|^2_{L^2(0,|\mathcal{G}_{n-1}|)} + \|(v_n^*)'\|^2_{L^2(0,|e_n|)}\\ &\leq \|u'\|^2_{L^2(\mathcal{G}_{n-1})} + \|u'\|^2_{L^2(e_n)}\\ &= \|u'\|^2_{L^2(\mathcal{G}_n)}.\end{aligned}$$

We consider the sequence $(w_n)_{n\geq 1} \subseteq L^2(0,|\mathcal{G}|)$ defined by[5]

$$w_n(x) := \begin{cases}(u_n^*)'(x) & \text{if } 0 < x < |\mathcal{G}_n|,\\ 0 & \text{otherwise.}\end{cases}$$

We remark that the restriction of w_n to $(0,|\mathcal{G}_n|)$, which is u_n^*, belongs to $H^1(0,|\mathcal{G}_n|)$.

Inequality (B.14) implies that, for all $n \geq 1$, we have

$$\|w_n\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

Hence, $(w_n)_{n\geq 1}$ is bounded in $L^2(0,|\mathcal{G}|)$ and, up to a subsequence, it converges weakly to some function $w \in L^2(0,|\mathcal{G}|)$. By weak lower semi-continuity, one has

$$\|w\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}. \tag{B.19}$$

According to (B.13), $(u_n^*)_{n\geq 1}$ converges to u^* in $L^2(0,|\mathcal{G}|)$. It remains to show that w is the weak derivative of u^*. To do so, we need to show that for every $a \in (0,|\mathcal{G}|)$, the restriction of w to $(0,a)$ is a weak derivative of u^* on $(0,a)$. This is true since for n large enough, one has $|\mathcal{G}_n| > a$ (since $|\mathcal{G}_n| \xrightarrow[n\to\infty]{} |\mathcal{G}|$), so that w_n and u_n have equal restrictions on $(0,a)$, and we conclude since u_n belongs to $H^1(0,|\mathcal{G}_n|)$. Finally, (B.19), implies that

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} = \|w\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}$$

which ends the proof. □

The following lemma will be crucial in the proof of Theorem B.23.

[5]Note that w_n may be discontinuous at $x = |\mathcal{G}_n|$ since $u_n^*(|\mathcal{G}_n|)$ may be nonzero.

Lemma B.28. *Let $\mathcal{G}$ be a metric graph. Let $u \in H^1(\mathcal{G})$ be nonnegative and let us define m and M by*

$$m := \inf_{\mathcal{G}} u = \inf_{(0,|\mathcal{G}|)} u^* \quad \text{and} \quad M := \sup_{\mathcal{G}} u = \sup_{(0,|\mathcal{G}|)} u^*.$$

Then, for almost every $t \in (m, M)$, all three sums or series

$$\sum_{x \in u^{-1}(\{t\})} \frac{1}{|u'(x)|}, \qquad \sum_{x \in (u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|}, \qquad \sum_{x \in (u^*)^{-1}(\{t\})} |(u^*)'(x)|$$

belong to $(0, +\infty)$, and one has

$$\sum_{x \in u^{-1}(\{t\})} \frac{1}{|u'(x)|} = \sum_{x \in (u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|} = \left(\sum_{x \in (u^*)^{-1}(\{t\})} |(u^*)'(x)| \right)^{-1}.$$

Remark B.29. Assuming some regularity assumptions, one can show that

$$\sum_{x \in u^{-1}(\{t\})} \frac{1}{|u'(x)|} = -\mu_u'(t), \tag{B.20}$$

where

$$\mu_u(t) := |\{x \in \mathcal{G} \mid u(x) > t\}|$$

see [158, equality (2.5)] (where sets $\{u < t\}$ are considered instead of $\{u > t\}$, which is not convenient when $|\mathcal{G}| = +\infty$ as they all have infinite measure). Equality (B.20) explains why this expression should be preserved by rearrangement, as we have $\mu_u = \mu_{u^*}$.

Proof of Lemma B.28. Using Proposition A.14, we know that for almost every $t \in \mathbb{R}$, it holds that for all $x \in u^{-1}(\{t\}), u'(x) \neq 0$. Therefore, for almost all $t \in (m, M)$, the three previous sums are positive, as they contain at least one positive term.

Since the function u^* is decreasing, the set $(u^*)^{-1}(\{t\})$ is a singleton for almost every $t \in (m, M)$. So the sums $\sum_{x \in (u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|}$ and $\sum_{x \in (u^*)^{-1}(\{t\})} |(u^*)'(x)|$ are made of only one term for almost every t and in particular obviously converge.

We want to prove that the positive measurable maps

$$(m, M) \to [0, +\infty] : t \mapsto \sum_{x \in (u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|} \tag{B.21}$$

and

$$(m, M) \to [0, +\infty] : t \mapsto \sum_{x \in u^{-1}(\{t\})} \frac{1}{|u'(x)|} \tag{B.22}$$

are equal almost everywhere.

Lemma B.25 ensures that u^* belongs to $H^1(0,|\mathcal{G}|)$. Moreover, Theorem A.10 and Proposition A.14 imply that the set

$$T := \Big\{ t \in (m,M) \mid u^{-1}(\{t\}) \text{ and } (u^*)^{-1}(\{t\}) \text{ are finite or countable,} \\ (u^*)'(x) \neq 0 \text{ for all } x \in (u^*)^{-1}(\{t\}) \text{ and } u'(x) \neq 0 \text{ for all } x \in u^{-1}(\{t\}) \Big\} \quad \text{(B.23)}$$

has measure $M-m$. In particular, the sums (or series) only contain terms in $(0,+\infty)$ for almost every t.

Using a standard technique in measure theory, the two functions (B.21) and (B.22) are equal almost everywhere if and only if for every nonnegative measurable map $h : (m,M) \to [0,+\infty]$, one has[6]

$$\int_m^M h(t) \left(\sum_{x \in (u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|} \right) \mathrm{d}t = \int_m^M h(t) \left(\sum_{x \in u^{-1}(\{t\})} \frac{1}{|u'(x)|} \right) \mathrm{d}t,$$

where both integrals are well defined in $[0,+\infty]$. Up to modifying h on a set of measure zero, we can assume that $h(t) = 0$ if $t \in (m,M) \setminus T$. Equivalently, we need to show that

$$\int_m^M \sum_{x \in (u^*)^{-1}(\{t\})} g_1(x)\,\mathrm{d}t = \int_m^M \sum_{x \in u^{-1}(\{t\})} g_2(x)\,\mathrm{d}t, \quad \text{(B.24)}$$

where

$$g_1(x) = \begin{cases} \frac{h(u^*(x))}{|(u^*)'(x)|} & \text{if } (u^*)(x) \notin T, \\ 0 & \text{if } (u^*)(x) \in T \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} \frac{h(u(x))}{|u'(x)|} & \text{if } u(x) \notin T, \\ 0 & \text{if } u(x) \in T. \end{cases}$$

The definition of T ensures that g_1 and g_2 are well defined. Since we assumed h to vanish on T, the previous definition ensures that $g_1(x)|(u^*)'(x)| = h(u^*(x))$ for all $x \in (0,|\mathcal{G}|)$ and that $g_2(x)|u'(x)| = h(u(x))$ for all $x \in \mathcal{G}$. Using the coarea formula (Theorem A.10) with $f = u^* \in H^1(0,|\mathcal{G}|)$ and $g = g_1$, the left hand side of (B.24) is equal to

$$\int_0^{|\mathcal{G}|} h(u^*(x))\,\mathrm{d}x.$$

Using again Theorem A.10 with $f = u \in H^1(\mathcal{G})$ and $g(x) = g_2$, the right hand side of (B.24) is equal to

$$\int_{\mathcal{G}} h(u(x))\,\mathrm{d}x.$$

[6]Let us prove that $f = g$ a.e. if for every mesurable $h \geq 0$, one has $\int hf = \int hg$ (the other implication is clear). Taking $h = \mathbb{1}_{\{f \geq g\}}$, one has $\int \mathbb{1}_{\{f\geq g\}}(f-g) = 0$, where the integrated function is nonnegative, thus $\mathbb{1}_{\{f\geq g\}}(f-g) = 0$ a.e. Similarly, one shows that $\mathbb{1}_{\{f<g\}}(f-g) = 0$ a.e. Summing the two previous equalities, we deduce that $f - g = 0$ a.e., which proves the claim.

It therefore remains to show that

$$\int_0^{|\mathcal{G}|} h(u^*(x))\,\mathrm{d}x = \int_{\mathcal{G}} h(u(x))\,\mathrm{d}x,$$

which is true using Proposition B.11 as $h(0) = 0$ (since h was defined on (m, M) and extended to $\mathbb{R}$ by 0) and as u^* vanishes on $(|\mathcal{G}|, +\infty)$ so $\int_{|\mathcal{G}|}^{+\infty} h(u^*(x))\,\mathrm{d}x = 0$. This ends the proof of the almost everywhere equality of maps (B.21) and (B.22).

Finally, equality

$$\sum_{x\in(u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|} = \left(\sum_{x\in(u^*)^{-1}(\{t\})} |(u^*)'(x)|\right)^{-1}$$

is clearly true for all t for which $(u^*)^{-1}(\{t\})$ is a singleton, which is the case for a.e. $t \in (m, M)$. The almost everywhere finiteness of the three expressions follows since the sums $\sum_{x\in(u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|}$ and $\sum_{x\in(u^*)^{-1}(\{t\})} |(u^*)'(x)|$ converge for a.e. $t \in (m, M)$ as they only contain one term, which belongs to $(0, +\infty)$. □

Let us prove Theorem B.23. We adapt [158, Proof of Lemma 3] to our setting.

Proof of Theorem B.23. Using the coarea formula for graphs (Theorem A.12), with $f = u$ and $g = \mathbb{1}_{u^{-1}(A)}|u'|$, we obtain

$$\int_{u^{-1}(A)} |u'(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \sum_{x\in u^{-1}(\{t\})} \mathbb{1}_{u^{-1}(A)}(x)|u'(x)|\,\mathrm{d}t = \int_A \sum_{x\in u^{-1}(\{t\})} |u'(x)|\,\mathrm{d}t. \tag{B.25}$$

In particular, for almost every $t \in A$, one has

$$0 < \sum_{x\in u^{-1}(\{t\})} |u'(x)| < +\infty, \tag{B.26}$$

where the positivity comes from Proposition A.14 and the fact that $u^{-1}(\{t\})$ is nonempty for a.e. $t \in A$ (see Remark B.24), and the finiteness from equality (B.25) (since $\int_{u^{-1}(A)} |u'(x)|^2\,\mathrm{d}x \le \int_{\mathcal{G}} |u'(x)|^2\,\mathrm{d}x < +\infty$). Similarly, taking $f = u^*$ and $g = \mathbb{1}_{(u^*)^{-1}(A)}|(u^*)'|$, we have

$$\int_{(u^*)^{-1}(A)} |(u^*)'(x)|^2\,\mathrm{d}x = \int_A \sum_{x\in(u^*)^{-1}(\{t\})} |(u^*)'(x)|\,\mathrm{d}t. \tag{B.27}$$

Let us consider $t \in A$ such that $\#u^{-1}(\{t\}) \ge N$ (which is possible for a.e. $t \in A$ by Lemma B.28 and by assumption on A) and such that the conclusions of Lemma B.28 hold, namely that the three expressions

$$\sum_{x\in u^{-1}(\{t\})} \frac{1}{|u'(x)|} = \sum_{x\in(u^*)^{-1}(\{t\})} \frac{1}{|(u^*)'(x)|} = \left(\sum_{x\in(u^*)^{-1}(\{t\})} |(u^*)'(x)|\right)^{-1} \tag{B.28}$$

are equal and belong to $(0, +\infty)$.

The Cauchy-Schwarz inequality implies that one has

$$\left(\sum_{x\in u^{-1}(\{t\})} |u'(x)|\right)\left(\sum_{x\in u^{-1}(\{t\})} \frac{1}{|u'(x)|}\right) \geq N^2. \tag{B.29}$$

Equality in inequality (B.29) implies that $\#u^{-1}(\{t\}) = N$.

Using equality (B.25) and inequality (B.29), we obtain

$$\int_{u^{-1}(A)} |u'(x)|^2 \,\mathrm{d}x \geq N^2 \int_A \left(\sum_{x\in u^{-1}(\{t\})} \frac{1}{|u'(x)|}\right)^{-1} \mathrm{d}t,$$

and if equality holds, then $\#u^{-1}(\{t\}) = N$ for a.e. $t \in A$, thanks to the equality case in equality (B.29) and since (B.26) implies that $\sum_{x\in u^{-1}(\{t\})} \frac{1}{|u'(x)|}$ is nonzero for a.e. $t \in A$.

Using equalities (B.27) and (B.28), we obtain

$$\int_{u^{-1}(A)} |u'(x)|^2 \,\mathrm{d}x \geq N^2 \int_A \sum_{x\in(u^*)^{-1}(\{t\})} |(u^*)'(x)| \,\mathrm{d}t = N^2 \int_{(u^*)^{-1}(A)} |(u^*)'(x)|^2 \,\mathrm{d}x,$$

where equality implies that for almost all $t \in A$, $\#u^{-1}(\{t\}) = N$. □

B.3 Symmetric rearrangement on the line

It is also possible to rearrange sets and functions symmetrically on the real line. In this case, the symmetric rearrangement of a measurable set $A \in \mathcal{A}$ is defined by $\hat{A} := (-\mu(A)/2, \mu(A))$. We get the following analogue of Theorem B.8.

Theorem B.30. *Let $u : \Omega \to [0, +\infty]$ be an admissible function. There exists a unique nonnegative even function $\hat{u} : (-\mu(\Omega)/2, \mu(\Omega)/2) \to \mathbb{R}$ decreasing on the interval $[0, \mu(\Omega)/2)$ whose superlevel sets are given for all $t \in \mathbb{R}$ by $\{\hat{u} > t\} = \widehat{\{u > t\}}$.*

The natural analogues of Propositions B.11 and B.17 are true as well for the symmetric rearrangement of functions on the real line and can be proved similarly. They can be deduce from the results we proved before. Indeed, defining

$$\hat{u}(x) := u^*(2|x|),$$

one can check that $\hat{u}$ does define the symmetrized function from Theorem B.30 and deduce properties of $\hat{u}$ from those of u^*. In particular, there exists a Pólya-Szegő inequality for the symmetric rearrangement.

Theorem B.31 (Pólya-Szegő inequality for the symmetric rearrangement). *Let $\mathcal{G}$ be a metric graph, $u \in H^1(\mathcal{G})$ be a nonnegative function and $A \subseteq [0, +\infty)$ be a measurable set. Let $N \geq 1$ be an integer so that*

$$\#u^{-1}(\{t\}) \geq N \qquad \text{for a.e. } t \in A.$$

Then, the symmetric rearrangement

$$\hat{u} : (-|\mathcal{G}|/2, |\mathcal{G}|/2) \to \mathbb{R}$$

belongs to $H^1(-|\mathcal{G}|/2, |\mathcal{G}|/2)$ and is such that

$$\|\hat{u}'\|_{L^2(\hat{u}^{-1}(A))} \leq \tfrac{2}{N} \|u'\|_{L^2(u^{-1}(A))}.$$

Moreover, if the equality

$$\|\hat{u}'\|_{L^2(\hat{u}^{-1}(A))} = \tfrac{2}{N} \|u'\|_{L^2(u^{-1}(A))}$$

holds, then $\#u^{-1}(\{t\}) = N$ for a.e. $t \in A$.

A typical corollary of the previous theorem, used in the study of the nonlinear Schrödinger equation on metric graphs[7], is the following.

Corollary B.32. *Let $\mathcal{G}$ be a metric graph and $u \in H^1(\mathcal{G})$ be a nonnegative function so that $\#u^{-1}(\{t\}) \geq 2$ for a.e. $t \in (\inf_{\mathcal{G}} u, \sup_{\mathcal{G}} u)$. Then,*

$$\|\hat{u}'\|_{L^2(-|\mathcal{G}|/2, |\mathcal{G}|/2)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

[7] See e.g. [19, Proposition 3.1] and the arguments using the hypotheses (H), (H0) or (H1) in the two first chapters of the thesis.

Appendix C
The soliton on the real line

In dimension one, the equation (NLS) takes the form

$$-u'' + \lambda u = |u|^{p-2}u \tag{$\mathrm{NLS}_{\mathrm{ODE}}$}$$

where λ and $p > 2$ are two real parameters. Let us present some properties of the ordinary differential equation (ODE).

In particular, we will investigate the properties of the *soliton* $\phi_{\lambda,p}$, the unique nonzero solution of ($\mathrm{NLS}_{\mathrm{ODE}}$) in $H^1(\mathbb{R})$, up to sign and translations.

C.1 The ODE energy and the phase plane

In the analysis of ($\mathrm{NLS}_{\mathrm{ODE}}$), it is useful to consider the *potential* $V_\lambda \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ given by

$$V_\lambda(u) := \frac{|u|^p}{p} - \frac{\lambda |u|^2}{2},$$

so that ($\mathrm{NLS}_{\mathrm{ODE}}$) can be written as $-u'' = V'_\lambda(u)$.

When $\lambda > 0$, the potential has the following geometry.

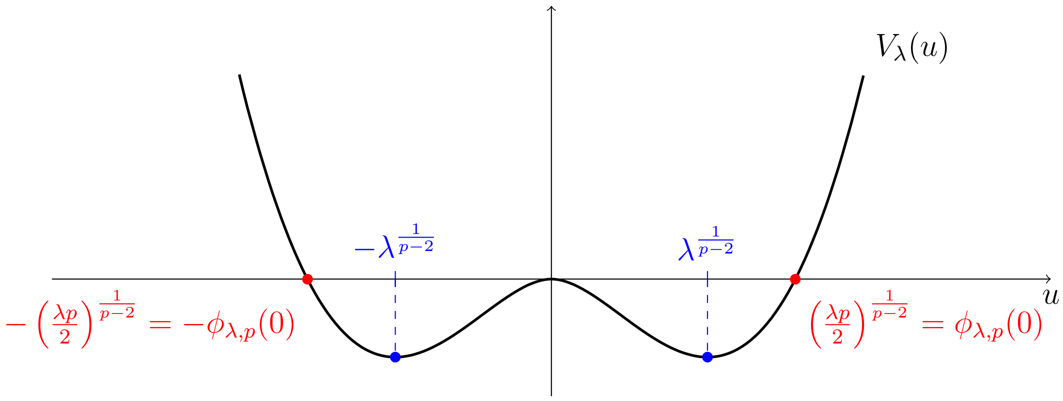


Figure C.1: The potential V_λ when $\lambda > 0$

For every x in the domain of u, we also define the *ODE energy*[1]

$$H_u(x) := \frac{1}{2}(u'(x))^2 + V_\lambda(u(x)). \tag{C.1}$$

[1] Not to be confused with the energy functional $E(u) := \frac{1}{2}\|u'\|^2_{L^2} - \frac{1}{p}\|u\|^p_{L^p}$.

Proposition C.1. *If u is a solution of* ($\mathrm{NLS}_{\mathrm{ODE}}$) *on some interval $I \subseteq \mathbb{R}$, then the function $x \mapsto H_u(x)$ is constant on I.*

Proof. Differentiating (C.1), we obtain

$$H_u'(x) = u'(x) \cdot \Big(u''(x) + V_\lambda'(u(x))\Big) = 0. \qquad \square$$

Thus, all solutions of ($\mathrm{NLS}_{\mathrm{ODE}}$) are such that the curves $x \mapsto (u(x), u'(x))$ in the *phase plane*[2] stay on level curves of the energy, as shown in the following phase portrait.

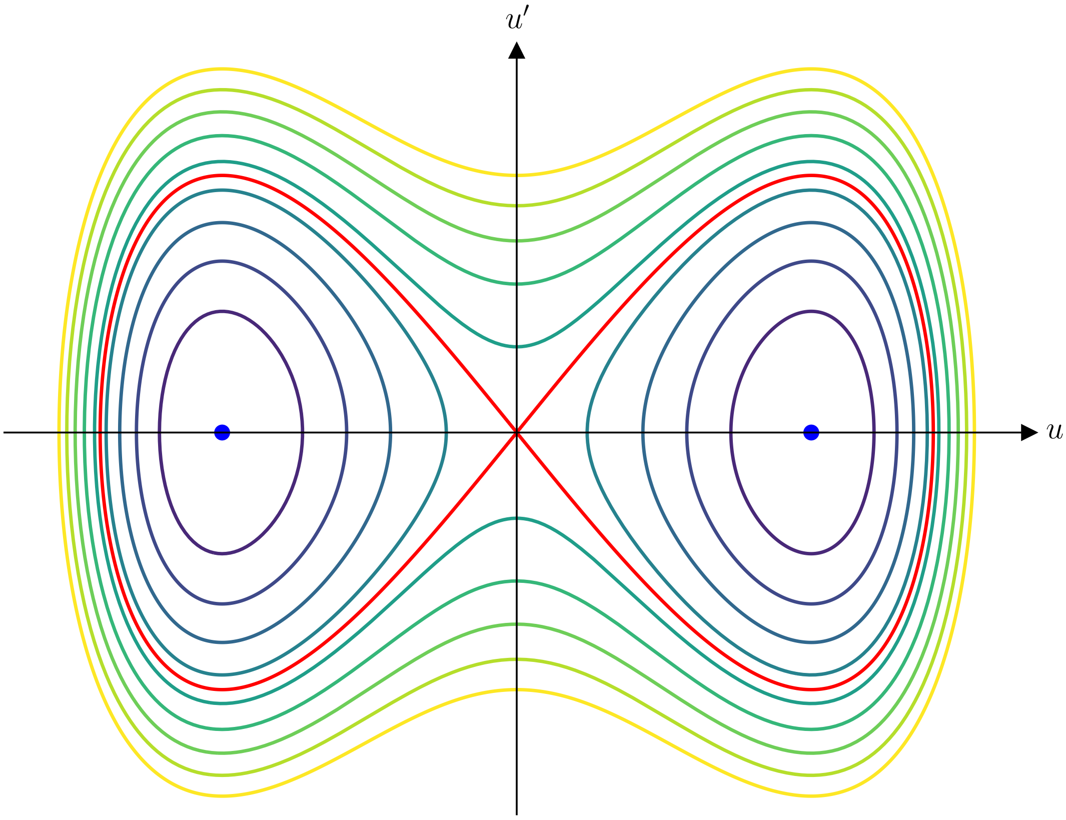


Figure C.2: Phase portrait of the ODE when $\lambda > 0$

The minimal value of the ODE energy is attained for $c := \lambda^{\frac{1}{p-2}}$ and $-c$, shown in blue in the phase plane. It corresponds to constant solutions of the ODE.

When $\lambda > 0$, there are only two level curves of the energy at level 0, joining the origin of the axis to itself. They are shown in red.

Finally, let us remark that all nonzero solutions u which vanish at some point are such that $H_u > 0$.

[2]See e.g. [33, Chapter 2] for a presentation of the phase plane method.

Proposition C.2. *Let $p > 2$ and λ be two real numbers.*
Assume that $u : [0, +\infty) \to \mathbb{R}$ is a $\mathcal{C}^2$ solution of $(\mathrm{NLS_{ODE}})$ *such that*

$$u(x) \xrightarrow[x\to+\infty]{} 0.$$

Then,

- *if $\lambda \leq 0$, then $u \equiv 0$;*
- *if $\lambda > 0$, then there exist $\varepsilon \in \{-1, 0, 1\}$ and $a \in \mathbb{R}$ such that for all $x \geq 0$,*

$$u(x) = \varepsilon \phi_{\lambda,p}(x - a)$$

*where $\phi_{\lambda,p}$ is the **soliton** defined by*

$$\phi_{\lambda,p}(x) := \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-2}{p-2}}.$$

Proof. Checking that the soliton $\phi_{\lambda,p}$ is a solution to $(\mathrm{NLS_{ODE}})$ follows from the explicit computations hereunder.

$$\begin{aligned}
\phi'_{\lambda,p}(x) &= \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-2}{p-2}-1} \frac{-2}{p-2} \sinh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right) \frac{p-2}{2}\lambda^{\frac{1}{2}} \\
&= -\lambda^{\frac{1}{2}} \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-p}{p-2}} \sinh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right),
\end{aligned}$$

$$\begin{aligned}
\phi''_{\lambda,p}(x) &= -\lambda^{\frac{1}{2}} \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-p}{p-2}-1} \frac{-p}{p-2} \sinh^2\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right) \frac{p-2}{2}\lambda^{\frac{1}{2}} \\
&\quad - \lambda^{\frac{1}{2}} \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-p}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right) \frac{p-2}{2}\lambda^{\frac{1}{2}} \\
&= \lambda \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{2-2p}{p-2}} \left[\cosh^2\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right) - \frac{p}{2}\right],
\end{aligned}$$

so that

$$\begin{aligned}
-\phi''_{\lambda,p}(x) + \lambda\phi_{\lambda,p}(x) &= \lambda \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{2-2p}{p-2}} \frac{p}{2} \\
&= \left[\left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}} \cosh\left(\frac{p-2}{2}\lambda^{\frac{1}{2}} x\right)^{\frac{-2}{p-2}}\right]^{p-1} \\
&= \phi_{\lambda,p}(x)^{p-1}.
\end{aligned}$$

Moreover, one easily checks that $\phi_{\lambda,p}$ belongs to $\mathcal{C}^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$.

Now, Proposition C.1 shows that the ODE energy $H_u(x)$ defined by (C.1) does not depend on x, so that the equality

$$H_u(0) = \frac{1}{2}(u'(x))^2 + V_\lambda(u(x))$$

holds for all x. When $x \to +\infty$, $V_\lambda(u(x)) \to 0$, thus $(u'(x))^2 \to 2H_u(0)$.

If $H_u(0) \neq 0$, then $u'(x)$ converges to a nonzero constant as $x \to +\infty$, which contradicts the fact that $u(x) \to 0$. Therefore, $H_u(0) = H_u(x) = 0$ for all x.

When $\lambda \leq 0$, this implies that $u \equiv 0$ since $V_\lambda(u) \geq 0$ for all $u \in \mathbb{R}$.

When $\lambda > 0$, the phase plane analysis shows that the zero level set of the energy is the disjoint union of $\{(0,0)\}$ with two curves connecting the origin of the phase plane to itself (see the red curves in Figure C.2). Those curves correspond to $x \mapsto (\phi_{\lambda,p}(x), \phi'_{\lambda,p}(x))$ and $x \mapsto (-\phi_{\lambda,p}(x), -\phi'_{\lambda,p}(x))$ which ends the proof. □

Reasoning as above, we easily deduce the following result.

Corollary C.3. *Given two real numbers λ and $p > 2$, the equation* (NLS$_{\mathrm{ODE}}$)*:*

- *has no $H^1(\mathbb{R})$ solutions when $\lambda \leq 0$;*
- *has a set of solutions given by*

$$\left\{\varepsilon\phi_{\lambda,p}(x-a) \mid \varepsilon \in \{-1,0,1\}, a \in \mathbb{R}\right\}$$

  when $\lambda > 0$.

C.2 Action level and mass of the soliton

Explicit computations based on the expression of the soliton show that the following proposition holds.

Proposition C.4. *Let $p > 2$ and $\lambda > 0$ be two real numbers. The action level and the L^2-mass of $\phi_{\lambda,p}$ are respectively given by*

$$s_{\lambda,p} := J_\lambda(\phi_{\lambda,p}) = \lambda^{\frac{p+2}{2(p-2)}} s_{1,p} \quad \text{and} \quad \|\phi_{\lambda,p}\|_{L^2} = \lambda^{\frac{6-p}{4(p-2)}} \|\phi_{1,p}\|_{L^2}.$$

Remark C.5. Let us remark that the derivative of

$$(0,+\infty) \to \mathbb{R} : \lambda \mapsto s_{\lambda,p}$$

is a multiple of

$$(0,+\infty) \to \mathbb{R} : \lambda \mapsto \|\phi_{\lambda,p}\|_{L^2}^2.$$

This fact is very general, as can be seen on compact domains in Chapter 3.

C.3 Energy ground states on the real line

Proposition C.4 shows that the exponent $p = 6$ plays a special role when studying L^2-masses of solitons. Indeed:

- when $2 < p < 6$, the map $(0, +\infty) \to \mathbb{R} : \lambda \mapsto \|\phi_{\lambda,p}\|_{L^2}$ is increasing;
- when $p = 6$, all solitons have the same L^2-mass;
- when $p > 6$, the map $(0, +\infty) \to \mathbb{R} : \lambda \mapsto \|\phi_{\lambda,p}\|_{L^2}$ is decreasing.

This leads us to the following definition.

Definition C.6. Let $2 < p < 6$ and $\mu > 0$ be two real numbers. The *mass-μ soliton* $\widehat{\phi}_{\mu,p}$ is defined as the unique element of the set $\{\phi_{\lambda,p} \mid \lambda \in (0, +\infty)\}$ such that $\|\phi_{\lambda,p}\|^2_{L^2(\mathbb{R})} = \mu$.

The mass-μ solitons satisfy the following variational characterization.

Proposition C.7. *Let $2 < p < 6$ be a real number. The energy functional $E : H^1(\mathbb{R}) \to \mathbb{R}$ defined by*

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})}$$

is minimized by $\widehat{\phi}_{\mu,p}$ on the mass constraint

$$H^1_\mu(\mathbb{R}) := \Big\{u \in H^1(\mathbb{R}) \mid \|u\|^2_{L^2(\mathbb{R})} = \mu\Big\}.$$

Moreover, the set of constrained minima of E on $H^1_\mu(\mathbb{R})$ is given by

$$\Big\{\Big(x \mapsto \varepsilon\widehat{\phi}_{\mu,p}(x-a)\Big) \;\Big|\; a \in \mathbb{R}, \varepsilon \in \{-1, 1\}\Big\}.$$

Remark C.8. Using the terminology presented in section II.5, this means that the solitons of mass μ are the energy ground states for $(\mathrm{NLS}_{\mathrm{ODE}})$ on the real line.

Proof. The fact that E admits a constrained minimum on $H^1_\mu(\mathbb{R})$ follows from concentration-compactness methods (see e.g. [99, Theorem II.1]).

The Euler-Lagrange equation associated to a constrained minimum is given by $(\mathrm{NLS}_{\mathrm{ODE}})$, where $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier associated to the mass constraint[3]. Corollary C.3 shows that such a minimum is necessarily (up to translations and sign) a mass-μ soliton, which ends the proof. □

[3] The reasoning to obtain the associated ODE is similar to the one presented for the action functional in section II.3.5.

C.4 Energy ground states on the half-line

Let us present the equivalent of Proposition C.7 on the half-line.

Proposition C.9. *Let $2 < p < 6$ be a real number. The energy functional $E : H^1(\mathbb{R}^+) \to \mathbb{R}$ defined by*

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R}^+)}$$

is minimized by $\widehat{\phi}_{\mu,p}$ on the mass constraint

$$H^1_\mu(\mathbb{R}^+) := \Big\{u \in H^1(\mathbb{R}^+) \mid \|u\|^2_{L^2(\mathbb{R}^+)} = \mu\Big\}.$$

Moreover, the set of constrained minima of E on $H^1_\mu(\mathbb{R}^+)$ is given by

$$\Big\{\Big(x \mapsto \varepsilon\widehat{\phi}_{2\mu,p}(x)\Big) \;\Big|\; \varepsilon \in \{-1, 1\}\Big\}.$$

Namely, an energy ground state of mass μ on $[0, +\infty)$ is obtained by "cutting $\widehat{\phi}_{2\mu}$ in half at its maximum point".

Proof. Once again, we take the existence of energy ground states for granted[4].

As in the proof of Proposition C.7, we show that if u is a constrained minimum then u solves ($\mathrm{NLS}_{\mathrm{ODE}}$), but in this case we also deduce[5] that $u'(0) = 0$. We end the proof like the one of Proposition C.7 remarking that, since we only consider solitons ϕ_λ due to the condition $u'(0) = 0$ (and not their translates $x \mapsto \phi_\lambda(x-a)$), thus having a mass μ on $\mathbb{R}^+$ is equivalent to having a mass 2μ on $\mathbb{R}$. □

C.5 Uniqueness of positive solutions of ($\mathrm{NLS}_{\mathrm{ODE}}$) on intervals with Dirichlet conditions

The aim of this section is to prove the following result.

Proposition C.10. *Let $\lambda > 0$, $p > 2$ and $L > 0$ be three real numbers. Then, the boundary value problem*

$$\begin{cases} -u'' + \lambda u = |u|^{p-2}u & \text{on } (0, L), \\ u(0) = u(L) = 0, \end{cases}$$

has a unique positive solution u.

[4] It is a classical result, stated e.g. in [16, Page 4].

[5] As shown for the action functional in section II.3.5.

Proof. Since we are looking for solutions with Dirichlet boundary conditions, they must be portions of periodic solutions, as can be seen in the phase portrait (Figure C.2). We distinguish them by their maximum value v with $v > v_* := \left(\frac{\lambda p}{2}\right)^{\frac{1}{p-2}}$.

The period of such a solution is given by

$$T(v) = 2\int_{-v}^{v} \frac{\mathrm{d}x}{\sqrt{2(V_\lambda(v) - V_\lambda(x))}} = 2\sqrt{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{\frac{v^{p-2}}{p}(1-t^p) - \frac{\lambda}{2}(1-t^2)}}.$$

Step 1. The period function

$$T : (v_*, +\infty) \to (0, +\infty) : v \mapsto T(v)$$

is well-defined and is a continuous, decreasing bijection.

First observe that, for any $0 \le t \le 1$, we have that

$$\frac{v^{p-2}}{p}(1-t^p) - \frac{\lambda}{2}(1-t^2) \ge \left(\frac{v^{p-2}}{p} - \frac{\lambda}{2}\right)(1-t^2), \tag{C.2}$$

so, since $\frac{v^{p-2}}{p} - \frac{\lambda}{2} > 0$, this proves that the function in the definition of $T(v)$ is integrable. It is obvious that T is continuous and decreasing.

Let us study its limits as $v \to v_*$ and $v \to +\infty$.

- *Behavior of T for $v \approx v_*$.* We have seen that for any $t \in [0, 1]$, the function

$$v \mapsto \frac{1}{\sqrt{\frac{v^{p-2}}{p}(1-t^p) - \frac{\lambda}{2}(1-t^2)}}$$

  is decreasing. Therefore, the monotone convergence theorem implies that when $v \to v_*$, $T(v)$ converges towards the Lebesgue integral (taking values in $[0, +\infty]$)

$$2\sqrt{2}\int_0^1 \frac{\mathrm{d}t}{\sqrt{\frac{\lambda}{2}(t^2 - t^p)}}.$$

  However, near $t = 0$ the integrand behaves like $\frac{1}{t}$, which is not integrable near $t = 0$. Thus,

$$\lim_{v \to v_*} T(v) = +\infty.$$

- *Behavior of T for $v \to +\infty$.* Using (C.2), we obtain

$$T(v) \le \frac{2\sqrt{2}}{\sqrt{\frac{v^{p-2}}{p} - \frac{\lambda}{2}}}\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}},$$

  so that (since the integral is a finite constant) $\lim_{v\to+\infty} T(v) = 0$.

Step 2. End of the proof.

Positive solutions of ($\mathrm{NLS_{ODE}}$) are obtained exactly when $T(v) = 2L$. The previous properties about T imply that there exists exactly one such solution, which concludes the proof. □

Appendix D
Maximum principles

Maximum principles are a useful tool in nonlinear analysis. We refer to [283] for a reference in the classical setting of intervals of the real line or domains of $\mathbb{R}^N$.

Such principles have already been used on metric graphs in the literature, see for instance [19, Proposition 3.3].

In a joint undergoing project[1] with Pablo Carrillo, Colette De Coster, Louis Jeanjean and Christophe Troestler, we had to further study maximum principles on metric graphs.

We present the results we obtained hereunder. A maximum principle can be proved in a quite general setting (Proposition D.1). Under extra hypotheses, it can be refined to obtain a *strong maximum principle* (Proposition D.2).

Proposition D.1 (Maximum principle)**.** *Let $\mathcal{G}$ be a graph with a finite number of vertices, $V \in \mathcal{C}(\mathcal{G})$ be a nonnegative function, and $\varphi \in \mathcal{C}(\mathcal{G})$ be a function of class $\mathcal{C}^1$ on each edge of $\mathcal{G}$ (up to its boundary) and twice differentiable in the interior of each edge. Set $L(\varphi) := -\varphi'' + V(x)\varphi$. Assume that*

(1) $L(\varphi) \geq 0$ *on every edge* e *of* $\mathcal{G}$*;*

(2) for every vertex $v \in \mathbb{V}$, $\quad \varphi(v) < 0 \;\Rightarrow\; \sum_{\mathrm{e} \succ v} \dfrac{\mathrm{d}\varphi_{\mathrm{e}}}{\mathrm{d}x}(v) \leq 0.$

Then, one of the following three possibilities applies:

(a) $\varphi \geq 0$ *on* $\mathcal{G}$*;*

(b) there exists $C < 0$ *such that* $\varphi \equiv C$ *and* $V \equiv 0$ *on* $\mathcal{G}$*;*

(c) $\inf_{\mathcal{G}} \varphi = -\infty$*.*

In the third case, $\mathcal{G}$ *contains at least one half-line along which* $\lim\limits_{x \to \infty} \varphi = -\infty$*.*

[1] Devoted to a blow-up analysis of solutions for nonlinear Schrödinger equations on graphs in the asymptotic regime $\lambda \to +\infty$, see Lemma 4.32 and the discussion below it.

Proof. Assume by contradiction that φ is non constant and that $-\infty < \inf_{\mathcal{G}} \varphi < 0$.

Step 1. For every half-line e_0, if $\inf_{e_0} \varphi < 0$*, then* $\inf_{e_0} \varphi = \varphi(0)$.

Otherwise, there exists an half-line e and $\bar{x} \in e$ such that $\varphi(\bar{x}) < \min(0, \varphi(0))$ and $\varphi'(\bar{x}) < 0$ (with the parametrization of e starting at the vertex 0).

Let $I := \{x > \bar{x} \mid \varphi(x) < 0\}$. By assumption (1), we have that, for all $x \in I$, $\varphi''(x) \leq 0$ so that, for all $x \in I$, $\varphi'(x) \leq \varphi(\bar{x}) < 0$. This implies that $I = (\bar{x}, +\infty)$ and, along the edge e, we have $\lim_{x\to\infty} \varphi(x) = -\infty$, which contradicts $\inf_{\mathcal{G}} \varphi > -\infty$.

Step 2. For every bounded edge e, if $\inf_e \varphi < 0$ *and the vertices of e are v_1, v_2 then* $\inf_e \varphi = \min(\varphi(v_1), \varphi(v_2))$.

Let e be a bounded edge with $\inf_e \varphi < 0$. As e is compact, we have $x_0 \in e$ such that $\varphi(x_0) = \min_e \varphi < 0$. Let us prove that x_0 is a vertex of e.

Otherwise, if x_0 belongs to the interior of e, there exist $x_1 < x_0 < x_2$ such that, for all $x \in (x_1, x_2)$, $\varphi(x) < 0$. By assumption (1), this implies that the inequality

$$\varphi(x_0) \leq \varphi(x) \leq \varphi(x_0) + \int_{x_0}^{x} \int_{x_0}^{t} V(s)\varphi(s)\,\mathrm{d}s\,\mathrm{d}t \leq \varphi(x_0)$$

holds for all $x \in (x_1, x_2)$, which implies that φ is constant on (x_1, x_2). Thus, $(x_1, x_2) = e$ and $V \equiv 0$ on e. In particular, $\min_e \varphi = \varphi(v_1) = \varphi(v_2)$.

Step 3. There exists $x_0 \in \mathcal{G}$ such that $\inf_{\mathcal{G}} \varphi = \min_{\mathcal{G}} \varphi = \varphi(x_0)$.

Assume that $\inf_{\mathcal{G}} \varphi$ is not attained. Let $(x_n)_n$ and $(i_n)_n$ be two sequences such that $x_n \in e_{i_n}$, $\varphi(x_n) < 0$, $\varphi(x_n) \to \inf_{\mathcal{G}} \varphi$ and $(\varphi(x_n))_n$ is decreasing.

By Steps 1 and 2, we can consider that $(x_n)_n$ is a sequences of vertices. This contradicts the fact that the graph has finitely many vertices.

Conclusion.

By Step 3, there exists $x_0 \in \mathcal{G}$ such that $\inf_{\mathcal{G}} \varphi = \min_{\mathcal{G}} \varphi = \varphi(x_0)$.

By Steps 1 and 2, we can consider that x_0 is a vertex of $\mathcal{G}$. As $\varphi(x_0) = \min_{\mathcal{G}} \varphi$, we have $\frac{\mathrm{d}\varphi_e}{\mathrm{d}x}(x_0) \geq 0$ for all $e \succ x_0$. By assumption (2), this implies that for all $e \succ x_0$, $\frac{\mathrm{d}\varphi_e}{\mathrm{d}x}(x_0) = 0$. As in Step 2, this implies that $\varphi(x) \equiv \varphi(x_0)$ and $V \equiv 0$ on all the edges $e \succ x_0$. Iterating the procedure, we deduce that $\varphi(x) \equiv \varphi(x_0)$ on $\mathcal{G}$.

It remains to prove the last point concerning the case $\inf_{\mathcal{G}} = -\infty$. This can be deduced as in Step 1 if there exists an half-line e and $x_0 \in e$ such that $\varphi(x_0) < 0$ and $\varphi'(x_0) < 0$.

Assume it is not the case, namely that for every half-line e and every $x \in e$ such that $\varphi(x) < 0$, we have $\varphi'(x) \geq 0$. Then, we have

$$\inf_{\cup\{\text{half-lines}\}} \varphi \geq \min\Big\{\min(\varphi(v_i), 0) \mid v_i \text{ is a vertex of some half-line}\Big\}.$$

As in Step 3, we deduce that there exists $x_0 \in \mathcal{G}$ such that $\inf_{\mathcal{G}} \varphi = \min_{\mathcal{G}} \varphi = \varphi(x_0)$, which contradicts the assumption $\inf_{\mathcal{G}} = -\infty$. $\square$

Here is the statement of the strong maximum principle, taking into account the possible presence of Dirichlet vertices.

Proposition D.2 (Strong Maximum Principle)**.** *Let $\mathcal{G}$ be a graph, $V \in \mathcal{C}(\mathcal{G})$ be a nonnegative function, and $\varphi \in \mathcal{C}(\mathcal{G})$ be a function of class $\mathcal{C}^1$ on each edge of $\mathcal{G}$ (up to its boundary) and twice differentiable in the interior of each edge. Let Z be a (possibly empty) set of vertices of $\mathcal{G}$ having degree one. Set $L(\varphi) := -\varphi'' + V(x)\varphi$. Assume that*

(1) $L(\varphi) \geq 0$ in every edge e of $\mathcal{G}$;

(2) for every vertex $v \in \mathbb{V} \setminus Z$, $\quad \varphi(v) = 0 \;\Rightarrow\; \sum_{\mathrm{e} \succ v} \frac{\mathrm{d}\varphi_{\mathrm{e}}}{\mathrm{d}x}(v) \leq 0;$

(3) $\varphi \geq 0$ on $\mathcal{G}$.

Then, either $\varphi \equiv 0$ or, for all $x \in \mathcal{G} \setminus Z$, $\varphi(x) > 0$.

Proof. Otherwise, there exists an edge $\bar{e}$ and two points $x_0, x_1 \notin Z$ of $\bar{e}$ such that $\varphi(x_0) = 0$ and $\varphi(x_1) > 0$. If x_0 belongs to the interior of $\bar{e}$ then, since x_0 is a local minimum, $\varphi'(x_0) = 0$. If x_0, then (using again that $x_0 \notin Z$ is a local minimum) for every $\mathrm{e} \succ x_0$, $\frac{\mathrm{d}\varphi_{\mathrm{e}}}{\mathrm{d}x}(x_0) \geq 0$ and by hypothesis (2), we get $\frac{\mathrm{d}\varphi_{\bar{e}}}{\mathrm{d}x}(x_0) = 0$.

Up to parameterizing the edge in the reverse direction, we may assume that $x_0 < x_1$. Then, the restriction of φ to the interval $[x_0, x_1]$ satisfies $\varphi(x_0) = 0$, $\varphi'(x_0) = 0$ and $\varphi(x_1) > 0$.

Now[2], we consider z, the solution of the Cauchy problem

$$\begin{cases} -z'' + V(t)z = 1, \\ z(x_0) = 0, \;\; z'(x_0) = -1, \end{cases}$$

and we consider the function

$$w := \varphi + \epsilon z,$$

where $\epsilon > 0$ is small enough so that $w(x_1) > 0$. Let us remark that

$$L(w) > 0 \text{ on } [x_0, x_1], \quad w(x_0) = 0 \;\text{ and }\; w(x_1) > 0.$$

Using the maximum principle on $[x_0, x_1]$ (see e.g. [122, Appendix, Theorem 5.1]), we deduce that $w(x) \geq 0$ on $[x_0, x_1]$. This inequality contradicts the fact that $w(x_0) = 0$ and $w'(x_0) = -\epsilon < 0$, which ends the proof. □

[2]From there on, we proceed as in the proof of [122, Appendix, Theorem 5.1].

Appendix E
Implicit function theorems (IFT)

In this appendix, we state and prove several results related to the implicit function theorem (IFT) under rather weak regularity assumptions. They will be used in Chapter 5.

Even though those kinds of results and their proofs are quite standard, we chose to include them for completeness and to use minimal hypotheses.

This appendix is based on notes written by Christophe Troestler and then reworked by Colette De Coster and the author.

E.1 Topological IFT

Theorem E.1. *Let X be a topological space and Y, Z be Banach spaces. Let us also consider $F : X \times Y \to Z : (x,y) \mapsto F(x,y)$ and $(x_0, y_0) \in X \times Y$ satisfying*

1. $F(x_0, y_0) = 0$*;*
2. *the map $X \to Z : x \mapsto F(x, y_0)$ is continuous at x_0;*
3. *$\partial_y F$ exists in a neighborhood of (x_0, y_0) and is continuous at (x_0, y_0);*
4. *$\partial_y F(x_0, y_0) \in \mathcal{L}(Y; Z)$ is an isomorphism.*

Then there exists V, a neighborhood of x_0, W, a neighborhood of y_0, and a map $\eta : V \to W$ such that

- $\forall (x,y) \in V \times W,\ F(x,y) = 0 \Leftrightarrow y = \eta(x)$*;*
- *η is continuous at x_0.*

Remark E.2. In particular, we have $\eta(x_0) = y_0$.

Remark E.3. Under the above assumptions, F is continuous at (x_0, y_0). Indeed,

$$\begin{aligned} \|F(x,y) - F(x_0,y_0)\|_Z &\le \|F(x,y) - F(x,y_0)\|_Z + \|F(x,y_0) - F(x_0,y_0)\|_Z \\ &\le \sup_{z \in (y_0,y)} \|\partial_y F(x,z)\| \, \|y - y_0\|_Y + \|F(x,y_0) - F(x_0,y_0)\|_Z \end{aligned}$$

Since $\partial_y F$ is continuous at (x_0, y_0), it is locally bounded and so the supremum is finite when y is close enough to y_0. The result follows then from assumption (2).

Proof. Define $G : X \times Y \to Y$ by

$$G(x,y) := y - \Big(\partial_y F(x_0,y_0)\Big)^{-1} F(x,y).$$

A direct computation using the continuity of $\partial_y F$ shows that

$$\partial_y G(x,y) = \mathbb{1} - \Big(\partial_y F(x_0,y_0)\Big)^{-1} \partial_y F(x,y) \xrightarrow[(x,y)\to(x_0,y_0)]{} 0.$$

Therefore, there exist V, a neighborhood of x_0, and $W = B(y_0,R)$ a neighborhood of y_0 such that

$$\forall (x,y) \in V \times W, \quad \|\partial_y G(x,y)\| \le 1/2. \tag{E.1}$$

Possibly making V smaller, we can also assume that

$$\forall x \in V, \quad \|G(x,y_0) - y_0\|_Y \le R/2.$$

Thus, for all $x \in V$, $G(x,\cdot)$ maps $B[y_0,R]$ into itself. Indeed,

$$\begin{aligned} \|G(x,y) - y_0\|_Y &\le \|G(x,y) - G(x,y_0)\|_Y + \|G(x,y_0) - y_0\|_Y \\ &\le \sup_{z\in(y_0,y)} \|\partial_y G(x,z)\| \, \|y - y_0\|_Y + R/2 \\ &\le R/2 + R/2. \end{aligned}$$

Given (E.1), for all $x \in V$, $G(x,\cdot)$ is a contraction on the complete metric space $B[y_0,R]$. The contraction mapping theorem then implies the existence of a map $\eta : V \to B[y_0,R]$ such that

$$\forall x \in V, \quad \eta(x) \text{ is the unique fixpoint of } G(x,\cdot) \text{ in } B[y_0,R].$$

Moreover, η is continuous at x_0. Indeed,

$$\begin{aligned} \|\eta(x) - \eta(x_0)\|_Y &= \|\eta(x) - y_0\|_Y \\ &= \|G(x,\eta(x)) - G(x_0,y_0)\|_Y \\ &\le \|G(x,\eta(x)) - G(x,y_0)\|_Y + \|G(x,y_0) - G(x_0,y_0)\|_Y \\ &\le \sup_{z\in(y_0,\eta(x))} \|\partial_y G(x,z)\| \, \|\eta(x) - y_0\|_Y + \|G(x,y_0) - G(x_0,y_0)\|_Y \\ &\le \tfrac{1}{2}\|\eta(x) - y_0\|_Y + \|G(x,y_0) - G(x_0,y_0)\|_Y. \end{aligned}$$

Thus, we obtain

$$\forall x \in V, \quad \|\eta(x) - \eta(x_0)\|_Y \le 2\|G(x,y_0) - G(x_0,y_0)\|_Y, \tag{E.2}$$

which implies the continuity of η at x_0 since $x \mapsto G(x,y_0)$ is continuous at x_0. □

Remark E.4. Let $x_1 \in V$ and $y_1 = \eta(x_1)$. If $x \mapsto F(x,y_1)$ is continuous at $x_1 \in V$, we can adapt the argument at the end of the proof to show that η is continuous at x_1. Note that the continuity of $\partial_y F$ at (x_1,y_1) is not required.

E.2 Differentiable IFT

Theorem E.5. *Under the assumptions of Theorem E.1, if further X is a part of a normed space and if $\partial_x F(x_0, y_0)$ exists, then $\partial_x \eta(x_0)$ exists and is given by*

$$\partial_x \eta(x_0) = -\Big(\partial_y F(x_0, y_0)\Big)^{-1} \partial_x F(x_0, y_0). \tag{E.3}$$

Proof. We use the notations used in the proof of Theorem E.1.

Step 1. If $\partial_x \eta(x_0)$ exists, it is given by (E.3).

If $\partial_x \eta(x_0)$ exists, it must be the unique $y_0' \in \mathcal{L}(X; Y)$ such that the equalities

$$y_0' = \partial_x G(x_0, y_0) + \partial_y G(x_0, y_0) \circ y_0' = \partial_x G(x_0, y_0)$$

hold. This is equivalent to (E.3).

Step 2. There exists $C > 0$ such that $\|\eta(x) - \eta(x_0)\| \leq C\|x - x_0\|$.

Notice that the differentiability of $x \mapsto G(x, y_0)$ and the inequality (E.2) imply (possibly taking V smaller) that, for all $x \in V$,

$$\|\eta(x) - \eta(x_0)\| \leq 2\Big(\|\partial_x G(x_0, y_0)\| + 1\Big)\|x - x_0\|$$

Step 3. The map G is Fréchet differentiable at (x_0, y_0).

Given (x, y), we have

$$\begin{aligned}
&\Big\|G(x, y) - G(x_0, y_0) - \partial_x G(x_0, y_0)[x - x_0] - \partial_y G(x_0, y_0)[y - y_0]\Big\| \\
= &\Big\|G(x, y) - G(x, y_0) + G(x, y_0) - G(x_0, y_0) \\
&\quad - \partial_x G(x_0, y_0)[x - x_0] - \partial_y G(x_0, y_0)[y - y_0]\Big\| \\
\leq &\Big\|G(x, y) - G(x, y_0) - \partial_y G(x, y_0)[y - y_0]\Big\| \\
+ &\Big\|\partial_y G(x, y_0) - \partial_y G(x_0, y_0)\Big\| \, \|y - y_0\| \\
+ &\Big\|G(x, y_0) - G(x_0, y_0) - \partial_x G(x_0, y_0)[x - x_0]\Big\|.
\end{aligned}$$

We recall that $\partial_x G(x_0, y_0)$ exists and that $\partial_y G$ exists in a neighborhood of (x_0, y_0) and is continuous at (x_0, y_0). Therefore, we deduce that for every $\epsilon > 0$, taking V smaller if necessary, the inequalities

$$\Big\|\partial_y G(x, y_0) - \partial_y G(x_0, y_0)\Big\| \, \|y - y_0\| \leq \frac{\epsilon}{3}(\|x - x_0\| + \|y - y_0\|)$$

and

$$\Big\|G(x, y_0) - G(x_0, y_0) - \partial_x G(x_0, y_0)[x - x_0]\Big\| \leq \frac{\epsilon}{3}(\|x - x_0\| + \|y - y_0\|)$$

hold for all $x \in V$.

Fixing $x \in V$, again taking W smaller if necessary (in particular, assuming it is convex), for every $y \in W$, the mean value theorem implies that

$$\left\|G(x,y) - G(x,y_0) - \partial_y G(x,y_0)[y-y_0]\right\| \leq \|y-y_0\| \sup_{z\in W}\left\|\partial_y G(x,z) - \partial_y G(x,y_0)\right\|.$$

Since $(x,y) \mapsto \partial_y G(x,y)$ is continuous at (x_0,y_0), we deduce that, if W is small enough, the inequality

$$\left\|G(x,y) - G(x,y_0) - \partial_y G(x,y_0)[y-y_0]\right\| \leq \frac{\epsilon}{3}\|y-y_0\|$$

holds for all $y \in W$. This proves the Fréchet differentiability of G at (x_0,y_0).

Conclusion.

We want to show that

$$\lim_{x\to x_0} \frac{\left\|\eta(x) - \eta(x_0) - \partial_x G(x_0,y_0)[x-x_0]\right\|}{\|x-x_0\|} = 0.$$

Using the fact that $\eta(x)$ and y_0 are fixpoints and the equality $\partial_y G(x_0,y_0) = 0$, this will be the case if we prove that, for all $\epsilon > 0$, up to restricting again V, the inequality

$$\begin{aligned}&\left\|G(x,\eta(x)) - G(x_0,y_0) - \partial_x G(x_0,y_0)[x-x_0] - \partial_y G(x_0,y_0)[\eta(x)-y_0]\right\| \\ &\leq \epsilon\|x-x_0\| \end{aligned} \tag{E.4}$$

holds for all $x \in V$. This ends the proof since (E.4) follows from the differentiability of G at (x_0,y_0) and Step 2. □

Theorem E.6. *Under the assumptions of Theorem E.1, let $x_1 \in V$ and define $y_1 := \eta(x_1)$. If further X is a part of a normed space, $\partial_x F(x_1,y_1)$ exists and $\partial_y F$ is[a] continuous at (x_1,y_1), then $\partial_x \eta(x_1)$ exists and is given by*

$$\partial_x \eta(x_1) = -\Big(\partial_y F(x_1,y_1)\Big)^{-1} \partial_x F(x_1,y_1). \tag{E.5}$$

[a]The map $\partial_y F$ is supposed to exist in a neighborhood of (x_0,y_0) in Theorem E.1.

Proof. The first three points of the previous proof still hold with the exception that the derivative $\partial_y G(x_1,y_1)$, with $y_1 := \eta(x_1)$, may not vanish. But we still have that $\|\partial_y G(x_1,y_1)\| \leq 1/2$ and hence $\mathbb{1} - \partial_y G(x_1,y_1)$ is invertible. This implies that the equation

$$y_1' = \partial_x G(x_1,y_1) + \partial_y G(x_1,y_1) \circ y_1' \tag{E.6}$$

has a unique solution $y_1' \in \mathcal{L}(X;Y)$.

We want to show that

$$\lim_{x \to x_1} \frac{\|\eta(x) - \eta(x_1) - y_1'[x - x_1]\|}{\|x - x_1\|} = 0. \tag{E.7}$$

Using the equality $\eta(x) = G(\eta(x))$ and replacing y_1' according to (E.6), we obtain

$$\begin{aligned} &\eta(x) - \eta(x_1) - y_1'[x - x_1] \\ &= G(x, \eta(x)) - G(x_1, y_1) - \partial_x G(x_1, y_1)[x - x_1] - \partial_y G(x_1, y_1)\Big[y_1'[x - x_1]\Big] \\ &= G(x, \eta(x)) - G(x_1, y_1) - \partial_x G(x_1, y_1)[x - x_1] - \partial_y G(x_1, y_1)\Big[\eta(x) - y_1\Big] \\ &\quad + \partial_y G(x_1, y_1)\Big[\eta(x) - y_1 - y_1'[x - x_1]\Big], \end{aligned}$$

so that

$$\begin{aligned} &\Big(\mathbb{1} - \partial_y G(x_1, y_1)\Big)\Big[\eta(x) - \eta(x_1) - y_1'[x - x_1]\Big] \\ &= G(x, \eta(x)) - G(x_1, y_1) - \partial_x G(x_1, y_1)[x - x_1] - \partial_y G(x_1, y_1)[\eta(x) - y_1]. \end{aligned}$$

As in the previous proof, we show that

$$\lim_{x \to x_1} \frac{\Big\|G(x, \eta(x)) - G(x_1, y_1) - \partial_x G(x_1, y_1)[x - x_1] - \partial_y G(x_1, y_1)[\eta(x) - y_1]\Big\|}{\|x - x_1\|} = 0$$

and the result follows since $\mathbb{1} - \partial_y G(x_1, y_1)$ is invertible. □

Finally, we deduce the following result.

Corollary E.7. *Under the assumptions of Theorem E.1, if further X is a part of a normed space and, for all $x \in V$, $\partial_x F(x, y)$ exists where $y := \eta(x)$, and $\partial_y F$ is continuous at (x, y), then η is differentiable.*

If moreover $\partial_x F$ is continuous at $(x, \eta(x))$, then $\partial_x \eta$ is continuous at x.

Appendix F
The evolution equation
Blow-up phenomena vs stationary waves

Even though this thesis only studies *elliptic* equations, we think it is important to highlight the link between our problems and questions related to the study of the nonlinear Schrödinger *evolution* equation. Below, we will present it on $\mathbb{R}^N$, but it could be studied more generally on domains of $\mathbb{R}^N$ or on metric graphs.

Remark. This appendix relies on the introduction of my Master's thesis [159].

F.1 The Cauchy problem

Let us consider the Cauchy problem associated to the evolution equation

$$\boldsymbol{i}\partial_t\Psi(t,x)+\Delta\Psi(t,x)+|\Psi(t,x)|^{p-2}\Psi(t,x)=0,\quad (t,x)\in[0,+\infty)\times\mathbb{R}^N, \quad (\mathrm{NLS_{evol}})$$

with the initial condition

$$\Psi(0,x)=\Psi_0(x),\quad x\in\mathbb{R}^N,$$

where $\Psi_0\in H^1(\mathbb{R}^N;\mathbb{C})$, $2<p<2^*$ and

$$2^*:=\begin{cases}\frac{2N}{N-2} & \text{si } N\geq 3,\\ +\infty & \text{si } N\in\{1,2\}\end{cases}$$

is the critical Sobolev exponent.

A result of J. Ginibre and G. Velo (see [163]) implies that ($\mathrm{NLS_{evol}}$) is well-posed in $H^1(\mathbb{R}^N;\mathbb{C})$. Thus, for every function $\Psi_0\in H^1(\mathbb{R}^N;\mathbb{C})$, there exists a time $T\in(0,+\infty]$ and a unique maximal solution $\Psi\in\mathcal{C}\Big([0,T);H^1(\mathbb{R}^N;\mathbb{C})\Big)$ to the Cauchy problem associated to ($\mathrm{NLS_{evol}}$) with the initial condition Ψ_0.

The well-posedness results are similar to those of ordinary differential equations in finite dimension. In particular, there exists a "blow-up alternative" for ($\mathrm{NLS_{evol}}$). Namely:

- if $T=+\infty$, the solution is said to be *global*;
- if $T<+\infty$, then
$$\lim_{t\to T}\|\nabla u(t,\cdot)\|_{L^2}=+\infty$$
and we say that the solution *blows up in finite time* or that it is *explosive*.

Remark F.1. The existence of explosive solutions is linked to the nonlinearity of the equation. Indeed, all solutions of the linear Schrödinger equation $\boldsymbol{i}\partial_t u = -\Delta u$ are global.

Solutions of ($\mathrm{NLS}_{\mathrm{evol}}$) satisfy conservation laws associated to the symmetries of the problem. In particular, the L^2 *mass*

$$\|\Psi(t,\cdot)\|_{L^2} := \left(\int_{\mathbb{R}^N} |\Psi(t,x)|^2 \,\mathrm{d}x\right)^{1/2}$$

and the *energy*

$$E(\Psi(t,\cdot)) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\Psi(t,x)|^2 \,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |\Psi(t,x)|^p \,\mathrm{d}x$$

are preserved during the evolution. In other words, if Ψ is a solution to ($\mathrm{NLS}_{\mathrm{evol}}$), then the equalities $\|\Psi(t,\cdot)\|_{L^2} = \|\Psi_0\|_{L^2}$ and $E(\Psi(t,\cdot)) = E(\Psi_0)$ are satisfied for all $t \in [0,T)$.

The explanations above show that the mass and the energy introduced in the section II.5.1 play a role in the study of ($\mathrm{NLS}_{\mathrm{evol}}$). It is also the case for the *mass-critical exponent* $p_{\mathrm{crit}} = 2 + \frac{4}{N}$, as we will see in the following section.

F.2 Stationary waves

Unlike explosion phenomena, a *stationary wave* of the problem ($\mathrm{NLS}_{\mathrm{evol}}$) is a solution of the form

$$\Psi(t,x) = e^{\boldsymbol{i}\lambda t}\Psi_0(x).$$

The initial conditions Ψ_0 associated to stationary waves are those for which Ψ_0 is a (complex-valued) solution to the elliptic equation

$$-\Delta u + \lambda u = |u|^{p-2}. \tag{NLS}$$

Those solutions belong to the Nehari manifold (see Section II.4), which means that

$$\|\nabla\Psi_0\|_2^2 + \lambda\|\Psi_0\|_2^2 = \|\Psi_0\|_p^p. \tag{F.1}$$

Moreover, the Pohožaev[1] identity claims that the solutions of (NLS) satisfy

$$(N-2)\|\nabla\Psi_0\|_2^2 + \lambda N\|\Psi_0\|_2^2 = \frac{2N}{p}\|\Psi_0\|_p^p. \tag{F.2}$$

Among the nonzero solutions of the equation (NLS) on $\mathbb{R}^N$, only one is radial and positive: the *soliton* ϕ_1.

[1]Proved originally in [280]. A proof in the case of $\mathbb{R}^N$ can for instance be found in [335, Theorem B.3].

M.I. Weinstein showed in [332] that the interpolation inequality

$$\|u\|_{L^p} \leq \left(\frac{p}{2}\right)^{\frac{1}{p}} \frac{\|u\|_{L^2}^{1-s}\|\nabla u\|_{L^2}^{s}}{\|\phi_1\|_{L^2}^{p-2}} \text{ where } s := \frac{(p-2)N}{2p} \tag{F.3}$$

is satisfied for every function $u \in H^1(\mathbb{R}^N)$, which means[2] that ϕ_1 realises the equality case in the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p}^p \leq K_p \|u\|_{L^2}^{p-N\left(\frac{p}{2}-1\right)} \|\nabla u\|_{L^2}^{N\left(\frac{p}{2}-1\right)}$$

if the optimal constant K_p is used.

We will see in the following section that the inequality (F.3) allows in particular to exclude the explosion of solutions in the mass-subcritical regime.

The mass-critical exponent plays also a role when studying the sign of the energy of the solitary waves. Indeed, the energy of the solution $\Psi(t,x) = e^{i\lambda t}\Psi_0(x)$ associated to Ψ_0 is given by

$$E\Big(\Psi(t,\cdot)\Big) = E(\Psi_0) = \frac{1}{2}\|\nabla\Psi_0\|_{L^2}^2 - \frac{1}{p}\|\Psi_0\|_{L^p}^p = \frac{N(p-2)-4}{4p}\|\Psi_0\|_{L^p}^p,$$

where we used the law of the conservation of the enery and then the identities (F.1) and (F.2).

In other words, the energy of the solitary waves is negative if $p < 2 + \frac{4}{N}$, equal to zero if $p = 2 + \frac{4}{N}$ and positive if $p > 2 + \frac{4}{N}$.

F.3 Non-explosion in the mass-subcritical regime $\left(2 < p < 2 + \frac{4}{N}\right)$

Let $\Psi \in \mathcal{C}\Big([0,T); H^1(\mathbb{R}^N;\mathbb{C})\Big)$ be a solution of (NLS$_{\text{evol}}$) with initial condition Ψ_0 (i.e. so that $\Psi(0) = \Psi_0$). The conservation laws of mass and energy and the inequality (F.3) imply that for all $t \in [0,T)$, we have

$$\begin{aligned}
\|\nabla\Psi(t,\cdot)\|_{L^2}^2 &= 2E(\Psi(t,\cdot)) + \frac{2}{p}\|\Psi(t,\cdot)\|_{L^q}^p \\
&= 2E(\Psi_0) + \frac{2}{p}\|\Psi(t,\cdot)\|_{L^q}^p \\
&\leq 2E(\Psi_0) + \frac{\|\Psi_0\|_{L^2}^{p(1-s)}\ \|\nabla\Psi(t,\cdot)\|_{L^2}^{ps}}{\|\phi_1\|_{L^2}^{p-2}}.
\end{aligned}$$

If $p < 2 + \frac{4}{N}$, then $ps < 2$ (since $s := \frac{(p-2)N}{2p}$), so $\|\nabla\Psi(t,\cdot)\|_{L^2}^2$ is bounded uniformly in t and there is no explosion according to the blow-up alternative.

[2]Using a few computations based on the identities (F.1) and (F.2), see e.g. [159, 332] for more details.

F.4 An explosion threshold in the mass-critical regime ($p = 2 + \frac{4}{N}$)

If $p = 2 + \frac{4}{N}$, we can rewrite the inequality

$$\|\nabla\Psi(t,\cdot)\|_{L^2}^2 \leq 2E(\Psi_0) + \frac{\|\Psi_0\|_{L^2}^{p(1-s)} \, \|\nabla\Psi(t,\cdot)\|_{L^2}^{ps}}{\|\phi_1\|_{L^2}^{p-2}}$$

obtained above as

$$\|\nabla\Psi(t,\cdot)\|_{L^2}^2 \left(1 - \frac{\|\Psi_0\|_{L^2}^{4/N}}{\|\phi_1\|_{L^2}^{4/N}}\right) \leq 2E(\Psi_0).$$

Therefore, if $\|\Psi_0\|_{L^2} < \|\phi_1\|_{L^2}$, we also obtain a bound on $\|\nabla\Psi(t,\cdot)\|_{L^2}$ which is uniform in t and there is no explosion.

It turns out that when $p = 2 + \frac{4}{N}$, explosion is possible for a L^2 norm equal to $\|\phi_1\|_{L^2}$, as shown by the explicit solution (see e.g. [238, Page 430], [153, Page 2])

$$s_T(t,x) := \left(\tfrac{T}{T-t}\right)^{N/2} \phi_1\left(\tfrac{xT}{T-t}\right) \exp\left(\boldsymbol{i}\left(\tfrac{Tt}{T-t} - \tfrac{|x|^2}{4(T-t)}\right)\right), \tag{F.4}$$

obtained by the "pseudo-conformal transformation" and which explodes at time $t = T$.

Remark. The complex exponential in the expression (F.4) plays a big role in the explosion phenomenon. Indeed, for all $x \in \mathbb{R}^N$, the equality $|s_T(0,x)| = |\phi_1(x)|$ holds, while the initial condition $\Psi_0 = \phi_1$ leads to the solitary wave $e^{\boldsymbol{i}t}\,\phi_1(x)$ which does not blow-up.

F.5 Explosion in the mass-supercritical regime ($2 + \frac{4}{N} < p < 2^*$)

In the L^2-supercritical regime, we do not know any *explicit*[3] explosive solution such as (F.4). Nevertheless, an argument due to R.T. Glassey [164], known as the *virial argument* (see [319, Section 3.8]), shows that explosion is possible. We will not present the historical details and refer to the original article of Glassey and to [98, Section 6.5] for the technical aspects .

We consider an initial condition $\Psi_0 \in H^1(\mathbb{R}^N;\mathbb{C})$ such that $|x| \cdot \Psi_0$ belongs to $L^2(\mathbb{R}^N;\mathbb{C})$. Let $\Psi \in \mathcal{C}\Big([0,T); H^1(\mathbb{R}^N;\mathbb{C})\Big)$ be the associated maximal solution (where T belongs to $(0,+\infty]$). Then, one can show that the *variance* of $|\Psi(t,x)|^2$, given by

$$V(t) := \int_{\mathbb{R}^N} |x|^2 |\Psi(t,x)|^2 \,\mathrm{d}x$$

is well defined for all $t \in [0,T)$.

[3]The solution (F.4) is linked to extra invariances of the problem when $p = 2 + \frac{4}{N}$.

Integrating by parts and using the evolution equation, one can show that

$$\partial_{tt} V(t) = 16E(\Psi_0) - \frac{4(N(p-2)-4)}{p} \|\Psi\|_{L^p}^p.$$

Since $p \geq 2 + \frac{4}{N}$, we obtain

$$\partial_{tt} V(t) \leq 16E(\Psi_0).$$

Therefore, the function $[0, T) \to [0, +\infty) : t \mapsto V(t)$ is positive and strictly concave because $\partial_{tt} V(t) \leq E(\Psi_0) < 0$. This gives a contradiction if $T = +\infty$ because there does not exist any function which is positive and strictly concave on $[0, +\infty)$. Therefore, T is finite and the solution Ψ blows up.

F.6 Dynamics of the explosion when $p = 2 + \frac{4}{N}$

Many works studying the dynamical properties of $(\text{NLS}_{\text{evol}})$ exist. For instance, a result of F. Merle [238] shows that solutions of the form (F.4) are the only explosive solutions of $(\text{NLS}_{\text{evol}})$ having minimal L^2-mass when $p = 2 + \frac{4}{N}$, up to the symmetries of the equation.

In this mass-critical case, a detailed understanding of the behavior of explosive solutions of the equation was obtained by F. Merle and P. Raphaël in [239, 240, 241] (modulo the verification of a "spectral property" verified solely by numerical methods, see [153, 337]). The interested reader is invited to consult the overview articles [88, 97].

Now, let us focus on the question of *orbital stability* of solitary waves, opposed to the explosion phenomena and related to the study of normalized solutions as performed in the chapters 3 and 4 of the thesis.

F.7 Orbital stability and links with normalized solutions

As T. Cazenave and P.L. Lions [99], we say that a solution $\Psi_0 \in H^1(\mathbb{R}^N; \mathbb{C})$ of (NLS), associated to the solitary wave $e^{it}\, \Psi_0(x)$, is *orbitally stable* if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|\Phi_0 - e^{i\theta}\, \Phi_0(\cdot + y)\|_{H^1(\mathbb{R}^N)} < \delta,$$

then the solution $\Phi(t, x)$ of $(\text{NLS}_{\text{evol}})$ is global and satisfies the inequality

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|\Phi(t, \cdot) - e^{i\theta}\, \Phi_0(\cdot + y)\|_{H^1(\mathbb{R}^N)} < \varepsilon$$

for all $t \geq 0$.

The contribution of Cazenave and Lions[4] consists in showing that energy ground states exist on $\mathbb{R}^N$ in the mass-subcritical case and that such minima of the energy on the mass constraint are orbitally stable.

Remark. Other methods to study the orbital stability exist. Let us mention the celebrated method of M.I. Weinstein, M. Grillakis, J. Shatah and W. Strauss [172, 173, 330, 331] based on the properties of the linearized equation around the stationary solution.

In the mass-subcritical regime, energy ground states exist and are orbitally stable (see section II.5).

In the mass-supercritical regime, there are no more energy ground states since the energy is not anymore bounded from below on the mass constraint (see the discussion in section II.5). Then, a result of H. Berestycki and T. Cazenave [76, Theorem 1] implies that all action ground states[5]are *unstable by blow-up.* More precisely, the authors show that when $2 + \frac{4}{N} < p < 2^*$, if Ψ_0 is an action ground state of (NLS), then for all $\varepsilon > 0$, there exists[6] $\tilde{\Psi}_0 \in H^2(\mathbb{R}^N; \mathbb{C})$ such that

$$\|\Psi_0 - \tilde{\Psi}_0\|_{H^2(\mathbb{R}^N)} < \varepsilon$$

and such that the solution of ($\mathrm{NLS_{evol}}$) corresponding to the initial condition $\tilde{\Psi}_0$ blows up. In particular, Ψ_0 is not orbitally stable.

F.8 A few words on integrability in dimension one in the case $p = 4$

The equation ($\mathrm{NLS_{evol}}$) possesses *integrability* properties in dimension one when $p = 4$. In this case, the equation is given by

$$i\partial_t \Psi(t,x) + \partial_{xx}\Psi(t,x) + |\Psi(t,x)|^2\Psi(t,x) = 0, \tag{F.5}$$

where $(t,x) \in [0,+\infty) \times \mathbb{R}^N$.

For finite dimensional Hamiltonian systems such as those coming from classical mechanics, the notion of integrability is essentially translated by the presence of enough independent first integrals of the equations. The reader can consult [205, Section 14.1], [306, Chapter 4], [33, Section 49], [34], or [323] for more information.

[4]For ($\mathrm{NLS_{evol}}$). Other evolution equations are also considered in [99].

[5]Let us remark that the notion of action ground state is well-defined even when $p > 2 + \frac{4}{N}$. Let us also mention that energy ground states, when they exist, are necessarily action ground states (see [129, Theorem 1.3] and section II.10.3). Therefore, action ground states are the most natural candidates to replace energy ground states when moving from the subcritical to the supercritical regime.

[6]Even though we mostly use $H^1(\mathbb{R}^N)$ in our explanations, elliptic regularity results imply that solutions of (NLS) belong to $H^2(\mathbb{R}^N)$. For more details about the regularity of solutions to (NLS), see e.g. [98, Theorem 8.1.1].

When studying the dynamics of partial differential evolution equations, there exist several notions of integrability not always easy to define[7].

In 1971, V.E. Zakharov and A.B. Shabat have shown (see [338]) that (F.5) possesses integrability properties. For this, those authors have used the *inverse scattering transform*[8] method.

We will not detail precisely how this method operates. Nevertheless, let us mention that this technique allows for instance to construct[9] "N-solitons" (see [338, Section 4]). The case $p = 4$ in dimension one is thus the "most understood" in the theory of the nonlinear Schrödinger evolution equation. Therefore, it is not surprising that the case $p = 4$ was the first to be studied on metric graphs, as seen in section II.3.3.

To conclude, let us mention that integrability techniques as the one of Zakharov and Shabat have also been developed for metric graphs, see [96].

[7]The theory of integrability is very vast and entire books are devoted to it. We refer to [144, 262] for more information.

[8]Thanks to Prof. Yves Brihaye for his course on *solitons* in which we discovered the inverse scattering method for the Korteweg–de Vries equation.

[9]The inverse scattering method and integrability properties are also useful in order to study the *soliton resolution* of equation (F.5). This notoriously delicate question amounts to show that, generically and in large time, solutions of the equation decompose into a part made of translating solitons and a "scattering" part which has a rather "linear" behavior. We refer to [320] for a discussion on this vast subject and to [79] for an application of the inverse scattering method to study the soliton resolution for (F.5).

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